Commun. math. Phys. 9, 142—145 (1968)

A Generalization of a Theorem by Wightman

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Received December 10, 1967

Abstract. It is shown for some class of sets in the Minkowski space that the intersection of local Haag algebras assigned to open neighbourhoods of such a set contains only multiples of the identity.

In the paper [1] A. S. WIGHTMAN has shown for one point subset K of the Minkowski space, that the intersection of local Haag algebras [2] assigned to open neighbourhoods of the set K contains only multiples of the identity. In the present paper, this theorem is proved for a larger class of sets K. The similar results are contained in the paper [3].

For any open subset $\mathcal{O}$ in the Minkowski space X, $R(\mathcal{O})$ will mean the v. Neumann algebra of operators in the Hilbert space H, associated with $\mathcal{O}$. Let $U(\cdot)$ be the representation of the translation group. $R(\cdot)$ and $U(\cdot)$ have the following properties:

1. Translation invariance:

$$U(a)\,R(\mathcal{O})\,U(a)^{-1} = R(\mathcal{O}+a)\,.$$

2. Locality: If $\mathcal{O}$ is spacelike to $\mathcal{O}_1$, then

$$R(\mathcal{O}) \subset [R(\mathcal{O}_1)]'\,.$$

3. Spectral condition:

$$U(a) = \int e^{-iap}\,dE(p)$$

where $dE(\cdot)$ is a spectral measure with the support contained in the set $\{p\colon p^2 \geqq 0,\ p_0 \geqq 0\}$.

4. Uniqueness of the vacuum state: There is one and only one vector $\Omega \in H$ such that, for all translations $a \in X$: $U(a)\,\Omega = \Omega$.

5. Cyclicity of the vacuum state: The set

$$\{A\Omega : A \in R(\mathcal{O})\,,\ \mathcal{O} \subset X\}$$

is dense in H.

We shall say that a compact set $K \subset X$ fulfills the condition C, if there exist vectors $w_1, w_2 \in X$ such that: $w_1^2 = w_2^2 = -1$, $(w_1 - w_2)^2 \geqq 0$ and for all $a, b \in K$: $(a-b)^2 \leqq 0$, $|(a-b)\,w_i| \leqq \sqrt{-(a-b)^2}$ $\quad i = 1, 2$.

Theorem I. *If the compact set K fulfills the condition C, then*

$$\bigcap_{\mathcal{O}\supset K} R(\mathcal{O}) = \{\lambda I : \lambda \in C^1\}$$

where $\mathcal{O}$ runs over all open neighbourhoods of K.

Proof. One can choose the Lorentz coordinate system such that

$$w_1 = \left(\frac{v}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}}, 0, 0\right)$$

$$w_2 = \left(\frac{-v}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}}, 0, 0\right)$$

where $0 < |v| < 1$.

The following lemma is the simple conclusion of the condition C.

Lemma. *If* $|t| < |vx|$, *then the sets* K *and* $K + (t, x, 0, 0)$ *have neighbourhoods* $\mathcal{O}$ *and* $\mathcal{O}_1$ *such that* $\mathcal{O}$ *is spacelike to* $\mathcal{O}_1$.

Let

$$B \in \bigcap_{\mathcal{O} \supset K} R(\mathcal{O})$$

and

$$W(t, x) = (B\Omega \mid U(t, x, 0, 0)\, B\Omega)$$

$$W_*(t, x) = (B^*\Omega \mid U(t, x, 0, 0)\, B^*\Omega)$$

Using the spectral conditions, we have

$$W(t, x) = \int e^{-i(Et - px)}\, d\mu(E, p)$$

$$W_*(t, x) = \int e^{-i(Et - px)}\, d\mu_*(E, p)\,,$$

where $d\mu(\cdot)$ and $d\mu_*(\cdot)$ are finite positive measure with supports in $\{(E, p) : E \geqq |p|\}$.

The translation invariance implies

$$U(t, x, 0, 0)\, B\, U(-t, -x, 0, 0) \in \bigcap_{\mathcal{O} \supset K + (t, x, 0, 0)} R(\mathcal{O})\,.$$

Let $|t| < |vx|$. From the lemma and the locality condition it follows, that

$$B^* U(t, x, 0, 0)\, B\, U(-t, -x, 0, 0) = U(t, x, 0, 0)\, B\, U(-t, -x, 0, 0) B^*.$$

In this case (see 4.)

$$\begin{aligned} W(t, x) &= (\Omega \mid B^* U(t, x, 0, 0)\, B\, U(-t, -x, 0, 0)\, \Omega) \\ &= (\Omega \mid U(t, x, 0, 0)\, B\, U(-t, -x, 0, 0)\, B^*\Omega) \\ &= (\Omega \mid B\, U(-t, -x, 0, 0)\, B^*\Omega) = W^*(-t, -x)\,. \end{aligned}$$

This means that $W(t, x) = W_*(-t, -x)$ for (t, x) such that $|t| \leqq |vx|$ (because W and W_* are continuous functions).

Let

$$M(t, x) = W(t, x) - W_*(-t, -x)\,.$$

The function M has the following properties:

(*) $$M(t, x) = \int e^{-i(Et - px)}\, d\sigma(E, p)$$

where $d\sigma(E, p) = d\mu(E, p) - d\mu_*(-E, -p)$ is a positive measure for $E > 0$ and negative one for $E < 0$.

(**) $$M(t, x) = 0 \quad \text{if} \quad |t| \leqq |vx|\,.$$

Let

$$A_{\pm} = \{(t, x) : t = \pm vx\} .$$

From $(*)$ it follows that the restriction to the axis A_+ (resp. A_-) of the function M is the Fourier transform of the projection of the measure $d\sigma(\cdot)$ on A_+ (resp. A_-). The property $(**)$ tells us that the projection of the measure $d\sigma(\cdot)$ is zero. This means, that:

$$\int_{S_{p_0}^+} d\sigma(E, p) = 0 , \quad \int_{S_{p_0}^-} d\sigma(E, p) = 0 ,$$

where

$$S_{p_0}^{\pm} = \{(E, p) : \pm Ev > p - p_0\} .$$

Subtracting left hand sides of these last equations we have

$$0 = \Big(\int_{S_{p_0}^+} - \int_{S_{p_0}^-} \Big) d\sigma(E, p) = \Big(\int_{I_{p_0}^+} - \int_{I_{p_0}^-} \Big) d\sigma(E, p) ,$$

where

$$I_{p_0}^+ = S_{p_0}^+ - S_{p_0}^- \quad \text{and} \quad I_{p_0}^- = S_{p_0}^- - S_{p_0}^+ .$$

But the measure $d\sigma(\cdot)$ is positive on $I_{p_0}^+$ and negative on $I_{p_0}^-$.

By this reason:

$$0 = \int_{I_{p_0}^+} d\sigma(E, p) = \int_{I_{p_0}^+} d\mu(E, p) .$$

The semiplane $\{(E, p) : E > 0\}$ can be covered by countable family of sets $I_{p_0}^+$ (p_0 – rational). This means, that the measure $d\mu(\cdot)$ is concentrated at the point $(0, 0)$ and

$$W(t, x) = \int e^{-(Et - px)} \, d\mu(E, p) = \text{const} .$$

According to the definition of W:

$$(B\Omega \mid U(t, x, 0, 0) \, B\Omega) = (B\Omega \mid B\Omega) .$$

It is easily seen, that the last equation implies the following one:

$$U(t, x, 0, 0) \, B\Omega = B\Omega .$$

Using the spectral condition one can prove, that any vector invariant under time translation is invariant under all translations. Now, from the uniqueness of the vacuum state it follows, that

$$B\Omega = \lambda\Omega$$

where $\lambda \in C^1$. Since Ω is a separating vector of the algebra $R(\mathcal{O})$ ($\mathcal{O}$ is a open neighbourhood of K), then

$$B = \lambda I .$$

This completes the proof.

The next theorem gives us interesting examples of sets fulfilling the condition C.

Theorem II. *Let M be a smooth (C^1 class) two-dimensional submanifold of X. Let $p \in M$ such that the tangent plane $\mathcal{T}_p(M)$ lies outside the light cone ($\mathcal{T}_p(M)$ contains neither timelike nor isotropic vectors). Then p has the compact neighbourhood K in M which fulfills the condition C.*

Proof. We can assume that $p = 0$ and $\mathcal{T}_p(M) = \{(0, 0, y, x) : y, z \in \mathbb{R}^1\}$. Points of a sufficiently small neighbourhood of the point p in M fulfil the following equations:

$$t = \tau(y, z)\,, \qquad x = \xi(y, z)\,,$$

where τ, $\xi \in C^1(\mathcal{O})$; $\mathcal{O}$ is some neighbourhood of zero in $\mathbb{R}^2$. We have also

$$\frac{\partial \tau}{\partial y} = \frac{\partial \tau}{\partial z} = \frac{\partial \xi}{\partial y} = \frac{\partial \xi}{\partial z} = 0 \quad \text{for} \quad (y, z) = (0, 0)\,.$$

Let U be a compact, convex neighbourhood of zero in $\mathbb{R}^2$ such that

$$\left|\frac{\partial \tau}{\partial y}\right| \leqq \frac{1}{2\sqrt{3}}\,, \quad \left|\frac{\partial \tau}{\partial z}\right| \leqq \frac{1}{2\sqrt{3}}\,, \quad \left|\frac{\partial \xi}{\partial y}\right| \leqq \frac{1}{2\sqrt{3}}\,, \quad \left|\frac{\partial \xi}{\partial z}\right| \leqq \frac{1}{2\sqrt{3}}$$

for $(y, z) \in U$.

Let

$$K = \{(\tau(y, z)\,,\, \xi(y, z), y, z) : (y, z) \in U\}$$

and

$$w_1 = \left(\frac{3}{4}, \frac{5}{4}, 0, 0\right)$$

$$w_2 = \left(-\frac{3}{4}, \frac{5}{4}, 0, 0\right).$$

Using the mean value theorem one can prove, that

$$(a - b)^2 \leqq 0 \quad \text{and} \quad |(a - b)\, w_i| \leqq \sqrt{-(a - b)^2} \quad i = 1, 2$$

for $a, b \in K$. This means, that K fulfills the condition C.

The autor is very grateful to Professor K. Maurin for his help and encouragement in this work.

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