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A Generalisation of Tverberg's Theorem

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Abstract We will prove the following generalisation of Tverberg's Theorem: given a set $S \subset \mathbb{R}^d$ of $(r+1)(k-1)(d+1)+1$ points, there is a partition of S in k sets $A_1, A_2, \dots, A_k$ such that for any $C \subset S$ of at most r points, the convex hulls of $A_1 \backslash C, A_2 \backslash C, \dots, A_k \backslash C$ are intersecting. This was conjectured first by Natalia García-Colín (Ph.D. thesis, University College of London, 2007).

Keywords Tverberg partition · Tolerance

1 Introduction

In 1921 Johann Radon published a proof of Helly's theorem in which he used what is known today as Radon's lemma (or Radon's theorem): *for any* $d+2$ *points in* $\mathbb{R}^d$ *there is always a Radon partition* [7]; that is, there is a pair of disjoint subsets of those points whose convex hulls are intersecting. This result has been the basis of a wide number of generalisations [3]. One of these is due to Helge Tverberg in 1966 [11]. He proved that *for every* $(k-1)(d+1)+1$ *points in* $\mathbb{R}^d$ *there is a* k*-Tverberg partition*; that is, there are k disjoint subsets of those points whose convex hulls are intersecting. In both these theorems, the given numbers were proven to be tight.

The "tolerated" versions of these theorems started to appear in 1972 when David Larman solved a nice problem proposed to him by Peter McMullen [5]. The problem translates, via Gale's diagrams, to the following: *for any* $2d+3$ *points in* $\mathbb{R}^d$ *there is*

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a Radon partition with tolerance 1. That is, for any $2d+3$ points there is partition of them in two parts A and B, such that for any point x, the convex hulls of $A\backslash\{x\}$ and $B\backslash\{x\}$ are intersecting. This result was later generalised by Natalia García-Colin in her Ph.D. thesis [4] with Larman as her supervisor, saying that *for any* $(r+1)\times(d+1)+1$ *points in* $\mathbb{R}^d$ *there is a Radon partition with tolerance* r. It is interesting that García-Colin's proof of the tolerated Radon theorem is very similar to Tverberg's original proof.

In both proofs one finds a suitable set of points that has a k-Tverberg (a tolerated Radon) partition, and starts to move continuously one point. When the k-Tverberg (the tolerated Radon) partition stops working, one can prove that there is a suitable rearrangement of the partition that solves the corresponding problem.

Larman's result has been proven to be optimal only for $d \leq 4$ (see also [2]). The best lower bound so far is $3+\lceil\frac{5d}{3}\rceil$ which was proven by Jorge Ramírez-Alfonsín in [8].

Other tolerated theorems have begun to appear recently, such as the tolerated versions of Helly's and Carathéodory's theorems (see [6] and the references therein).

In [4] the author conjectures a tolerated version of Tverberg's theorem. In this paper we give a positive answer to that conjecture. The main theorem reads as follows:

Theorem 1 *Let* $k\geq 2, d\geq 1, r\geq 0$ *be integers. If* $S\subset\mathbb{R}^d$ *is a set of at least* $(r+1)(k-1)(d+1)+1$ *points, there is a k-Tverberg partition of S with tolerance r.*

2 Preliminaries

Given a set $S\subset\mathbb{R}^d$ we denote by $\langle S\rangle := \operatorname{conv}(S)$ its convex hull. A Radon partition is a pair of disjoint sets A, B such that $\langle A\rangle\cap\langle B\rangle\neq\emptyset$. A k-Tverberg partition of S is a partition of S in k sets $A_1, A_2, \dots, A_k$ such that $\bigcap_{i=1}^k\langle A_i\rangle\neq\emptyset$ (a Radon partition is a 2-Tverberg partition). A k-Tverberg partition with tolerance r, or simply an r-tolerated k-Tverberg partition, of S will be a k-Tverberg partition such that the convex hulls of the parts have a point of common even after we remove any r points of S. We will say S captures the origin with tolerance r if $0\in\langle S\backslash C\rangle$ for any $C\subset S$ of at most r points.

Definition 1 *Given two sets* $S\subset S'\subset\mathbb{R}^d$ *of points, and a group* G, *we will say that an action of* G *in* S' *is compatible with* S *if the following two properties hold.*

- *If* $A\subset S'$ *captures the origin, then* gA *captures the origin for any* $g\in G$.
- *Given a point* $x\in S$, *then* Gx *captures the origin.*

It is clear that an action of G that is compatible with S is also compatible with any subset of S.

$$\begin{array}{c} G \\ \Big\downarrow \end{array}\quad \left.\begin{array}{l} \left.\begin{array}{ccccc} g_1s_1 & g_1s_2 & g_1s_3 & \cdots & g_1s_n \end{array}\right\} S \\ \begin{array}{ccccc} g_2s_1 & g_2s_2 & g_2s_3 & \cdots & g_2s_n \\ \vdots & \vdots & \vdots & & \vdots \\ g_ts_1 & g_ts_2 & g_ts_3 & \cdots & g_ts_n \end{array} \end{array}\right\} S'$$

The following lemma is the core of the main theorem's proof.

Lemma 1 *Let $p \geq 1$ and $r \geq 0$ be integers, $S \subset \mathbb{R}^p$ a set of $n = p(r+1)+1$ points $a_1, a_2, \dots, a_n$ and G a group with $|G| \leq p$. If there is an action of G in a set $S' \subset \mathbb{R}^p$ which is compatible with $S \subset S'$, then for each a_i there is a $g_i \in G$ such that the set $\{g_1 a_1, g_2 a_2, \dots, g_n a_n\}$ captures the origin with tolerance r.*

Proof We proceed by induction on r. For $r = 0$, the lemma is a direct consequence of the Bárány–Lovász generalisation of Carathéodory theorem [1], taking $Ga_1, Ga_2, \dots, Ga_n$ as the colour classes.

Searching for a contradiction, suppose the lemma is true for $r-1$ and false for r. Let $\{g_1, g_2, \dots, g_t\}$ be the elements of G with $t \leq p$. Given any vector $\alpha = (g_1, g_2, \dots, g_n)$ in G^n, let $\alpha \cdot S = \{g_1 a_1, g_2 a_2, \dots, g_n a_n\}$. Since we are supposing that the lemma is false, for any α there is a subset $C \subset \alpha \cdot S$ of r points such that $(\alpha \cdot S) \backslash C$ does not capture the origin. For each α, let

$$P(\alpha) = \max_{|C|=r} \operatorname{dist}\big(\big\langle(\alpha \cdot S)\backslash C\big\rangle, 0\big).$$

Observe that $P(\alpha) > 0$ for all α.

Let α_0 be the vector in G^n such that $P(\alpha_0)$ is minimal, and let $C_0 \subset \alpha_0 \cdot S$ be a set of r points such that $\operatorname{dist}(\langle(\alpha_0 \cdot S)\backslash C_0\rangle, 0) = P(\alpha_0)$. If p_0 is the point of $\langle(\alpha_0 \cdot S)\backslash C_0\rangle$ closest to the origin, p_0 must be in a face of $\langle(\alpha_0 \cdot S)\backslash C_0\rangle$. Thus, there is a set $A \subset (\alpha_0 \cdot S)\backslash C_0$ of at most p points such that p_0 is in the relative interior of $\langle A\rangle$. Let $B = (\alpha_0 \cdot S)\backslash A$ and H be a hyperplane that contains A and leaves the origin in one of its open half-spaces H^-.

By induction, since the action of G is compatible with B and B has at least $pr+1$ points, there is a vector β of $G^{|B|}$ such that $\beta \cdot B$ captures the origin with tolerance $r-1$. Since Gx captures the origin for all $x \in B$, for each $b \in B$ there is a g_i such that $g_i b \in H^-$. Consider the sets $(g_1\beta) \cdot B, (g_2\beta) \cdot B, \dots, (g_t\beta) \cdot B$. Using $|G| \leq p$ we can apply the pigeonhole principle to find a $g \in G$ such that $(g\beta) \cdot B$ contains at least $r+1$ points in H^-.

Let α_1 be the vector in G^n that results in changing in α_0 the elements corresponding to B for those of $(g\beta) \cdot B$. We claim that $P(\alpha_1) < P(\alpha_0)$. Let $C' \subset \alpha_1 \cdot S$ be a set of r points. If in those r points there are at most $r-1$ of $(g\beta) \cdot B$, then $(\alpha_1 \cdot S)\backslash C'$ captures the origin. If $C' \subset (g\beta) \cdot B$, then there is a point $x \in H^- \cap ((g\beta) \cdot B)$ that is not in C'. It follows that $\langle A \cup \{x\}\rangle$ is closer to the origin than $\langle A\rangle$; thus $P(\alpha_1) < P(\alpha_0)$, which contradicts the minimality of $P(\alpha_0)$. □

3 The Main Theorem

Using Lemma 1 instead of the Bárány–Lovász theorem, the proof resembles Karanbir Sarkaria's proof of Tverberg's theorem [9].

Proof of Theorem 1 Let $n = (r+1)(k-1)(d+1)+1$, $p = (k-1)(d+1)$ and $S_0 = \{a_1, a_2, \dots, a_n\} \subset \mathbb{R}^d$. Let $u_0, u_1, \dots, u_{k-1}$ be the k vertices in $\mathbb{R}^{k-1}$ of a regular simplex centred at the origin. Thus $\alpha_0 u_0 + \alpha_1 u_1 + \cdots + \alpha_{k-1} u_{k-1} = 0$ if and only if $\alpha_0 = \alpha_1 = \cdots = \alpha_{k-1}$. For $1 \le i \le n$, let $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$ be the vectors in $\mathbb{R}^{d+1}$ that result in adding a coordinate 1 to each a_i.

Let $S' := \{u_i \otimes b_j \mid 0 \le i \le k-1,\ 1 \le j \le n\} \subset \mathbb{R}^p$ and $S := \{u_0 \otimes b_j \mid 1 \le j \le n\}$. There is a natural action of $\mathbb{Z}_k$ in S' defined as $m(u_i \otimes b_j) = u_{i+m} \otimes b_j$, where the sum is taken mod k.

$$
\begin{array}{c|ccccc}
\mathbb{Z}_k & u_0 \otimes b_1 & u_0 \otimes b_2 & u_0 \otimes b_3 & \dots & u_0 \otimes b_n \\
\downarrow & u_1 \otimes b_1 & u_1 \otimes b_2 & u_1 \otimes b_3 & \dots & u_1 \otimes b_n \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & u_{k-1} \otimes b_1 & u_{k-1} \otimes b_2 & u_{k-1} \otimes b_3 & \dots & u_{k-1} \otimes b_n
\end{array}
$$

Since the set $\{u_0, u_1, \dots, u_{k-1}\}$ captures the origin in $\mathbb{R}^{k-1}$, we have that $\mathbb{Z}_k a$ captures the origin in $\mathbb{R}^p$ for every $a \in S$.

Given a subset $A \subset S'$ that captures the origin, since the simplex with vertices $\{u_0, u_1, \dots, u_{k-1}\}$ is regular, the coefficients of the convex combination of A that gave 0 work in gA to give 0 again for any $g \in \mathbb{Z}_k$. So gA also captures the origin. Observe that $p \ge k$. Thus, applying Lemma 1, one can choose elements $g_1, g_2, \dots, g_n$ in $\mathbb{Z}_k$ such that the set $\{g_1(u_0 \otimes b_1), g_2(u_0 \otimes b_2), \dots, g_n(u_0 \otimes b_n)\}$ captures the origin with tolerance r.

We will prove that the sets $A_i = \{a_j \mid g_j = i\}$ for $0 \le i \le k-1$ form the k-Tverberg partition we are looking for. Consider $H_i = \{j \mid g_j = i\}$ the set of indices of A_i.

Let C be a subset of $\{1, 2, \dots, n\}$ of r elements, and consider the sets $A'_i = \{a_j \mid j \in H_i \backslash C\}$. We want to prove that $\bigcap_{i=1}^k \langle A'_i \rangle \ne \emptyset$.

Since the set $\{u_{g_j} \otimes b_j \mid j \notin C\}$ captures the origin, there are nonnegative real numbers not all zero $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\sum_{i \notin C} \alpha_i (u_{g_i} \otimes b_i) = 0$. The last coordinate of the b_i is 1, so we have that $\sum_{i \notin C} \alpha_i u_{g_i} = 0$. Thus there is a positive T such that:

$$T = \sum_{i \in H_1 \backslash C} \alpha_i = \sum_{i \in H_2 \backslash C} \alpha_i = \cdots = \sum_{i \in H_k \backslash C} \alpha_i.$$

We also know that for all $1 \le t \le d$, $\sum_{i \notin C} \alpha_i (a_i)_t u_{g_i} = 0$, where $(a_i)_t$ denotes the tth coordinate of a_i. So, we obtain

$$\sum_{i \in H_1 \backslash C} \alpha_i (a_i)_t = \sum_{i \in H_2 \backslash C} \alpha_i (a_i)_t = \cdots = \sum_{i \in H_k \backslash C} \alpha_i (a_i)_t.$$

If we take $\beta_i = \frac{a_i}{T}$, we obtain that $\sum_{i \in H_j \backslash C} \beta_i = 1$ for all $1 \le j \le k$ and

$$\sum_{i \in H_1 \backslash C} \beta_i a_i = \sum_{i \in H_2 \backslash C} \beta_i a_i = \cdots = \sum_{i \in H_k \backslash C} \beta_i a_i.$$

Since these are convex combinations, we have found a point in $\bigcap_{i=1}^k \langle A'_i \rangle$.

We conjecture that the number of points in this theorem is tight; that is:

Conjecture 1 *For every triple of integers $k \geq 2, d \geq 1, r \geq 1$ there is a set S of $(r+1)(k-1)(d+1)$ points in $\mathbb{R}^d$ with no r-tolerated k-Tverberg partition.* □

4 Remarks on the Proof

The proof of the main theorem is made by extending Sarkaria's proof of Tverberg's Theorem for the tolerated case. It seems at first glance that instead of Lemma 1 one could try to extend the Bárány–Lovász theorem in a tolerated way. However, this turns out to be false.

False Claim *Given $d(r+1)+1$ colour classes that capture the origin in $\mathbb{R}^d$, there is a colourful choice that captures the origin with tolerance r.*

For, it suffices to take each colour class as the vertices of the same simplex in $\mathbb{R}^d$. Since a colourful choice has less than $(r+1)(d+1)$ points, there is a vertex that was used at most r times. Removing those points gives us a set contained in a face of the simplex, which does not capture the origin. This also shows that the Bárány–Lovász Theorem does not have a non-trivial tolerated version—$(r+1)(d+1)$ colour classes are necessary and sufficient—which is also interesting by itself.

If one tries to prove only the case $k=2$, it is easy to see that the only action of $\mathbb{Z}_2$ compatible with a set S is $(-1)x = -x$ except for positive factors. Also, the tensor product in the main proof is not necessary, since we only need to choose between b_i and $-b_i$ in $\mathbb{R}^{d+1}$ for each i. This yields a simpler proof of García–Colin's result.

Sarkaria's quick proof of Tverberg's Theorem can be seen in its most natural context perhaps in his subsequent paper [10]. Lemma 1 is essentially a tolerated generalisation of the "linear" Borsuk–Ulam theorem (2.4) of that paper—viz., *if an N-dimensional representation $\mathbb{E}$ of a group G with $|G| \leq N$ does not contain the trivial representation, then, under any linear G-map $E_{N(r+1)}(G) \to \mathbb{E}$, some simplex of $E_{N(r+1)}(G)$ shall capture the origin with tolerance r* (here $E_t(G)$ denotes the join of $t+1$ copies of G)—which yields Theorem 1 for the case of the matrix representation $\mathbb{E} = \mathbb{L}^{\perp}$ defined in [10]; and there may well also be a tolerated generalisation of its "continuous" theory.

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