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A generalization of Chaitin's halting probability Ω and halting self-similar sets

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Abstract. We generalize the concept of randomness in an infinite binary sequence in order to characterize the degree of randomness by a real number $D>0$. Chaitin's halting probability Ω is generalized to Ω^{D} whose degree of randomness is precisely D. On the basis of this generalization, we consider the degree of randomness of each point in Euclidean space through its base-two expansion. It is then shown that the maximum value of such a degree of randomness provides the Hausdorff dimension of a self-similar set that is computable in a certain sense. The class of such self-similar sets includes familiar fractal sets such as the Cantor set, von Koch curve, and Sierpiński gasket. Knowledge of the property of Ω^{D} allows us to show that the self-similar subset of $[0,1]$ defined by the halting set of a universal algorithm has a Hausdorff dimension of one.

Key words: algorithmic information theory, Kolmogorov complexity, randomness, Chaitin's Ω, Hausdorff dimension, self-similar set.

1. Introduction

The Kolmogorov complexity $H(s)$ of a finite binary sequence s is the size, in bits, of the shortest program for a universal algorithm U to calculate s. The concept of Kolmogorov complexity plays an important role in characterizing the randomness of an infinite binary sequence. In [C3], the four concepts of randomness in an infinite binary sequence (Chaitin, weak Chaitin, Martin-Löf, and Solovay randomness) are considered. These four concepts are shown to be equivalent to one another. In this paper, we first generalize these four concepts of randomness in order to deal with the degree of randomness of an infinite binary sequence. The degree of randomness is specified by a real number D with $0<D\leq 1$. As D becomes larger, the degree of randomness increases. In the case when $D=1$, the concept of the degree of randomness becomes the same as that of randomness. The relationship among the generalized concepts of randomness is investigated. Chaitin's halting probability Ω is an example of a random real number. We generalize Ω to Ω^{D} so that the degree of randomness of Ω^{D} is precisely D.

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Although the first n bits of Ω can solve the halting problem for a program of size not greater than n, the first n bits of Ω^D can solve the halting problem for a program of size not greater than Dn. Moreover, Ω^D is infinitely differentiable as a function of D, and each derivative $d^k\Omega^D/dD^k$ has the same properties as Ω^D.

On the basis of this generalization, we next study the relationship between the degree of randomness and Hausdorff dimension. Hausdorff dimension is closely related to Kolmogorov complexity, as studied by several researchers e.g., [R1], [R2], [St1], [CH], and [St2]. In these previous studies however, the normalized Kolmogorov complexity $\lim_{n\to\infty} H(x_n)/n$ of a real number x was considered, where x_n is the first n bits of the base-two expansion of x, and Hausdorff dimension was related to the normalized Kolmogorov complexity. That is to say, in [R1], [R2], [St1], and [St2], the Hausdorff dimension of a subset F of $\mathbb{R}$ was compared with the maximum value, or supremum, over the normalized Kolmogorov complexity $\lim_{n\to\infty} H(x_n)/n$ for all points x in F. (We recommend reading [St1] as a monograph.) On the other hand, [CH] considered the Hausdorff dimension of the graph of the normalized Kolmogorov complexity $\lim_{n\to\infty} H(x_n)/n$ as a function of x.

If an inifinite binary sequence is random, then its normalized Kolmogorov complexity is equal to one; however, the converse is not necessarily true. Thus, although the concept of normalized Kolmogorov complexity is related to randomness it alone cannot capture randomness. Corresponding to this fact, the concept of the degree of randomness which we introduce in this paper is more insightfull than that of the normalized Kolmogorov complexity. Consideration of the degree of randomness allows us to classify infinite sequences which have the same normalized Kolmogorov complexity. Hence, we study the relationship between Kolmogorov complexity and Hausdorff dimension using a more rigorous system than previous work.

We introduce six "algorithmic dimensions", 1st, 2nd, 3rd, 4th, upper, and lower algorithmic dimensions as fractal dimensions for a subset F of $\mathbb{R}^N$. These dimensions are defined by means of Kolmogorov complexity. On the one hand, the 3rd, 4th, upper, and lower algorithmic dimensions are related to the maximum value, or supremum, over the normalized Kolmogorov complexity for all points in F and were, in essence, researched by [R2] and [St1]. On the other hand, the 1st and 2nd algorithmic dimensions are related to the maximum value over the degree of randomness for all points in F. Therefore, they are stronger concepts with regard to the possibilities of their existence than the former four algorithmic dimensions. We show that all six algorithmic dimensions are equal to the Hausdorff dimension for any self-similar set that is computable in a certain sense. The class of such self-similar sets includes familiar fractal sets such as the Cantor set, von Koch curve, and Sierpiński gasket.

Based on the relationship between the definition of Ω^D and the mathematical theory of self-similar sets (e.g., [Hu], [Ha]), we define the self-similar subset F_{halt} of $[0,1]$ by using the halting set of a universal algorithm U. We may regard F_{halt} as the set of an endless succession of coded messages sent through a noiseless binary communication channel. From the property of Ω^D, it is shown that F_{halt} has a Hausdorff dimension of one and a zero-Lebesgue measure.

The paper is organized as follows. In the next section, we review some basic concepts from algorithmic information theory. We then treat the definition of Hausdorff dimension. Section 3 is devoted to a generalization of the concepts of randomness in an infinite binary sequence through the introduction of a real number D. Chaitin's halting probability Ω is also generalized. In Section 4, the six algorithmic dimensions for a subset of $\mathbb{R}^N$ are defined by means of Kolmogorov complexity, and their properties are investigated. The halting self-similar set F_{halt} is introduced in Section 5. The Hausdorff dimension and all six algorithmic dimensions of F_{halt} are evaluated.

2. Preliminary definitions

In this section, we first recall some basic notations from algorithmic information theory or the theory of Kolmogorov complexity. According to [C1], we use some variant of Kolmogorov complexity, i.e., Kolmogorov complexity based on self-delimiting programs; we recommend reading [C1].

$\#S$ is the cardinality of S for any set S. $\mathbb{N} \equiv \{0,1,2,3,\dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Z}$ is the set of integers, and $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}^N$ denotes N-dimensional Euclidean space, where $\mathbb{R}^1 = \mathbb{R}$ is just the set of real numbers. $X \equiv \{\Lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$ is the set of finite binary sequences, and X is ordered as indicated. For any $s \in X$, $|s|$ is the length of s. X^∞ is the set of infinite binary sequences. For any $\alpha \in X^\infty$, α_n is the prefix of

α of length n, especially, α_0 is the empty word Λ. For any $S \subset X$, $\mathfrak{I}(S)$ denotes the set of infinite binary sequences beginning with a finite sequence that belongs to S, i.e.,

$$\mathfrak{I}(S) \equiv \{\alpha \in X^\infty \mid \exists n \in \mathbb{N}\ \alpha_n \in S\}. \tag{1}$$

We write "r.e." instead of "recursively enumerable."

A subset S of X is called a prefix-free set if no sequence in S is a prefix of another sequence in S. For any partial recursive function $C\colon X \to X$, the domain of C is denoted by $\operatorname{dom} C$, i.e., $\operatorname{dom} C \equiv \{p \in X \mid C(p) \text{ is defined}\}$. A computer is a partial recursive function $C\colon X \to X$ such that $\operatorname{dom} C$ is a prefix-free set. Let C be a computer. For any $s \in X$, $H_C(s)$ is defined as

$$H_C(s) \equiv \min\{|p| \mid p \in X \ \ \&\ \ C(p) = s\} \quad \text{(may be } \infty). \tag{2}$$

It is shown that there exists a computer U such that for each computer C there exists a constant $\operatorname{sim}(C)$ with the following property: if $p \in \operatorname{dom} C$, then there is a q for which $U(q) = C(p)$ and $|q| \leq |p| + \operatorname{sim}(C)$. We choose any one of such a computer U and define $H(s) \equiv H_U(s)$, which is referred to as the algorithmic information content of s, the program-size complexity of s, or the Kolmogorov complexity of s. Thus $H(s)$ has the following property:

$$\forall C: \text{computer} \quad H(s) \leq H_C(s) + \operatorname{sim}(C). \tag{3}$$

We see that there is $c \in \mathbb{N}$ such that for any $s \neq \Lambda$,

$$H(s) \leq |s| + 2\log_2 |s| + c. \tag{4}$$

For any $n \in \mathbb{N}$, $H(n)$ is defined to be $H(\text{the } n\text{-th element of } X)$.

Chaitin's halting probability Ω is defined as

$$\Omega \equiv \sum_{p \in \operatorname{dom} U} 2^{-|p|}. \tag{5}$$

It is then shown that $0 < \Omega < 1$.

Normally, $o(n)$ denotes any one function $f\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/n = 0$.

Let D be a real number. $D \bmod 1$ denotes $D - \lfloor D \rfloor$, where $\lfloor D \rfloor$ is the greatest integer less than or equal to D, and $D \bmod{}'1$ denotes $D - \lceil D \rceil + 1$, where $\lceil D \rceil$ is the smallest integer greater than or equal to D. Hence, $D \bmod 1 \in [0, 1)$ but $D \bmod{}'1 \in (0, 1]$. We say that D is computable if the base-two expansion of D can be generated by an algorithm, i.e., if there exists a total recursive function $f\colon \mathbb{N}^+ \to \{0,1\}$ such that

$$0.f(1)f(2)f(3)f(4)\dots\dots$$

is the base-two expansion of $D \bmod 1$. The following three conditions are equivalent to one another.

(a) D is a computable real number.

(b) If $f\colon \mathbb{N}^+ \to \mathbb{Z}$ with $f(n) = \lfloor Dn \rfloor$ then f is a total recursive function.

(c) There exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Z}$ such that $|D - f(n)/n| < 1/n$ for all $n \in \mathbb{N}^+$.

Let $x \in \mathbb{R}^N$ and use the coordinate form $x = (x^1, x^2, \dots, x^N)$. For each $i = 1, \dots, N$ we denote $x^i \bmod 1$ in base-two notation with infinitely many zeros:

$$x^i \bmod 1 = 0.x^i_1 x^i_2 x^i_3 \dots\dots\dots \tag{6}$$

We then define $\mathrm{code}_N\colon \mathbb{R}^N \to X^\infty$ as

$$\mathrm{code}_N(x) \equiv x^1_1 x^2_1 \dots x^N_1 x^1_2 x^2_2 \dots x^N_2 x^1_3 x^2_3 \dots x^N_3 \dots\dots\dots \tag{7}$$

Throughout the rest of the paper, where there is no likelihood of confusion, $\mathrm{code}_N(x)$ may be denoted simply by x. Thus x_n is the first n bits of the infinite binary sequence $\mathrm{code}_N(x)$ for any $x \in \mathbb{R}^N$. We will identify any point of $\mathbb{R}^N$ with an infinite binary sequence in this manner.

Definition 2.1 (Hausdorff dimension) If U is any non-empty subset of $\mathbb{R}^N$, the diameter of U is defined as $|U| \equiv \sup\big\{|x-y| \bigm| x, y \in U\big\}$. Suppose that $F \subset \mathbb{R}^N$ and $D \geq 0$. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F, i.e., $F \subset \bigcup_i U_i$ with $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. For any $\delta > 0$ we define

$$\mathcal{H}^D_\delta(F) \equiv \inf\left\{\sum_i |U_i|^D \;\middle|\; \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}. \tag{8}$$

We then define

$$\mathcal{H}^D(F) \equiv \lim_{\delta\to 0} \mathcal{H}^D_\delta(F). \tag{9}$$

This limit exists for any subset F of $\mathbb{R}^N$, though the limiting value can be 0 or ∞. It is shown that $\mathcal{H}^D$ is an outer measure on $\mathbb{R}^N$. $\mathcal{H}^D$ is called

D-dimensional Hausdorff outer measure. Finally, the Hausdorff dimension $\dim_H F$ of F is defined as

$$\dim_H F \equiv \inf \left\{ D \geq 0 \mid \mathcal{H}^D(F) = 0 \right\}. \tag{10}$$

See e.g., the book [F2] for a treatment of the mathematics of Hausdorff dimension and self-similar sets.

3. D-Randomness

This section is, for all intents and purposes, a generalization of Chapter 7 in [C3].

Definition 3.1 (weakly Chaitin D-random) Let D be a real number and $D \geq 0$, and let $\alpha \in X^\infty$. α is called weakly Chaitin D-random if

$$\exists c \in \mathbb{R} \quad \forall n \in \mathbb{N} \quad Dn - c \leq H(\alpha_n). \tag{11}$$

If $\mathcal{T}$ is a subset of $\mathbb{N} \times X$ and $i \in \mathbb{N}$, we write $\mathcal{T}_i \equiv \left\{ s \mid (i,s) \in \mathcal{T} \right\}$.

Definition 3.2 (Martin-Löf D-test) Let $\mathcal{T} \subset \mathbb{N} \times X$ and $D \geq 0$. $\mathcal{T}$ is called Martin-Löf D-test if $\mathcal{T}$ is an r.e. set and

$$\forall i \in \mathbb{N} \quad \sum_{s \in \mathcal{T}_i} 2^{-D|s|} \leq 2^{-i}. \tag{12}$$

Definition 3.3 (Martin-Löf D-random) Let $D \geq 0$ and $\alpha \in X^\infty$. We say α is Martin-Löf D-random if

$$\forall \mathcal{T} : \text{Martin-Löf } D\text{-test} \quad \exists i \in \mathbb{N} \quad \alpha \notin \mathfrak{I}(\mathcal{T}_i). \tag{13}$$

In the case where $D = 1$, the weak Chaitin D-randomness and Martin-Löf D-randomness result in weak Chaitin randomness and Martin-Löf randomness respectively, which are defined in [C3].

Remark 3.1 Suppose that D is a computable real number and $D \geq 0$. Then there exists a universal Martin-Löf D-test $\mathcal{U}^D$, i.e.,

$$\exists \mathcal{U}^D : \text{Martin-Löf } D\text{-test} \quad \forall \mathcal{T} : \text{Martin-Löf } D\text{-test}$$

$$\bigcap_{i \in \mathbb{N}} \mathfrak{I}(\mathcal{T}_i) \subset \bigcap_{i \in \mathbb{N}} \mathfrak{I}(\mathcal{U}^D_i).$$

Thus, α is not Martin-Löf D-random if and only if $\alpha \in \bigcap_{i \in \mathbb{N}} \mathfrak{I}(\mathcal{U}^D_i)$.

Theorem 3.1 *Let D be a computable real number and $D \geq 0$. For any $\alpha \in X^\infty$, α is weakly Chaitin D-random $\Longleftrightarrow$ α is Martin-Löf D-random.*

The proof of Theorem 3.1 is given in Appendix A.1.

Definition 3.4 (D-compressible) Let $\alpha \in X^\infty$ and $D \geq 0$. We say that α is D-compressible if

$$H(\alpha_n) \leq Dn + o(n), \tag{14}$$

which is equivalent to

$$\varlimsup_{n\to\infty} \frac{H(\alpha_n)}{n} \leq D. \tag{15}$$

We generalize Chaitin's halting probability Ω as follows.

Definition 3.5 (Generalized halting probability)

$$\Omega^D \equiv \sum_{p \in \operatorname{dom} U} 2^{-\frac{|p|}{D}} \qquad (D > 0). \tag{16}$$

Thus, $\Omega = \Omega^1$. If $0 < D \leq 1$, then Ω^D converges and $0 < \Omega^D < 1$, since $\Omega^D \leq \Omega < 1$.

Theorem 3.2 *Let D be a real number.*
(a) *If $0 < D \leq 1$ and D is computable, then Ω^D is weakly Chaitin D-random and D-compressible.*
(b) *If $1 < D$, then Ω^D diverges to infinity.*

Proof. (a) Suppose that $0 < D \leq 1$ and D is a computable real number.

We first show that Ω^D is weakly Chaitin D-random. The proof is a straightforward generalization of Chaitin's original proof that Ω is weakly Chaitin random. Let $p_1, p_2, p_3, \ldots$ be a recursive enumeration of the r.e. set $\operatorname{dom} U$. Let α be the infinite binary sequence such that $0.\alpha$ is the base-two expansion of Ω^D with infinitely many ones. Then, since D is a computable real number, there exists a partial recursive function $\xi\colon X \to \mathbb{N}^+$ with the property that

$$0.\alpha_n < \sum_{i=1}^{\xi(\alpha_n)} 2^{-\frac{|p_i|}{D}}. \tag{17}$$

It is then easy to see that $Dn < |p_i|$ for all $i > \xi(\alpha_n)$ (i.e., given α_n, one can calculate all programs p of size not greater than $\lfloor Dn \rfloor$ such that $U(p)$ is defined). Hence, $Dn < H(s)$ for an arbitrary $s \in X$ such that $s \neq U(p_i)$ for all $i \leq \xi(\alpha_n)$. Therefore, given α_n, by calculating the set $\{U(p_i) \mid i \leq \xi(\alpha_n)\}$ and picking any finite binary sequence that is not in this set, one can obtain an $s \in X$ such that $Dn < H(s)$.

Thus, there exists a partial recursive function $\Psi\colon X \to X$ with the property that

$$Dn < H(\Psi(\alpha_n)). \tag{18}$$

Using (3), there is a natural number c_Ψ such that

$$H(\Psi(\alpha_n)) < H(\alpha_n) + c_\Psi. \tag{19}$$

Therefore, α is weakly Chaitin D-random. It follows that α has infinitely many zeros, which implies that $\alpha = \mathrm{code}_1(\Omega^D)$. Thus, Ω^D (i.e., $\mathrm{code}_1(\Omega^D)$) is weakly Chaitin D-random.

Next, we prove that Ω^D is D-compressible. We note that there exists a total recursive function $f\colon \mathbb{N}^+ \times \mathbb{N} \to \mathbb{N}$ such that

$$\left| \sum_{i=1}^{k} 2^{-\frac{|p_i|}{D}} - 2^{-n} f(k,n) \right| < 2^{-n}. \tag{20}$$

Let β be the infinite binary sequence such that $0.\beta$ is the base-two expansion of the halting probability Ω.

Given n and $\beta_{\lceil Dn \rceil}$ (i.e., the first $\lceil Dn \rceil$ bits of β), one can find a k_0 with the property that

$$0.\beta_{\lceil Dn \rceil} < \sum_{i=1}^{k_0} 2^{-|p_i|}. \tag{21}$$

It is then easy to see that

$$\sum_{i=k_0+1}^{\infty} 2^{-|p_i|} < 2^{-Dn}. \tag{22}$$

Using the inequality for real numbers $a^c + b^c \leq (a+b)^c$ $(a, b > 0,\ c \geq 1)$, it follows that

$$\left| \Omega^D - \sum_{i=1}^{k_0} 2^{-\frac{|p_i|}{D}} \right| < 2^{-n}. \tag{23}$$

From (20), (23), and $\left|\Omega^D - 0.\alpha_n\right| < 2^{-n}$ it is shown that

$$\left|\, 0.\alpha_n - 2^{-n} f(k_0, n) \,\right| < 3 \cdot 2^{-n}. \tag{24}$$

Hence

$$\alpha_n = f(k_0, n),\ f(k_0, n) \pm 1,\ f(k_0, n) \pm 2, \tag{25}$$

where α_n is regarded as a dyadic integer. Based on this, one is left with five possibilities of α_n, so that one needs only 3 bits more in order to determine α_n.

Thus, there exists a partial recursive function $\Phi\colon \mathbb{N}^+ \times X \times X \to X$ such that

$$\forall n \in \mathbb{N}^+ \quad \exists s \in X \quad |s| = 3 \ \& \ \Phi(n, \beta_{\lceil Dn \rceil}, s) = \alpha_n. \tag{26}$$

From (4) it follows that

$$H(\alpha_n) \leq |\beta_{\lceil Dn \rceil}| + o(n) \leq Dn + o(n), \tag{27}$$

which implies that Ω^D is D-compressible.

(b) Suppose that $D > 1$. We then choose a computable real number d satisfying $D \geq d > 1$. Let us first assume that Ω^d converges. Based on an argument similar to the first half of the proof of Theorem 3.2 (a), it is easy to show that Ω^d is weakly Chaitin d-random, i.e., there exists $c \in \mathbb{R}$ such that $dn - c \leq H((\Omega^d)_n)$. It follows from (4) that $dn - c \leq n + o(n)$. Dividing by n and letting $n \to \infty$ we have $d \leq 1$, which contradicts the fact $d > 1$. Thus, Ω^d diverges to infinity. By noting $\Omega^d \leq \Omega^D$ it is shown that Ω^D diverges to infinity. □

Suppose that $0 < D \leq 1$ and D is a computable real number. From Theorem 3.2 (a) it follows that $\lim_{n\to\infty} H((\Omega^D)_n)/n = D$. Also, $(\Omega^D)_n$ solves the halting problem for a program of size not greater than Dn, as is shown in the proof of Theorem 3.2 (a).

Moreover, as shown in the following theorem, Ω^D is infinitely differentiable as a function of $D \in (0, 1)$, and each derived function $d^k\Omega^D/dD^k$ has the same properties as Ω^D.

Theorem 3.3 *Let* $f\colon (0,1) \to \mathbb{R}$ *with* $f(D) = \Omega^D$. *For any* $p \in \operatorname{dom} U$, *let* $f_p\colon (0,1) \to \mathbb{R}$ *with* $f_p(D) = 2^{-|p|/D}$.

(a) *f is a function of class C^∞, and for each $k \in \mathbb{N}^+$,*

$$\forall D \in (0,1) \quad f^{(k)}(D) = \sum_{p \in \operatorname{dom} U} f_p^{(k)}(D) \tag{28}$$

where $f^{(k)}$ and $f_p^{(k)}$ are the k-th derived functions of f and f_p respectively.

(b) *Let $k \in \mathbb{N}^+$ and D be a computable real number in $(0,1)$. Then $f^{(k)}(D)$ is weakly Chaitin D-random and D-compressible.*

Proof. It is shown that for each $k \in \mathbb{N}^+$,

$$f_p^{(k)}(D) = \frac{1}{D^k}\, Q_k\!\left(\frac{|p| \ln 2}{D}\right) 2^{-\frac{|p|}{D}} \tag{29}$$

where $Q_k(z)$ is the polynomial of degree k with integer coefficients such that $Q_k(z) = z^k - k(k-1)z^{k-1} + \cdots + (-1)^{k-1}k!z$.

(a) We note that for each k there exists L such that if $|p| \geq L$ then for any $D \in (0,1)$, $f_p^{(k)}(D) > 0$. We wish to show by induction on k that the k-th derived function $f^{(k)}$ of f exists and (28) holds. The result is obvious for $k = 0$ from the definition of Ω^D. Suppose that the hypothesis is true for $k = i$. We see that there is L such that if $|p| \geq L$ then $f_p^{(i)}(D)$, $f_p^{(i+1)}(D)$, $f_p^{(i+2)}(D) > 0$ for any $D \in (0,1)$. Let $D \in (0,1)$. We then choose D_0 so that $D < D_0 < 1$. Using the mean value theorem, it is shown that if $|p| \geq L$ then

$$f_p^{(i+1)}(D) < \frac{f_p^{(i)}(D_0) - f_p^{(i)}(D)}{D_0 - D} < \frac{f_p^{(i)}(D_0)}{D_0 - D}. \tag{30}$$

Hence

$$\sum_{|p| \geq L} f_p^{(i+1)}(D) \leq \frac{1}{D_0 - D} \sum_{|p| \geq L} f_p^{(i)}(D_0). \tag{31}$$

By the inductive hypothesis, $\sum_p f_p^{(i)}(D_0)$ is convergent. Thus $\sum_p f_p^{(i+1)}(D)$ is convergent for any $D \in (0,1)$. Since $f_p^{(i+1)}$ is a monotone increasing function for any p with $|p| \geq L$, it is easy to see that $\sum_p f_p^{(i+1)}(D)$ is uniformly convergent on $(0,1)$ in the wider sense. Therefore, $\sum_p f_p^{(i)}(D)$ is termwise differentiable, which implies that the hypothesis is true for $k = i + 1$ as desired.

(b) Let $k \in \mathbb{N}^+$. We then note that there exists $L \in \mathbb{N}$ such that if $|p| > L$ then

$$\forall D \in (0,1) \quad 1 \leq \frac{1}{D^k} Q_k\!\left(\frac{|p| \ln 2}{D}\right) \tag{32}$$

and

$$\left[\frac{1}{D^k} Q_k\!\left(\frac{|p| \ln 2}{D}\right)\right]^D \tag{33}$$

is a monotone increasing function of $D \in (0,1)$. Let $p_1, p_2, p_3, \dots$ be a recursive enumeration of the r.e. set $\{p \mid p \in \operatorname{dom} U \;\&\; |p| > L\}$. Also, let $S_L = \{p \mid p \in \operatorname{dom} U \;\&\; |p| \leq L\}$, which is a finite set. The proof is similar to the case of Ω^D.

Suppose that D is a computable real number in $(0,1)$. We then note that $\sum_{p \in S_L} f_p^{(k)}(D)$ is also a computable real number.

We begin by showing that $f^{(k)}(D)$ is weakly Chaitin D-random. Let α be the infinite binary sequence such that $0.\alpha$ is the base-two expansion of $f^{(k)}(D) \bmod{}' 1$ with infinitely many ones.

Given α_n, one can find a $G \in \mathbb{N}$ with the property that

$$\lceil f^{(k)}(D) \rceil - 1 + 0.\alpha_n < \sum_{p \in S_L} f_p^{(k)}(D) + \sum_{i=1}^{G} f_{p_i}^{(k)}(D). \tag{34}$$

It is then easy to see that

$$\sum_{i=G+1}^{\infty} f_{p_i}^{(k)}(D) < 2^{-n}. \tag{35}$$

Hence, from (32), $Dn < |p_i|$ for all $i > G$. One can then calculate the set

$$\{U(p) \mid p \in S_L\} \cup \{U(p_i) \mid i \leq G\} \tag{36}$$

and therefore pick an $s \in X$ that is not in this set. It follows that $Dn < H(s)$.

Thus, there exists a partial recursive function $\Psi\colon X \to X$ such that

$$Dn < H(\Psi(\alpha_n)). \tag{37}$$

Based on an argument similar to the case of Ω^D, we see that α is weakly Chaitin D-random. Since $f^{(k)}(D) \bmod 1 = f^{(k)}(D) \bmod{}' 1 = 0.\alpha$, it follows that $f^{(k)}(D)$ is weakly Chaitin D-random.

Next, we prove that $f^{(k)}(D)$ is D-compressible. We note that there exists a total recursive function $g\colon \mathbb{N}^+ \times \mathbb{N} \to \mathbb{Z}$ such that

$$\left| \sum_{p\in S_L} f_p^{(k)}(D) - \lfloor f^{(k)}(D) \rfloor + \sum_{i=1}^{m} f_{p_i}^{(k)}(D) - 2^{-n} g(m,n) \right| < 2^{-n}. \tag{38}$$

Let d be any computable real number with $D < d < 1$, and let β be the infinite binary sequence such that $0.\beta$ is the base-two expansion of $f^{(k)}(d) \bmod 1$. We then note that $\sum_{p\in S_L} f_p^{(k)}(d)$ is a computable real number.

Given n and $\beta_{\lceil Dn/d \rceil}$ (i.e., the first $\lceil Dn/d \rceil$ bits of β), one can find an $M \in \mathbb{N}$ with the property that

$$\lfloor f^{(k)}(d) \rfloor + 0.\beta_{\lceil Dn/d \rceil} < \sum_{p\in S_L} f_p^{(k)}(d) + \sum_{i=1}^{M} f_{p_i}^{(k)}(d). \tag{39}$$

It is then easy to see that

$$\sum_{i=M+1}^{\infty} f_{p_i}^{(k)}(d) < 2^{-Dn/d}. \tag{40}$$

Raising both sides of this inequality to the power d/D and noting the way of choosing L,

$$2^{-n} > \sum_{i=M+1}^{\infty} \left[\frac{1}{d^k}\, Q_k\!\left(\frac{|p_i| \ln 2}{d} \right) \right]^{d/D} 2^{-|p_i|/D} > \sum_{i=M+1}^{\infty} f_{p_i}^{(k)}(D). \tag{41}$$

It follows that

$$\left| \sum_{p\in S_L} f_p^{(k)}(D) + \sum_{i=1}^{M} f_{p_i}^{(k)}(D) - f^{(k)}(D) \right| < 2^{-n}. \tag{42}$$

From (38), (42), and

$$\left| \lfloor f^{(k)}(D) \rfloor + 0.\alpha_n - f^{(k)}(D) \right| < 2^{-n}, \tag{43}$$

it is shown that

$$|\, \alpha_n - g(M,n) \,| \le 2, \tag{44}$$

where α_n is regarded as a dyadic integer.

Thus, there exists a partial recursive function $\Phi\colon \mathbb{N}^+ \times X \times X \to X$ such that

$$\forall n \in \mathbb{N}^+ \quad \exists s \in X \quad |s| = 3 \ \& \ \Phi(n, \beta_{\lceil Dn/d \rceil}, s) = \alpha_n. \tag{45}$$

Using an argument similar to the case of Ω^D, we see that α is D/d-compressible. Since d is any computable real number with $D < d < 1$, it follows that $f^{(k)}(D)$ is D-compressible. □

Remark 3.2 Suppose that W is an infinite r.e. subset of X. Chaitin proved that both

$$\sum_{U(p)\in W} 2^{-|p|} \tag{46}$$

and

$$\sum_{s\in W} 2^{-H(s)} \tag{47}$$

are weakly Chaitin 1-random as Ω. Corresponding to this fact, it is shown that both

$$\sum_{U(p)\in W} 2^{-|p|/D} \tag{48}$$

and

$$\sum_{s\in W} 2^{-H(s)/D} \tag{49}$$

have the same properties as Ω^D, i.e., the following results hold: (i) If $D > 1$ then both (48) and (49) diverge to infinity. (ii) As a function of D, each of (48) and (49) is infinitely termwise differentiable on $(0, 1)$. (iii) If $k \in \mathbb{N}$ and D is a computable real number in $(0, 1)$ then, for each of (48) and (49), the value of its k-th derived function at D is weakly Chaitin D-random and D-compressible.

Definition 3.6 (Chaitin D-random) Let D be a real number and $D \geq 0$, and let $\alpha \in X^\infty$. α is called Chaitin D-random if

$$\lim_{n\to\infty} H(\alpha_n) - Dn = \infty. \tag{50}$$

Definition 3.7 (Solovay D-test) Let $\mathcal{T} \subset \mathbb{N} \times X$ and $D \geq 0$. $\mathcal{T}$ is called Solovay D-test if $\mathcal{T}$ is an r.e. set and

$$\sum_{(i,s)\in\mathcal{T}} 2^{-D|s|} < \infty, \tag{51}$$

where the sum is over all i and s such that $(i, s) \in \mathcal{T}$.

Definition 3.8 (Solovay D-random) Let $D \geq 0$ and $\alpha \in X^\infty$. We say that α is Solovay D-random if

$$\forall\,\mathcal{T} : \text{Solovay } D\text{-test} \quad \exists\, m \in \mathbb{N} \quad \forall\, i > m \quad \alpha \notin \mathfrak{I}(\mathcal{T}_i). \tag{52}$$

In the case where $D = 1$, the Chaitin D-randomness and Solovay D-randomness result in Chaitin randomness and Solovay randomness respectively, which are defined in [C3].

Theorem 3.4 *Let D be a computable real number and $D \geq 0$, and let $\alpha \in X^\infty$. Then α is Chaitin D-random $\Longleftrightarrow$ α is Solovay D-random.*

The proof of Theorem 3.4 is immediately obtained by generalizing the proof of Theorem R3 in [C3].

Theorem 3.5 *Let $D \geq 0$ and $\alpha \in X^\infty$. α is Chaitin D-random $\Longrightarrow$ α is weakly Chaitin D-random.*

Proof. This is immediately apparent from the definitions. □

Remark 3.3 The converse of Theorem 3.5 holds for $D = 1$, because all Martin-Löf 1-random sequences are Solovay 1-random, as is shown in [C3]. However, whether the converse of Theorem 3.5 also holds for any computable real number D with $D < 1$ is an open problem.

Definition 3.9 (semi D-random) Let $D \geq 0$ and $\alpha \in X^\infty$. We say α is semi D-random if

$$D \leq \varliminf_{n\to\infty} \frac{H(\alpha_n)}{n}. \tag{53}$$

Proposition 3.6 *α is weakly Chaitin D-random $\Longrightarrow$ α is semi D-random.*

Proof. This is obvious from the definitions. □

In general, the converse of Proposition 3.6 does not necessarily hold. For example, although the infinite binary sequence $r_1 r_2 r_3 \ldots$ considered in the proof of Theorem 5.1 is semi 1-random, it is not weakly Chaitin 1-random.

Proposition 3.7 *The following four conditions are equivalent to one another.*

(a) *α is semi D-random.*

(b) $Dn + o(n) \leq H(\alpha_n)$.

(c) $\forall d \in \mathbb{R}\ (0 \leq d < D \Longrightarrow \alpha$ *is Chaitin d-random*$)$.

(d) $\forall d \in \mathbb{R}\ (0 \leq d < D \Longrightarrow \alpha$ *is weakly Chaitin d-random*$)$.

Proof. The above equivalences follow immediately from the definitions. □

4. Algorithmic Dimensions

We introduce the six fractal dimensions which are related to the degree of randomness or the normalized Kolmogorov complexity.

Definition 4.1 (algorithmic dimensions) Let F be a subset of $\mathbb{R}^N$.

(a) The 1st algorithmic dimension of F, which is denoted by $\dim_{A1} F$, is defined as $D \in \mathbb{R}$ such that

$$\forall x \in F \quad x \text{ is } \frac{D}{N}\text{-compressible} \tag{54}$$

and

$$\exists x \in F \quad x \text{ is Chaitin } \frac{D}{N}\text{-random.} \tag{55}$$

(b) The 2nd algorithmic dimension of F, which is denoted by $\dim_{A2} F$, is defined as $D \in \mathbb{R}$ such that

$$\forall x \in F \quad x \text{ is } \frac{D}{N}\text{-compressible} \tag{56}$$

and

$$\exists x \in F \quad x \text{ is weakly Chaitin } \frac{D}{N}\text{-random.} \tag{57}$$

(c) The 3rd algorithmic dimension of F, which is denoted by $\dim_{A3} F$, is defined as $D \in \mathbb{R}$ such that

$$\forall x \in F \quad x \text{ is } \frac{D}{N}\text{-compressible} \tag{58}$$

and

$$\exists x \in F \quad x \text{ is semi } \frac{D}{N}\text{-random.} \tag{59}$$

(d) The 4th algorithmic dimension of F, which is denoted by $\dim_{A4} F$, is defined as $D \in \mathbb{R}$ such that

$$\forall x \in F \quad x \text{ is } \frac{D}{N}\text{-compressible} \tag{60}$$

and

$$\forall d < \frac{D}{N} \quad \exists x \in F \quad x \text{ is Chaitin } d\text{-random.} \tag{61}$$

(e) The lower and upper algorithmic dimensions of F are respectively defined as

$$\underline{\dim}_A F \equiv \sup\left\{ D \geq 0 \;\middle|\; \exists x \in F \quad x \text{ is Chaitin } \frac{D}{N}\text{-random} \right\} \tag{62}$$

$$= \sup_{x \in F} \varliminf_{n \to \infty} \frac{H(x_n)}{n/N} \tag{63}$$

and

$$\overline{\dim}_A F \equiv \min\left\{ D \geq 0 \;\middle|\; \forall x \in F \quad x \text{ is } \frac{D}{N}\text{-compressible} \right\} \tag{64}$$

$$= \sup_{x \in F} \varlimsup_{n \to \infty} \frac{H(x_n)}{n/N}. \tag{65}$$

Although the upper and lower algorithmic dimensions always exist unless F is the empty set, the existences of the 1st, 2nd, 3rd, and 4th algorithmic dimensions of F are nontrivial. However, the uniqueness of each algorithmic dimension of F is trivial for any non-empty set F. Note that the condition (61) in the definition of the 4th algorithmic dimension is equivalent to $D \leq \sup_{x \in F} \varliminf_{n \to \infty} H(x_n)/(n/N)$. Also, from Proposition 3.7, the condition "Chaitin d-random" in (61) can be equivalently replaced by "weakly Chaitin d-random" or "semi d-random". Thus, we need not consider the alternative definitions which are obtained by such replacements in the definition of the 4th algorithmic dimension. The 'dimension' N of Euclidean space $\mathbb{R}^N$ appears in the definition of each algorithmic dimension. If we identify any point in $\mathbb{R}^N$ with an infinite sequence over an alphabet that consists of 2^N elements instead of an infinite binary sequence, and redefine

Kolmogorov complexity using a computer whose range is the set of finite sequences over such an alphabet, then N vanishes from these definitions.

The properties of the 3rd, 4th, upper, and lower algorithmic dimensions were, in essence, studied by [R1] and [St1]. As more restrictive concepts, we introduce the 1st and 2nd algorithmic dimensions which are related to the degree of randomness instead of the normalized Kolmogorov complexity.

Proposition 4.1 *The algorithmic dimensions satisfy the following properties.*

(a) *For each* $k = 1, 2, 3, 4$, *if* $\dim_{Ak} F$ *exists then* $0 \leq \dim_{Ak} F \leq N$.

(b) *If* $\dim_{A1} F$ *exists then* $\dim_{A2} F$ *also exists and is equal to* $\dim_{A1} F$. *Similarly, for* $k = 2, 3$, *if* $\dim_{Ak} F$ *exists then* $\dim_{A(k+1)} F$ *also exists and is equal to* $\dim_{Ak} F$.

(c) *There is* $E \subset \mathbb{R}^N$ *such that* $\dim_{A4} E$ *exists and* $\dim_{A3} E$ *does not exist. Also, there is* $F \subset \mathbb{R}^N$ *such that* $\dim_{A3} F$ *exists and* $\dim_{A2} F$ *does not exist.*

(d) *For each* $k = 1, 2, 3, 4$, *if* $E \subset F$ *and both* $\dim_{Ak} E$ *and* $\dim_{Ak} F$ *exist then* $\dim_{Ak} E \leq \dim_{Ak} F$.

(e) *For each* $k = 1, 2, 3, 4$, *if both* $\dim_{Ak} E$ *and* $\dim_{Ak} F$ *exist then* $\dim_{Ak}(E \cup F)$ *also exists and is equal to* $\max\{\dim_{Ak} E, \dim_{Ak} F\}$.

(f) *If* $\dim_{A4} F_i$ *exists for all* $i \in \mathbb{N}^+$ *then* $\dim_{A4}(\bigcup_{i=1}^{\infty} F_i)$ *also exists and is equal to* $\sup_{1 \leq i < \infty} \dim_{A4} F_i$.

(g) *If* F *is an open subset of* $\mathbb{R}^N$ *then* $\dim_{Ak} F = N$ *for* $k = 1, 2, 3, 4$.

(h) *If* $0 < D \leq 1$ *and* D *is computable then* $\dim_{Ak}\{\Omega^D\} = D$ *for* $k = 2, 3, 4$.

(i) $0 \leq \underline{\dim}_A F \leq \overline{\dim}_A F \leq N$.

(j) *For each* $k = 1, 2, 3, 4$, *if* $\dim_{Ak} F$ *exists then* $\underline{\dim}_A F = \overline{\dim}_A F = \dim_{Ak} F$.

(k) *If* $\underline{\dim}_A F = \overline{\dim}_A F$ *then* $\dim_{A4} F$ *exists and these three algorithmic dimensions are equal to one another.*

Proof. These properties are obvious consequences of the definitions. The proof of Proposition 4.1 (c) is given as follows. Let $0 < D \leq 1$ and

$$E = \left\{\Omega^d \;\middle|\; 0 < d < D \text{ and } d \text{ is computable}\right\}. \tag{66}$$

Then $\dim_{A4} E = D$ but $\dim_{A3} E$ does not exist. Also, F_{halt} (introduced in the next section) is an example of a set F such that $\dim_{A3} F$ exists but $\dim_{A2} F$ does not. See Theorem 5.1. Proposition 4.1 (g) follows from the fact that for any $s \in X$ there is a Chaitin 1-random infinite binary sequence whose prefix is s. □

Corresponding to Remark 3.3, it is an open problem whether or not there is a set F such that $\dim_{A2} F$ exists and $\dim_{A1} F$ does not.

If $\dim_{A3} F$ exists, which follows from the existence of either $\dim_{A1} F$ or $\dim_{A2} F$, then $\dim_{A3} F = \max_{x \in F} \lim_{n\to\infty} H(x_n)/(n/N)$, where the maximum is over all $x \in F$ such that $\lim_{n\to\infty} H(x_n)/(n/N)$ exists. Note that from the definition of code_N, x_n corresponds to the first n/N digits of the base-two expansions of all components of $x \in \mathbb{R}^N$. This implies that $\dim_{A3} F$ is the maximum value over the program-size complexity per digit in base-two notation for all points in F.

Let A be a non-empty closed subset of $\mathbb{R}^N$. A transformation $S\colon A \to A$ is called a contraction on A if there is a number c with $0 < c < 1$ such that $|S(x) - S(y)| \le c\,|x - y|$ for all x, y in A. Let φ denote the class of all non-empty compact subsets of A. It is shown that the following theorem holds for contractions $S_1, \dots, S_m$ on A (for its proof, see e.g., [F2]).

Theorem 4.2 *Let $S_1, \dots, S_m$ be contractions on A. Then there exists a unique non-empty compact set F which satisfies*

$$F = \bigcup_{i=1}^{m} S_i(F). \tag{67}$$

Moreover, if we define a transformation $S\colon \varphi \to \varphi$ by

$$S(E) = \bigcup_{i=1}^{m} S_i(E) \tag{68}$$

and write S^k for the k-th iterate of S given by $S^0(E) = E$, $S^k(E) = S(S^{k-1}(E))$ for $k \ge 1$, then

$$F = \bigcap_{k=1}^{\infty} S^k(E) \tag{69}$$

for any set E in φ such that $S_i(E) \subset E$ for each i.

The unique non-empty compact set F satisfying (67) is called the invariant set of the contractions $S_1, \dots, S_m$.

A contraction S on A is called a similarity on A if there is a number c with $0 < c < 1$ such that $|S(x) - S(y)| = c\,|x - y|$ for all x, y in A (c is called the ratio of S). The invariant set of a collection of similarities is called a self-similar set.

Let $S_1, \dots, S_m$ be similarities on A. We say that $S_1, \dots, S_m$ satisfy the open set condition if there exists a non-empty bounded open set $V \subset A$ such that

$$V \supset \bigcup_{i=1}^{m} S_i(V) \tag{70}$$

and $S_i(V) \cap S_j(V) = \phi$ $(i \neq j)$.

Theorem 4.3 *Let $S_1, \dots, S_m$ be similarities on $\mathbb{R}^N$ with ratios $c_1, \dots, c_m$ respectively. We then note that each S_i is an affine transformation, i.e., for each i there exist $N \times N$ matrix M_i and $v_i \in \mathbb{R}^N$ such that $S_i(x) = M_i x + v_i$. We assume that all matrix elements of M_i and all components of v_i are computable real numbers for each i. Furthermore, suppose that the open set condition* (70) *holds for $S_1, \dots, S_m$. If F is the invariant set of $S_1, \dots, S_m$, then $\dim_{A1} F$ exists and $\dim_{A1} F = \dim_H F = D$, where D is given by*

$$\sum_{i=1}^{m} c_i^D = 1. \tag{71}$$

Therefore, all six algorithmic dimensions of F exist and are equal to $\dim_H F$.

The proof of Theorem 4.3 is given in Appendix A.2, and here we only present some examples of familiar self-similar sets F which are shown to satisfy $\dim_{A1} F = \dim_H F$ as a consequence of Theorem 4.3.

Example 4.1

(a) The middle-third Cantor set is the invariant set F of the similarities S_1, S_2 on $\mathbb{R}$ with ratios $1/3$, $1/3$ such that

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}. \tag{72}$$

The open set condition (70) holds for S_1, S_2 with V as the open interval $(0, 1)$. All of the real constants which appear in affine transformations

(72) (i.e., 0, 1/3, and 2/3) are computable real numbers. Thus, by Theorem 4.3, $\dim_{A1} F = \dim_H F = \log_3 2$, which is the solution of $(1/3)^D + (1/3)^D = 1$.

(b) The Sierpiński gasket with vertices at the points $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$ is the invariant set F of the similarities S_1, S_2, S_3 on $\mathbb{R}^2$ with ratios 1/2, 1/2, 1/2 such that

$$\begin{aligned} S_1\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \\ S_2\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \\ S_3\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}. \end{aligned} \tag{73}$$

The open set condition (70) holds for S_1, S_2, S_3, taking V as the interior of the equilateral triangle with vertices at $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$. All of the real constants which appear in affine transformations (73) (i.e., 0, 1/2, 1/4, and $\sqrt{3}/4$) are computable real numbers. It follows from Theorem 4.3 that $\dim_{A1} F = \dim_H F = \log_2 3$, which is the solution of $(1/2)^D + (1/2)^D + (1/2)^D = 1$.

(c) A modified von Koch curve $F \subset \mathbb{R}^2$ is constructed as follows. Fix a computable real number r with $0 < r \le 1/3$. Initially, consider a line segment which has endpoints (x_1, y_1) and (x_2, y_2) such that all of x_1, y_1, x_2, y_2 are computable real numbers. Construct a curve F by repeatedly replacing the middle proportion r of each line segment by the other two sides of an equilateral triangle. (In the case where $r = 1/3$, F results in the von Koch curve.) Then we can select the four similarities S_1, S_2, S_3, S_4 with ratios $\frac{1}{2}(1-r)$, r, r, $\frac{1}{2}(1-r)$ which have the following properties: (i) The curve F is the invariant set of $S_1, \dots, S_4$. (ii) The open set condition holds for $S_1, \dots, S_4$. (iii) All of the real constants which appear in each affine transformation S_i are computable real numbers. Thus, from Theorem 4.3 we see that $\dim_{A1} F = \dim_H F = D$, where D satisfies $2r^D + 2(\frac{1}{2}(1-r))^D = 1$.

5. Halting self-similar sets

The halting self-similar set F_{halt} is defined as

$$F_{\text{halt}} \equiv \left\{0.q_1q_2q_3\dots \;\middle|\; q_i \in \operatorname{dom} U \text{ for each } i\right\}. \tag{74}$$

F_{halt} is a compact subset of $[0,1]$. Let $S_p(x) = 2^{-|p|}x + 0.p$ for each $p \in \operatorname{dom} U$. Then F_{halt} satisfies

$$F_{\text{halt}} = \bigcup_{p\in\operatorname{dom} U} S_p(F_{\text{halt}}). \tag{75}$$

Thus, since $\operatorname{dom} U$ is a countably infinite set, F_{halt} is a self-similar set in the sense that F_{halt} is a union of a countably infinite number of smaller similar copies of itself. Also, since $\operatorname{dom} U$ is a prefix-free set, the function family $\{S_p\}$ satisfies an open set condition in the sense that there exists a non-empty bounded open set V (i.e., the open interval $(0,1)$) such that $V \supset \bigcup_p S_p(V)$ and $S_p(V) \cap S_q(V) = \phi$ $(p \neq q)$. Using the fact that $\operatorname{dom} U$ is an r.e. set and not a recursive set, it is easy to show that $\left\{s \in X \;\middle|\; I(s) \cap F_{\text{halt}} \neq \phi\right\}$ is also an r.e. set and not a recursive set, where $I(s) = [0.s, 0.s + 2^{-|s|})$.

Remark 5.1 As considered in [C1], think of U as decoding equipment at the receiving end of a noiseless binary communication channel. Regard its programs (i.e., finite binary sequences in $\operatorname{dom} U$) as code words and regard the result of the computation by U as the decoded message. Since $\operatorname{dom} U$ is a prefix-free set, such code words form what is called an "instantaneous code," so that successive messages sent through the channel can be separated. Then F_{halt} is the set of $x \in [0,1]$ such that the base-two expansion of x is an endless succession of coded messages sent through the channel.

Theorem 5.1 $\dim_H F_{\text{halt}} = 1$ *and* $\mathcal{L}^1(F_{\text{halt}}) = 0$, *where* $\mathcal{L}^1$ *is Lebesgue measure on* $\mathbb{R}$. *Neither* $\dim_{A1} F_{\text{halt}}$ *nor* $\dim_{A2} F_{\text{halt}}$ *exists, but* $\dim_{A3} F_{\text{halt}} = \dim_{A4} F_{\text{halt}} = 1$.

Proof. To begin with, we show that $\dim_H F_{\text{halt}} = 1$. Let $p_1, p_2, p_3, \dots$ be a recursive enumeration of the r.e. set $\operatorname{dom} U$, and let

$$P_m = \left\{0.q_1q_2q_3\dots \;\middle|\; q_i \in \{p_1, p_2, \dots, p_m\} \text{ for each } i\right\}. \tag{76}$$

Then P_m is the invariant set of $S_{p_1}, S_{p_2}, \dots, S_{p_m}$. Since the open set condition (70) holds for $S_{p_1}, S_{p_2}, \dots, S_{p_m}$, from Theorem A.10 in Appendix A.2

it is shown that $\dim_H P_m = D_m$, where D_m is given by

$$\sum_{i=1}^{m} 2^{-D_m|p_i|} = 1. \tag{77}$$

Now, from the definition of Ω^D,

$$\Omega^{\frac{1}{D}} = \sum_{i=1}^{\infty} 2^{-D|p_i|}. \tag{78}$$

From Theorem 3.2 (b), this sum diverges to infinity for each $D \in (0,1)$. Hence, given $\varepsilon > 0$, for all sufficiently large m

$$\sum_{i=1}^{m} 2^{-(1-\varepsilon)|p_i|} > 1 \tag{79}$$

and

$$\sum_{i=1}^{m} 2^{-1\cdot|p_i|} < \Omega^1 < 1, \tag{80}$$

which implies that $1 - \varepsilon < D_m < 1$. Thus, $\lim_{m\to\infty} D_m = 1$. Since $P_m \subset F_{\text{halt}}$, it follows that $\dim_H F_{\text{halt}} = 1$.

Second, we prove that $\mathcal{L}^1(F_{\text{halt}}) = 0$. We see that for each $n \in \mathbb{N}^+$,

$$\begin{aligned}\mathcal{L}^1(F_{\text{halt}}) &\le \mathcal{L}^1\left(\left\{0.q_1\dots q_n\alpha \;\middle|\; q_1,\dots,q_n \in \operatorname{dom} U \;\;\&\;\; \alpha \in X^\infty\right\}\right)\\ &= \sum_{q_1,\dots,q_n\in \operatorname{dom} U} 2^{-|q_1\dots q_n|} \\ &= (\Omega^1)^n.\end{aligned} \tag{81}$$

Since $0 < \Omega^1 < 1$, letting $n \to \infty$ gives $\mathcal{L}^1(F_{\text{halt}}) = 0$.

Third, we prove that $\dim_{A3} F_{\text{halt}} = \dim_{A4} F_{\text{halt}} = 1$. Fix a weakly Chaitin 1-random sequence β such as the base-two expansion of Ω. For each $k \in \mathbb{N}^+$, let r_k be any one of the shortest q such that $U(q)$ is equal to the k bits sequence from $(k-1)k/2+1$-th bit to $k(k+1)/2$-th bit of β. Also, let $y = 0.r_1r_2r_3\dots$. It follows that $y \in F_{\text{halt}}$.

Given y_n, one can find $r_1, r_2, r_3, \dots, r_m, t$ such that $y_n = r_1r_2r_3\dots r_m t$ and t is a proper prefix of r_{m+1}, possibly $t = \Lambda$. One can then calculate $\beta_{m(m+1)/2}$ from $r_1, r_2, r_3, \dots, r_m$. Hence, there exists a partial recursive function $\Psi\colon X \to X$ such that for each $n \in \mathbb{N}^+$, $\Psi(y_n) = \beta_{m(m+1)/2}$ where

m is the greatest integer with the property that $|r_1 r_2 r_3 \dots r_m| \leq n$. Using (3), it is easy to show that there is $d \in \mathbb{N}$ such that

$$\frac{m(m+1)}{2} - d \leq H(y_n). \tag{82}$$

However, using (4), $|r_k| \leq k + 2\log_2 k + c$ for any $k \in \mathbb{N}^+$. Thus

$$\begin{aligned} n &< |r_1 r_2 r_3 \dots r_m r_{m+1}| \\ &\leq \frac{(m+1)(m+2)}{2} + \log_2 (m+1)! + c(m+1). \end{aligned} \tag{83}$$

Since letting $n \to \infty$ implies $m \to \infty$, it follows that

$$1 \leq \varliminf_{n\to\infty} \frac{m(m+1)}{2n}. \tag{84}$$

Combining with (82) this implies that y is semi 1-random.

Now, it follows from (4) that x is 1-compressible for all $x \in \mathbb{R}$. Thus, $\dim_{A3} F_{\text{halt}} = 1$, which shows, from Proposition 4.1 (b), that $\dim_{A4} F_{\text{halt}} = 1$.

Finally, we show that neither $\dim_{A1} F_{\text{halt}}$ nor $\dim_{A2} F_{\text{halt}}$ exists. If $\dim_{A2} F_{\text{halt}}$ exists then, by Proposition 4.1 (b), $\dim_{A2} F_{\text{halt}} = \dim_{A3} F_{\text{halt}} = 1$, which implies that there is $x \in F_{\text{halt}}$ such that x is Martin-Löf 1-random. However, we will show that x is not Martin-Löf 1-random for any $x \in F_{\text{halt}}$.

Choosing $a \in \mathbb{Q}$ with $\Omega^1 < a < 1$ it follows that

$$\sum_{q_1,\dots,q_n \in \operatorname{dom} U} 2^{-|q_1 \dots q_n|} < a^n. \tag{85}$$

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a total recursive function such that $a^{f(i)} \leq 2^{-i}$ for all $i \in \mathbb{N}$. The r.e. set

$$\mathcal{T} \equiv \left\{ (i, q_1 q_2 \dots q_{f(i)}) \;\middle|\; i \in \mathbb{N} \;\&\; q_1, q_2, \dots, q_{f(i)} \in \operatorname{dom} U \right\} \tag{86}$$

is then Martin-Löf 1-test. For any $x \in F_{\text{halt}}$, $\forall i \in \mathbb{N}\ x \in \mathfrak{I}(\mathcal{T}_i)$ and hence x is not Martin-Löf 1-random.

Thus, $\dim_{A2} F_{\text{halt}}$ does not exist. From Proposition 4.1 (b), $\dim_{A1} F_{\text{halt}}$ also does not exist. □

We say that $f : X \to X$ is an optimal code if for each $s \in X$, $f(s)$ is one of the shortest program for U to calculate s, i.e., $U(f(s)) = s$ and $|f(s)| = H(s)$. For any optimal code f and $W \subset X$, $F_{\text{opt}}(f, W)$ is defined as

$$F_{\rm opt}(f, W) \equiv \left\{0.f(s_1)f(s_2)f(s_3)\dots\dots \;\middle|\; s_i \in W \text{ for each } i\right\}. \quad (87)$$

The following theorem, which is similar to Theorem 5.1, then holds.

Theorem 5.2 *Suppose that f is an optimal code and W is an infinite r.e. subset of X. Then*

$$\dim_{A3} F_{\rm opt}(f, W) = \dim_{A4} F_{\rm opt}(f, W) = \dim_H F_{\rm opt}(f, W) = 1$$

and $\mathcal{L}^1(F_{\rm opt}(f, W)) = 0$. *However, neither* $\dim_{A1} F_{\rm opt}(f, W)$ *nor* $\dim_{A2} F_{\rm opt}(f, W)$ *exists.*

Proof. The sum

$$\sum_{s \in W} 2^{-H(s)/D} \quad (88)$$

diverges to infinity for any $D > 1$, as we mentioned in Remark 3.2. Thus, using an argument similar to the case of $F_{\rm halt}$, we see that $\dim_H F_{\rm opt}(f, W) = 1$.

Note that $F_{\rm opt}(f, W) \subset F_{\rm halt}$. Thus, $\mathcal{L}^1(F_{\rm opt}(f, W)) \le \mathcal{L}^1(F_{\rm halt}) = 0$.

Next, we prove $\dim_{A3} F_{\rm opt}(f, W) = 1$. Fix a weakly Chaitin 1-random sequence β. For each $k \in \mathbb{N}^+$, let r_k be any one of the shortest q such that $U(q)$ is equal to the k bits sequence from $(k-1)k/2+1$-th bit to $k(k+1)/2$-th bit of β. Since W is an infinite r.e. set, there exists a one-to-one total recursive function $\xi\colon X \to W$. Let $y = 0.f(\xi(r_1))\, f(\xi(r_2))\, f(\xi(r_3))\dots$. It is then shown that $y \in F_{\rm opt}(f, W)$ and there is $c_\xi \in \mathbb{N}$ such that for any $k \in \mathbb{N}^+$, $|f(\xi(r_k))| \le k + 2\log_2 k + c_\xi$. Moreover, one can calculate $\beta_{m(m+1)/2}$ from $f(\xi(r_1)), f(\xi(r_2)), f(\xi(r_3)), \dots, f(\xi(r_m))$. Thus, making an argument similar to the case of $F_{\rm halt}$ it is shown that y is semi 1-random. Hence, $\dim_{A3} F_{\rm opt}(f, W) = 1$, and therefore $\dim_{A4} F_{\rm opt}(f, W) = 1$.

As was shown in the proof of Theorem 5.1, there is no Martin-Löf 1-random sequence in $F_{\rm halt}$. Since $F_{\rm opt}(f, W) \subset F_{\rm halt}$, there is no Martin-Löf 1-random sequence in $F_{\rm opt}(f, W)$. Thus, neither $\dim_{A1} F_{\rm opt}(f, W)$ nor $\dim_{A2} F_{\rm opt}(f, W)$ exists. □

For each $W \subset X$, we define

$$F_{\rm halt}(W) \equiv \left\{0.q_1q_2q_3\dots\dots \;\middle|\; U(q_i) \in W \text{ for each } i\right\}, \quad (89)$$

which is a generalization of F_{halt}, i.e., $F_{\text{halt}} = F_{\text{halt}}(X)$. Note that

$$F_{\text{opt}}(f, W) \subsetneqq F_{\text{halt}}(W) \subset F_{\text{halt}} \tag{90}$$

for any optimal code f and any infinite r.e. set $W \subset X$. The following generalization of Theorem 5.1 holds.

Theorem 5.3 *Suppose that W is an infinite r.e. subset of X. Then*

$$\dim_{A3} F_{\text{halt}}(W) = \dim_{A4} F_{\text{halt}}(W) = \dim_H F_{\text{halt}}(W) = 1$$

and $\mathcal{L}^1(F_{\text{halt}}(W)) = 0$. However, neither $\dim_{A1} F_{\text{halt}}(W)$ nor $\dim_{A2} F_{\text{halt}}(W)$ exists.

Proof. This follows immediately from Theorem 5.1, Theorem 5.2, and (90). □

Remark 5.2 [R1] showed that for any r.e. set $L \subset X$,

$$\underline{\dim}_A L^\infty = \dim_H L^\infty \tag{91}$$

where $L^\infty = \{0.l_1 l_2 l_3 \dots \mid l_i \in L \text{ for each } i\}$ (see also [St1]). Suppose that f is an optimal code and W is an infinite r.e. subset of X. Since $\{q \in X \mid U(q) \in W\}$ is an r.e. set, using (91) and $\dim_{A3} F_{\text{halt}}(W) = 1$ we immediately see that $\dim_H F_{\text{halt}}(W) = 1$. On the other hand, since $\{f(s) \mid s \in W\}$ is not an r.e. set, it would seem difficult to prove $\dim_H F_{\text{opt}}(f, W) = 1$ directly from (91) and $\dim_{A3} F_{\text{opt}}(f, W) = 1$. In Theorem 5.2, using the property of the sum (49), we proved $\dim_H F_{\text{opt}}(f, W) = 1$.

A. Appendix

A.1. The proof of Theorem 3.1

In the case where $D = 1$, Theorem 3.1 results in Theorem R1 in [C3]. The proof of Theorem 3.1 is a straightforward generalization of the proof of Theorem R1 given in [C3]. We need the following two theorems shown in [C1].

Theorem A.1 *Let both $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to \mathbb{N}$ be total recursive functions. Suppose that*

$$\sum_{n=0}^{\infty} 2^{-g(n)} \leq 1. \tag{92}$$

Then there exists a computer C such that

$$H_C(s) = \min_{f(n)=s} g(n). \tag{93}$$

Theorem A.2 *There is $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$\#\left\{s \in X \;\middle|\; |s| = n \;\&\; H(s) < k\right\} < 2^{k-H(n)+c}. \tag{94}$$

Theorem A.1 and Theorem A.2 are Theorem 3.2 and Theorem 4.2 (b) in [C1], respectively.

The proof of Theorem 3.1 is as follows.

Proof of Theorem 3.1. Suppose that D is a computable real number and $D \geq 0$. Let $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) = \lfloor Dn \rfloor$. Then f is a total recursive function.

($\neg$ (*weak Chaitin*) $\Longrightarrow$ $\neg$ *Martin-Löf*)

$\neg$ (weak Chaitin) says that for any $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $H(\alpha_n) < f(|\alpha_n|) - k$. Let $\mathcal{T} = \left\{(k, s) \in \mathbb{N} \times X \;\middle|\; H(s) < f(|s|) - k - c\right\}$ for the natural number c which is referred to in Theorem A.2. Then $\alpha \in \mathfrak{I}(\mathcal{T}_k)$ for any $k \in \mathbb{N}$.

However, it follows from Theorem A.2 that $\#\left\{s \in \mathcal{T}_k \;\middle|\; |s| = n\right\} \leq 2^{Dn-H(n)-k}$ for any $k, n \in \mathbb{N}$. Hence, for any $k \in \mathbb{N}$ we get

$$\begin{aligned}\sum_{s \in \mathcal{T}_k} 2^{-D|s|} &= \sum_{n=0}^{\infty} \#\left\{s \in \mathcal{T}_k \;\middle|\; |s| = n\right\} 2^{-Dn} \\ &\leq 2^{-k}\left(\sum_{n=0}^{\infty} 2^{-H(n)}\right) \leq 2^{-k}\Omega < 2^{-k}.\end{aligned} \tag{95}$$

Since f is a total recursive function, $\mathcal{T}$ is an r.e. set. Thus, $\mathcal{T}$ is Martin-Löf D-test, and hence α is not Martin-Löf D-random.

($\neg$ *Martin-Löf* $\Longrightarrow$ $\neg$ (*weak Chaitin*))

Suppose that there exists a Martin-Löf D-test $\mathcal{T}$ such that $\alpha \in \mathfrak{I}(\mathcal{T}_n)$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{n=2}^{\infty} \sum_{s \in \mathcal{T}_{n^2}} 2^{-[f(|s|)-n]} &\leq \sum_{n=2}^{\infty}\Bigl(2^{n+1} \sum_{s \in \mathcal{T}_{n^2}} 2^{-D|s|}\Bigr) \\ &\leq \sum_{n=2}^{\infty} 2^{-n^2+n+1} \leq 1.\end{aligned} \tag{96}$$

Since $\mathcal{T}$ is an r.e. set, there exists a bijective total recursive function g from $\mathbb{N}$ to the set $\{(n,s) \mid n \geq 2 \ \&\ s \in \mathcal{T}_{n^2}\}$. Let $n(k)$ and $s(k)$ be total recursive functions such that $g(k) = (n(k), s(k))$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=0}^{\infty} 2^{-[f(|s(k)|)-n(k)]} = \sum_{n=2}^{\infty} \sum_{s \in \mathcal{T}_{n^2}} 2^{-[f(|s|)-n]} \leq 1. \tag{97}$$

Since f is a total recursive function, by Theorem A.1, there is a computer C such that

$$H_C(s) = \min_{s(k)=s} \{f(|s(k)|) - n(k)\}. \tag{98}$$

Using (3), it follows that

$$n \geq 2 \ \&\ s \in \mathcal{T}_{n^2} \implies H(s) \leq D|s| - n + \mathrm{sim}(C). \tag{99}$$

Thus, since $\alpha \in \mathfrak{I}(\mathcal{T}_{n^2})$ for all $n \geq 2$, we see that for all $n \geq 2$ there exists $k \in \mathbb{N}$ such that

$$H(\alpha_k) \leq D\,|\alpha_k| - n + \mathrm{sim}(C) = Dk - n + \mathrm{sim}(C), \tag{100}$$

which implies that α is not weakly Chaitin D-random. □

A.2. The proof of Theorem 4.3

For each $D \geq 0$, we define T_D^1 and T_D^2 by

$$T_D^1 \equiv \left\{ x \in \mathbb{R}^N \;\middle|\; x \text{ is not Chaitin } \frac{D}{N}\text{-random} \right\} \tag{101}$$

$$T_D^2 \equiv \left\{ x \in \mathbb{R}^N \;\middle|\; x \text{ is not weakly Chaitin } \frac{D}{N}\text{-random} \right\}. \tag{102}$$

Theorem A.3 *Let $D \geq 0$. If D is a computable real number, then T_D^1 and T_D^2 are Borel sets and $\mathcal{H}^D(T_D^1) = \mathcal{H}^D(T_D^2) = 0$.*

Proof. In the case that $D = 0$, the results are obvious from the fact that $T_0^1 = T_0^2 = \phi$. Thus we assume that $D > 0$.

We first show that $\mathcal{H}^D(T_D^1) = 0$. Suppose that $\mathcal{T}$ is Solovay D/N-test. For any $s \in X$, we write $U(s) = \{x \in [0,1)^N \mid s \text{ is the prefix of } x\}$. It follows that $\sum_{(i,s)\in\mathcal{T}} |U(s)|^D < \infty$. Also, we let $\Gamma(\mathcal{T}) = \{x \in [0,1)^N \mid \forall m \in \mathbb{N}\ \exists i > m\ x \in \mathfrak{I}(\mathcal{T}_i)\}$. Then, for any $\delta, \varepsilon > 0$, it is shown that there exists a δ-cover $\{U(s_k)\}_k$ of $\Gamma(\mathcal{T})$ such that $\sum_k |U(s_k)|^D < \varepsilon$. Hence,

$\mathcal{H}^D_\delta(\Gamma(\mathcal{T})) < \varepsilon$. It follows that $\mathcal{H}^D(\Gamma(\mathcal{T})) = 0$.

On the other hand, Theorem 3.4 implies that $T^1_D \cap [0,1)^N = \bigcup \Gamma(\mathcal{T})$, where the union is over all Solovay D/N-test $\mathcal{T}$. Since there are only countably many Solovay D/N-tests, it follows that $\mathcal{H}^D(T^1_D \cap [0,1)^N) = 0$. By noting the fact that Hausdorff outer measures are translation invariant (i.e., $\mathcal{H}^D(F+z) = \mathcal{H}^D(F)$, where $F+z = \{x+z \mid x \in F\}$), we see that $\mathcal{H}^D(T^1_D) = 0$.

From Theorem 3.4 it follows that

$$T^1_D = \bigcup_{\mathcal{T}} \bigcap_{n=0}^{\infty} \bigcup_{i=n}^{\infty} \left\{x \in \mathbb{R}^N \mid \mathrm{code}_N(x) \in \mathfrak{I}(\mathcal{T}_i)\right\}, \tag{103}$$

where the leftmost union is over all Solovay D/N-test $\mathcal{T}$. Thus, T^1_D is a Borel set.

Similarly, using Theorem 3.1, it is shown that $\mathcal{H}^D(T^2_D) = 0$ and T^2_D is a Borel set. □

In the case where $N = 1$ and $D = 1$, $\mathcal{H}^D(T^1_D) = \mathcal{H}^D(T^2_D) = 0$ states the well-known fact that the set of all non-random real numbers has a zero-Lebesgue measure.

For each $D \geq 0$, we define T^3_D by

$$T^3_D \equiv \left\{x \in \mathbb{R}^N \;\middle|\; x \text{ is not semi } \frac{D}{N}\text{-random}\right\}. \tag{104}$$

Corollary A.4 (Staiger [St1], Cai and Hartmanis [CH]) *If $D \geq 0$ then T^3_D is a Borel set and $\mathcal{H}^D(T^3_D) = 0$.*

Proof. In the case that $D = 0$, the results are obvious from the fact that $T^3_0 = \phi$. Thus we assume that $D > 0$. Let $D_1, D_2, \dots$ be a sequence of computable real numbers such that $\lim_{n\to\infty} D_n = D$ and for any n, $D_n < D$. Using the equivalency between (a) and (c) in Proposition 3.7, we see that $T^3_D = \bigcup_{n=1}^{\infty} T^1_{D_n}$. Hence, by Theorem A.3, T^3_D is a Borel set. Since $\mathcal{H}^D$ is non-increasing with D, it follows that $\mathcal{H}^D(T^3_D) = 0$. □

Corollary A.5 (Ryabko [R1]) *$\dim_H F \leq \underline{\dim}_A F$, and for each $k = 1,2,3,4$, if $\dim_{Ak} F$ exists then $\dim_H F \leq \dim_{Ak} F$.*

Proof. Let $D = \dim_H F$. Since the results are trivial for $D = 0$, we assume that $D > 0$. For any $\varepsilon > 0$, we choose a computable real number d such that $D - \varepsilon \leq d < D$. From Theorem A.3 it follows that $\mathcal{H}^d(F \backslash T^1_d) = \mathcal{H}^d(F) >$

0. Hence, $F \backslash T_d^1 \neq \phi$ and therefore there is $x \in F$ such that x is Chaitin d/N-random. Thus, we see that $D - \varepsilon \leq \underline{\dim}_A F$ for any $\varepsilon > 0$, from which the results are easily produced. □

Definition A.1 (r.e. condition) Suppose that F is a subset of $\mathbb{R}^N$. We first define

$$F \bmod 1 \equiv \{(x^1 \bmod 1, x^2 \bmod 1, \ldots, x^N \bmod 1) \mid (x^1, x^2, \ldots, x^N) \in F\}. \tag{105}$$

For any $s \in X$, we define $I(s) \equiv [0.s, 0.s + 2^{-|s|})$ and $\hat{I}(s) \equiv [0.s - 2^{-|s|}, 0.s + 2^{-|s|} + 2^{-|s|}) \bmod 1$. We also generalize $I(s)$ and $\hat{I}(s)$ to intervals on $\mathbb{R}^N$ by the following manner.

$$I(s_1, s_2, \ldots, s_N) \equiv I(s_1) \times I(s_2) \times \cdots \times I(s_N), \tag{106}$$

$$\hat{I}(s_1, s_2, \ldots, s_N) \equiv \hat{I}(s_1) \times \hat{I}(s_2) \times \cdots \times \hat{I}(s_N). \tag{107}$$

Finally, we define

$$\mathfrak{M}(F) \equiv \{(s_1, \ldots, s_N) \mid |s_1| = \cdots = |s_N| \ \& \ I(s_1, \ldots, s_N) \cap (F \bmod 1) \neq \phi\} \tag{108}$$

and

$$\hat{\mathfrak{M}}(F) \equiv \{(s_1, \ldots, s_N) \mid |s_1| = \cdots = |s_N| \ \& \ \hat{I}(s_1, \ldots, s_N) \cap (F \bmod 1) \neq \phi\}. \tag{109}$$

We say that F satisfies the r.e. condition if there exists an r.e. set L such that $\mathfrak{M}(F) \subset L \subset \hat{\mathfrak{M}}(F)$.

The meaning of the r.e. condition is as follows. First we note that F mod $1 \subset [0,1)^N$. For all $n \in \mathbb{N}$, divide $[0,1)^N$ into 2^{Nn} pieces of N dimensional subintervals in the form $I(s_1, s_2, \ldots, s_N)$ with $|s_1| = |s_2| = \cdots = |s_N| = n$. Then, intuitively, the r.e. condition is that all of the subintervals I's intersecting F mod 1 and some of the subintervals neighboring to these I's form an r.e. set.

Let F be a bounded subset of $\mathbb{R}^N$, and let $N_\delta(F)$ be the smallest number of closed balls of radius δ that cover F. The upper box-counting dimension of F is defined as

$$\overline{\dim}_B F \equiv \varlimsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{110}$$

Theorem A.6 (Kolmogorov) *If F is a bounded subset of $\mathbb{R}^N$ and satisfies the r.e. condition, then x is $(\overline{\dim}_B F)/N$-compressible for any $x \in F$, i.e.,*

$$\forall x \in F \quad \forall n \in \mathbb{N} \quad H(x_n) \leq \frac{\overline{\dim}_B F}{N} n + o(n). \tag{111}$$

Proof. The essential part of the proof is due to Kolmogorov.

For each $m \in \mathbb{N}$, we consider the collection of cubes in the 2^{-m}-coordinate mesh of $\mathbb{R}^N$, i.e., the collection of sets of the form

$$[l_1 2^{-m}, (l_1+1)2^{-m}) \times \cdots \times [l_N 2^{-m}, (l_N+1)2^{-m}), \tag{112}$$

where $l_1, \dots, l_N$ are integers. Let $M_m(F)$ be the number of 2^{-m}-mesh cubes that intersect F. It is then shown that

$$\overline{\dim}_B F \equiv \varlimsup_{m \to \infty} \frac{\log_2 M_m(F)}{m} \tag{113}$$

(see e.g., [F2]).

Suppose that $x = (x^1, \dots, x^N)$ is any point in F. Since the r.e. condition holds for F, there exists an r.e. set L such that $\mathfrak{M}(F) \subset L \subset \hat{\mathfrak{M}}(F)$. We consider the following procedure in order to calculate x_n.

Given n, one enumerates all elements $(s_1, \dots, s_N)$ of L such that $|s_1| = \lceil n/N \rceil$. There then appears the element $(t_1, \dots, t_N)$ in the enumeration with the property that

$$(x^1 \bmod 1, \dots, x^N \bmod 1) \in I(t_1, \dots, t_N). \tag{114}$$

Assume this $(t_1, \dots, t_N)$ is the k_n-th element in the enumeration order. If one knows n and k_n, then one can calculate the first $\lceil n/N \rceil$ bits of the base-two expansion of each $x^i \bmod 1$ with infinitely many zeros and hence one can calculate x_n further.

Thus, since $k_n \leq 3^N M_{\lceil n/N \rceil}(F)$, we see that $H(x_n) \leq \log_2 M_{\lceil n/N \rceil}(F) + o(n)$. Using (113), the result is produced. □

Theorem A.6 immediately gives the following corollary.

Corollary A.7 *Suppose that F is a bounded subset of $\mathbb{R}^N$ and satisfies the r.e. condition. Then $\overline{\dim}_A F \leq \overline{\dim}_B F$. Moreover, for each $k = 1, 2, 3, 4$, if $\dim_{Ak} F$ exists then $\dim_{Ak} F \leq \overline{\dim}_B F$.*

Theorem A.8 *Let F be a bounded subset of $\mathbb{R}^N$. Suppose that F satisfies the r.e. condition and $\dim_H F = \overline{\dim}_B F$. Let $D = \dim_H F$.*

(a) *$\dim_{A4} F$ exists and $\dim_{A4} F = \dim_H F$.*

(b) *If $\mathcal{H}^D(F) > 0$, then $\dim_{A3} F$ exists and $\dim_{A3} F = \dim_H F$.*

(c) *If $\mathcal{H}^D(F) > 0$ and $\dim_H F$ is a computable real number, then both $\dim_{A1} F$ and $\dim_{A2} F$ exist and $\dim_{A1} F = \dim_{A2} F = \dim_H F$.*

Proof. It follows from Theorem A.6 that x is $(\dim_H F)/N$-compressible for any $x \in F$. Using Corollary A.5, we see that $\dim_{A4} F = \dim_H F$. If $\mathcal{H}^D(F) > 0$ then, from Corollary A.4, $\mathcal{H}^D(F \backslash T_D^3) = \mathcal{H}^D(F) > 0$. Hence, $F \backslash T_D^3 \neq \phi$, therefore there is $x \in F$ which is semi D/N-random. Thus, we see that $\dim_{A3} F = \dim_H F$. Moreover, if $\dim_H F$ is a computable real number, then using Theorem A.3 in a similar manner we see that $\dim_{A1} F = \dim_{A2} F = \dim_H F$. □

Theorem A.9 *Let $S_1, \ldots, S_m$ be contractions on $\mathbb{R}^N$. Suppose that each S_i is an affine transformation, i.e., for each i, $S_i(x) = M_i x + v_i$ where M_i is an $N \times N$ matrix and v_i is a vector in $\mathbb{R}^N$. If all matrix elements of M_i and all components of v_i are computable real numbers for each i, then the r.e. condition holds for the invariant set of $S_1, \ldots, S_m$.*

Proof. Let F be the invariant set of $S_1, \ldots, S_m$. Since $S_1, \ldots, S_m$ are contractions on $\mathbb{R}^N$, there exists $l \in \mathbb{N}$ such that $S_i(E) \subset E$ for each i where $E = \{x \in \mathbb{R}^N \mid |x| \leq l/2\}$. We write $S_{i_1,\ldots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}$. Using Theorem 4.2, for each k it is shown that

$$F \subset \bigcup_{i_1,\ldots,i_k} S_{i_1,\ldots,i_k}(E) \tag{115}$$

and $F \cap S_{i_1,\ldots,i_k}(E) \neq \phi$ for any $i_1, \ldots, i_k$. Let $c_1, \ldots, c_m$ be the ratios of $S_1, \ldots, S_m$ respectively. We choose $r \in \mathbb{Q}$ such that $c_i < r < 1$ for all i and choose $x_0 \in E \cap \mathbb{Q}^N$ such as $(0, 0, \ldots, 0)$. Then $|S_{i_1,\ldots,i_k}(E)| \leq l r^k$ for any k and $i_1, \ldots, i_k$. Since all matrix elements of M_i and all components of v_i are computable real numbers for all i, it follows that given k, $i_1, \ldots, i_k$, and $n \in \mathbb{N}$ one can find an $f(n; k; i_1, \ldots, i_k) \in \mathbb{Q}^N$ with the property that

$$|f(n;k;i_1,\dots,i_k) - S_{i_1,\dots,i_k}(x_0)| \le \frac{1}{n}. \tag{116}$$

It is then shown that $\mathfrak{M}(F) \subset L \subset \hat{\mathfrak{M}}(F)$ holds for the set L accepted by the following procedure, and hence F satisfies the r.e. condition.

Given $(s_1,\dots,s_N) \in X^N$, one checks whether or not $|s_1| = \cdots = |s_N|$ holds true. If this does not hold true, then one does not accept $(s_1,\dots,s_N)$. Otherwise when this does hold true, one chooses k such that $lr^k \le \delta/4$ and chooses n such that $1/n \le \delta/4$, where $\delta = 2^{-|s_1|}$. Let

$$\begin{aligned} &A^j(i_1,\dots,i_k) \\ &\quad = [\, f^j(n;k;i_1,\dots,i_k) - \delta/2,\ f^j(n;k;i_1,\dots,i_k) + \delta/2\,] \end{aligned} \tag{117}$$

where $f^j(n;k;i_1,\dots,i_k)$ is the j-th component of $f(n;k;i_1,\dots,i_k)$, and let

$$A(i_1,\dots,i_k) = A^1(i_1,\dots,i_k) \times \cdots \times A^N(i_1,\dots,i_k). \tag{118}$$

It is easy to see that

$$F \subset \bigcup_{i_1,\dots,i_k} A(i_1,\dots,i_k) \tag{119}$$

and $F \cap A(i_1,\dots,i_k) \ne \phi$ for any $i_1,\dots,i_k$. One then accepts $(s_1,\dots,s_N)$ if and only if one can find a $(i_1,\dots,i_k)$ such that $I(s_1,\dots,s_N) \cap (A(i_1,\dots,i_k) \bmod 1) \ne \phi$. □

We refer to the following familiar theorem on a self-similar set (e.g., Theorem 9.3 in [F2]).

Theorem A.10 *Suppose that the open set condition* (70) *holds for similarities $S_1,\dots,S_m$ on $\mathbb{R}^N$ with ratios $c_1,\dots,c_m$ respectively. If F is the invariant set of $S_1,\dots,S_m$, then* $\dim_H F = \overline{\dim}_B F = D$ *and* $0 < \mathcal{H}^D(F) < \infty$*, where D is given by*

$$\sum_{i=1}^{m} c_i^D = 1. \tag{120}$$

The proof of Theorem 4.3 is as follows.

Proof of Theorem 4.3. Suppose that the open set condition holds for similarities $S_1,\dots,S_m$ on $\mathbb{R}^N$ with ratios $c_1,\dots,c_m$ respectively. Let F be the invariant set of $S_1,\dots,S_m$. From Theorem A.10, it is shown that $\dim_H F = \overline{\dim}_B F = D$ and $0 < \mathcal{H}^D(F) < \infty$, where D satisfies (71). Furthermore, suppose that for each i there exist $N \times N$ matrix M_i and $v_i \in \mathbb{R}^N$ such that all matrix elements of M_i and all components of v_i are computable real numbers and $S_i(x) = M_i x + v_i$. Using Theorem A.9, we see that F satisfies the r.e. condition. Since each c_i is a computable real number, D satisfying (71) is also a computable real number. It follows from Theorem A.8 that $\dim_{A1} F$ exists and $\dim_{A1} F = \dim_H F$. □

Corollary A.11 *Let $P = \{p_1, p_2, \ldots, p_m\}$ be a finite prefix-free subset of X, and let F be the set of infinite binary sequences that consist of elements of P, i.e.,*

$$F = \left\{ q_1 q_2 q_3 \ldots \in X^\infty \,\middle|\, q_i \in P \text{ for each } i \right\}. \tag{121}$$

Then

$$\forall \alpha \in F \quad \forall n \in \mathbb{N} \quad H(\alpha_n) \le Dn + o(n), \tag{122}$$

$$\exists \alpha \in F \quad \lim_{n\to\infty} H(\alpha_n) - Dn = \infty, \tag{123}$$

where D is given by

$$\sum_{i=1}^{m} 2^{-D|p_i|} = 1. \tag{124}$$

Proof. For each i, let S_i be the similarity on $\mathbb{R}$ with $S_i(x) = 2^{-|p_i|}x + 0.p_i$. Since P is a prefix-free set, it is shown that $S_1, \ldots, S_m$ satisfy the open set condition. Note that all of $2^{-|p_i|}$ and $0.p_i$ are computable real numbers. Let $R(F) = \left\{ 0.\alpha \,\middle|\, \alpha \in F \right\}$. Then $R(F)$ is the invariant set of $S_1, \ldots, S_m$. From Theorem 4.3, we see that $\dim_{A1} R(F) = D$, and hence the results are produced. □

Proposition A.12 *Let D be a real number with $0 \le D \le N$.*
(a) $\dim_H T_D^3 = D$.
(b) *If D is a computable real number then* $\dim_H T_D^1 = \dim_H T_D^2 = D$.

Proof. Since the results are trivial for $D = 0$, we assume that $D > 0$.

(a) For any $\varepsilon > 0$, we choose a computable real number d such that $D - \varepsilon \le d < D$. Noting Theorem 4.3, we can construct similarities $S_1, \ldots, S_m$ on $\mathbb{R}^N$ such that $\dim_{A3} F = \dim_H F = d$ holds for the invariant set F of $S_1, \ldots, S_m$. Hence, $F \subset T_D^3$ and therefore $D - \varepsilon \le \dim_H T_D^3$. Thus, $D \le \dim_H T_D^3$ and using Corollary A.4 the result is produced.

(b) From the fact that $T_D^3 \subset T_D^2 \subset T_D^1$, it follows that $D \leq \dim_H T_D^2 \leq \dim_H T_D^1$. Since D is a computable real number, using Theorem A.3 we see that $\dim_H T_D^1 \leq D$. Thus, the result is produced. □

Note that Proposition A.12 (a) was derived by [R1].

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