Filomat 29:7 (2015), 1549–1556
DOI 10.2298/FIL1507549K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Generalization of Ćirić Fixed Point Theorems

Poom Kumam[a], Nguyen Van Dung[b], Kanokwan Sitthithakerngkiet[*,c]

[a]*Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand*
[b]*Institute of Research and Development, Duy Tan University, R.809, K7/25 Quang Trung, Da Nang, Viet Nam*
[c]*Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand*

Abstract.
In this paper, we state and prove a generalization of Ćirić fixed point theorems in metric space by using a new generalized quasi-contractive map. These theorems extend other well known fundamental metrical fixed point theorems in the literature (Banach [1], Kannan [11], Nadler [13], Reich [15], etc.) Moreover, a multi-valued version for generalized quasi-contraction is also established.

1. Introduction

The Banach's contraction principle [1] which was first appeared in 1922 is one of the most useful and important theorems in classical functional analysis. Its utility is not only to prove that, in a complete metric space X, the contraction map T (i.e., $d(Tx, Ty) \leq \alpha d(x, y)$ for some $0 \leq \alpha < 1$ and for all $x, y \in X$) has a unique fixed point but also to show that the Picard iteration converges to the fixed point. For the reason that the contraction must be continuous, there are many researchers establish the fixed point theorems on various classes of operators that are weaker than contractive conditions but are not continuous, see for example [11, 15].

One of the most well-known results in generalizations of Banach's contraction principle which the Picard iteration still converges to the fixed point of map is the Ćirić fixed point theorem [4]. Before providing the Ćirić fixed point theorem, we recall that a self-map T on a metric space (X, d), is said to be a *quasi-contraction* iff there exists a nonnegative number $q < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \quad \leq \quad q \cdot \max\Big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big\}. \tag{1}$$

The Ćirić fixed point theorem is given by the following theorem.

Theorem 1.1 ([4], Theorem 1). *Let the metric space X be T-orbitally complete and let T be a quasi-contraction. Then we have*

2010 *Mathematics Subject Classification*. Primary 47H10, 54H25; Secondary 54D99, 54E99
Keywords. Ćirić fixed point, metric space
Received: 30 December 2013; Accepted: 17 March 2014
Communicated by Ljubomir Ćirić
*The corresponding author: kanokwans@kmutnb.ac.th (K. Sitthithakerngkiet)
Email addresses: poom.kum@kmutt.ac.th (Poom Kumam), nvdung@dthu.edu.vn (Nguyen Van Dung), kanokwans@kmutnb.ac.th (Kanokwan Sitthithakerngkiet*)

1. *T has a unique fixed point x^* in X.*
2. $\lim_{n\to\infty} T^n x = x^*$ *for all* $x \in X$.
3. $d(T^n x, x^*) \leq \frac{q^n}{1-q} d(x, Tx)$ *for all* $x \in X$.

This result was generalized to many results, such as a common fixed point theorem of nonlinear contraction [17, Theorem 4], a generalized φ-contraction [2, Section 2.6], a Ćirić almost contraction [3, Theorem 3.2] and see also [5, 10, 12]. But from the well-known result of Rhoades [14] in 1977 to recent surveys, in Berinde [2] and Collaco and Silva [6] for instance, there were no any other value added to quasi-contraction condition. On the other hand, the Banach's contraction principle has been extended to multi-valued contractions by Nadler [13] and see also [7–9, 16].

In this paper, we define a new generalized quasi-contraction by adding four new values $d(T^2x, x)$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to a quasi-contraction condition. Also, an example is presented. After that we state and prove unique fixed point theorems which are the generalization of Ćirić fixed point theorem in [4]. Moreover, we also establish fixed point theorems for multi-valued generalized quasi-contraction.

2. Preliminaries

First, we recall some notions which will be used in what follows. Let (X, d) be a metric space and A, B be any two subsets of X. We denote

$$\begin{aligned} D(A,B) &= \inf\Big\{d(a,b) : a \in A, b \in B\Big\} \\ \rho(A,B) &= \sup\Big\{d(a,b) : a \in A, b \in B\Big\} \\ BN(X) &= \Big\{A : \emptyset \neq A \subset X \text{ and } \delta(A) < +\infty\Big\}, \end{aligned}$$

where $\delta(A) := \sup\Big\{d(a,b) : a, b \in A\Big\}$.

Definition 2.1 ([4]). *Let $T : X \longrightarrow X$ be a map on metric space. For each $x \in X$ and for any positive integer n, put*

$$O_T(x, n) = \{x, Tx, \ldots, T^n x\} \text{ and } O_T(x, +\infty) = \{x, Tx, \ldots, T^n x, \ldots\}.$$

The set $O_T(x, +\infty)$ is called the orbit *of T at x and the metric space X is called T*-orbitally complete *if every Cauchy sequence in $O_T(x, +\infty)$ is convergent in X.*

Note that every complete metric space is T-orbitally complete for all maps $T : X \longrightarrow X$. The following example shows that there exists a T-orbitally complete metric space but it is not complete.

Example 2.2. *Let (X, d) be a metric space which is not complete and $T : X \longrightarrow X$ be the map defined by $Tx = x_0$ for all $x \in X$ and some $x_0 \in X$. Then (X, d) is a T-orbitally complete metric space which is not complete.*

Definition 2.3 ([4]). *Let $F : X \longrightarrow BN(X)$ be a multi-valued mapping. Let $x_0 \in X$, an* orbit *of F at x_0 is a sequence*

$$\Big\{x_n : x_n \in Fx_{n-1}, n \in \mathbb{N}\Big\}.$$

A space X is called to be F-orbitally complete *if every Cauchy sequence which is a subsequence of an orbit of F at x for some $x \in X$, converges in X.*

Next, the definitions of generalized quasi-contraction for single-valued and multi-valued are given as follows;

Definition 2.4. *Let $T : X \longrightarrow X$ be a mapping on metric space X. The mapping T is said to be a* generalized quasi-contraction *iff there exists $q \in [0,1)$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx,Ty) \quad \le \quad & q \cdot \max\Big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ & d(T^2x,x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}. \end{aligned} \tag{2}$$

Example 2.5. *Let $X = \{1,2,3,4,5\}$ with d defined as*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x,y) \in \Big\{(1,4),(1,5),(4,1),(5,1)\Big\} \\ 1 & \text{otherwise.} \end{cases}$$

Let $T : X \longrightarrow X$ be defined by

$$T1 = T2 = T3 = 1, T4 = 2, T5 = 3.$$

Then, we have

$$d(Tx,Ty) = d(1,1) = 0 \;\; \text{if} \;\; x, y \in \{1,2,3\};$$

$$d(T1,T4) = d(T2,T4) = d(T3,T4) = d(1,2) = 1;$$

$$d(T1,4) = d(T2,4) = d(T3,4) = d(1,4) = 2;$$

$$d(T1,T5) = d(T2,T5) = d(T3,T5) = d(1,3) = 1;$$

$$d(T1,5) = d(T2,5) = d(T3,5) = d(1,5) = 2;$$

$$d(T4,T5) = d(2,3) = 1;$$

$$d(4,5) = d(4,T4) = d(5,T5) = d(4,T5) = d(5,T4) = 1;$$

$$d(T^24,4) = d(T2,4) = d(1,4) = 2;$$

$$d(T^25,5) = d(T3,5) = d(1,5) = 2.$$

The above calculations show that T is not quasi-contraction for $x = 4$ and $y = 5$ because there is no a nonnegative number $q < 1$ satisfying the equation (2). However, T is generalized quasi-contraction since the (2) holds for some $q \in \Big[0.5, 1\Big)$ and for all $x, y \in X$.

3. The Main Results

On the following results, we state and prove the new fixed point theorems which are general cases of the Ćirić fixed point theorem.

Theorem 3.1. *Let (X,d) be a metric space. Suppose that $T : X \longrightarrow X$ is a generalized quasi-contraction and X is T-orbitally complete. Then we have*

1. *T has a unique fixed point x^* in X.*
2. *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
3. *$d(T^n x, x^*) \le \dfrac{q^n}{1-q} d(x,Tx)$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. (1). **Step 1.** *T has a fixed point.* For each $x \in X$ and $1 \le i \le n-1$ and $1 \le j \le n$, we have

$$\begin{aligned} & d(T^i x, T^j x) \qquad (3)\\ = \ & d(TT^{i-1}x, TT^{j-1}x) \\ \le \ & q.\max\Big\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, TT^{i-1}x), d(T^{j-1}x, TT^{j-1}x), d(T^{i-1}x, TT^{j-1}x), \\ & d(T^{j-1}x, TT^{i-1}x), d(T^2T^{i-1}x, T^{i-1}x), d(T^2T^{i-1}x, TT^{i-1}x), d(T^2T^{i-1}x, T^{j-1}x) \\ & d(T^2T^{i-1}x, TT^{j-1}x)\Big\} \\ = \ & q.\max\Big\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x), d(T^{i-1}x, T^j x), \\ & d(T^{j-1}x, T^i x), d(T^{i+1}x, T^{i-1}x), d(T^{i+1}x, T^i x), d(T^{i+1}x, T^{j-1}x) \\ & d(T^{i+1}x, T^j x)\Big\} \\ \le \ & q.\delta\Big[O_T(x,n)\Big] \end{aligned}$$

where $\delta\Big[O_T(x,n)\Big] = \max\Big\{d(T^i x, T^j x) : 0 \le i, j \le n\Big\}$.

From (3), since $0 \le q < 1$, there exists $k_n(x) \le n$ such that

$$d(x, T^{k_n(x)}x) = \delta\Big[O_T(x,n)\Big]. \qquad (4)$$

Then we have

$$\begin{aligned} d(x, T^{k_n(x)}x) &\le d(x, Tx) + d(Tx, T^{k_n(x)}x) \\ &\le d(x, Tx) + q.\delta\Big[O_T(x,n)\Big] \\ &= d(x, Tx) + q.d(x, T^{k_n(x)}x). \end{aligned}$$

It implies that

$$\delta\Big[O_T(x,n)\Big] = d(x, T^{k_n(x)}x) \le \frac{1}{1-q} d(x, Tx). \qquad (5)$$

For all $n, m \le 1$ and $n < m$, it follows from the generalized quasi-contractive condition of T and (5) that

$$\begin{aligned} d(T^n x, T^m x) &= d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \qquad (6)\\ &\le q.\delta\Big[O_T(T^{n-1}x, m-n+1)\Big] \\ &= q.d(T^{n-1}x, T^{k_{m-n+1}(T^{n-1}x)}T^{n-1}x) \\ &= q.d(TT^{n-2}x, T^{k_{m-n+1}(T^{n-1}x)+1}T^{n-2}x) \\ &\le q^2.\delta\Big[O_T(T^{n-2}x, k_{m-n+1}(T^{n-1}x)+1)\Big] \\ &\le q^2.\delta\Big[O_T(T^{n-2}x, m-n+2)\Big] \\ &\le \ldots \\ &\le q^n.\delta\Big[O_T(x,m)\Big] \\ &\le \frac{q^n}{1-q} d(x, Tx). \end{aligned}$$

Since $\lim_{n\to\infty} q^n = 0$, $\{T^n x\}$ is a Cauchy sequence in X. Since X is T-orbitally complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} T^n x = x^*. \qquad (7)$$

By using the generalized quasi-contractive condition of T again, we have

$$\begin{aligned} & d(x^*, Tx^*) \\ \leq\ & d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*) \\ =\ & d(x^*, T^{n+1}x) + d(TT^n x, Tx^*) \\ \leq\ & d(x^*, T^{n+1}x) + q.\max\Big\{d(T^n x, x^*), d(T^n x, TT^n x), d(x^*, Tx^*), d(T^n x, Tx^*), \\ & d(x^*, TT^n x), d(T^2T^n x, T^n x), d(T^2T^n x, TT^n x), d(T^2T^n x, x^*), d(T^2T^n x, Tx^*)\Big\} \\ =\ & d(x^*, T^{n+1}x) + q.\max\Big\{d(T^n x, x^*), d(T^n x, T^{n+1}x), d(x^*, Tx^*), d(T^n x, Tx^*), \\ & d(x^*, T^{n+1}x), d(T^{n+2}x, T^n x), d(T^{n+2}x, T^{n+1}x), d(T^{n+2}x, x^*), d(T^{n+2}x, Tx^*)\Big\}. \end{aligned} \tag{8}$$

Taking the limit as $n \to \infty$ in (8), and using (7), we get $d(x^*, Tx^*) \leq qd(x^*, Tx^*)$. Since $q \in [0, 1)$, we obtain $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. Then T has a fixed point.

Step 2. *The fixed point of T is unique.* Let x^*, y^* be two fixed points of T. Since T is generalized quasi-contraction, we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq q.\max\Big\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(x^*, Ty^*), d(y^*, Tx^*), \\ &\quad d(T^2x^*, x^*), d(T^2x^*, Tx^*), d(T^2x^*, y^*), d(T^2x^*, Ty^*)\Big\} \\ &= qd(x^*, y^*). \end{aligned}$$

Since $q \in [0, 1)$, we obtain $d(x^*, y^*) = 0$. That is, $x^* = y^*$. Then the fixed point of T is unique.

(2). It is proved by (7).

(3). Taking the limit as $m \to \infty$ in (6), we get $d(T^n x, x^*) \leq \dfrac{q^n}{1-q} d(x, Tx)$. □

Corollary 3.2. *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map satisfying the following:*

1. *X is T-orbitally complete.*
2. *There exists $k \in \mathbb{N}$ and $q \in [0, 1)$ such that for all $x, y \in X$,*

$$\begin{aligned} d(T^k x, T^k y) \leq\ & q.\max\Big\{d(x, y), d(x, T^k x), d(y, T^k y), d(x, T^k y), d(y, T^k x), \\ & d(T^{2k}x, x), d(T^{2k}x, T^k x), d(T^{2k}x, y), d(T^{2k}x, T^k y)\Big\}. \end{aligned} \tag{9}$$

Then we have

1. *T has a unique fixed point x^* in X.*
2. *$d(T^n x, x^*) \leq \dfrac{q^m}{1-q}\max\Big\{d(T^i x, T^{i+k}x) : i = 0, 1, \ldots, k-1\Big\}$ for all $x \in X$ and $n \in \mathbb{N}$ where m is the greatest integer not exceeding $\dfrac{n}{k}$.*
3. *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*

Proof. (1). By the conclusion of Theorem 3.1, T^k has a unique fixed point x^* and $T^k(Tx^*) = T(T^k x^*) = Tx^*$. It implies that $Tx^* = x^*$, that is, T has a fixed point x^*. The uniqueness of the fixed point of T is easy to see.

(2). Let $n \in \mathbb{N}$. Then $n = mk + j$, $0 \leq j < k$ and for each $x \in X$, $T^n x = (T^k)^m T^j x$. It follows from Theorem 3.1.(3) that

$$\begin{aligned} d(T^n x, x^*) &\leq \frac{q^m}{1-q} d(T^j x, T^k T^j x) \\ &\leq \frac{q^m}{1-q}\max\Big\{d(T^i x, T^{i+k}x) : i = 0, 1, \ldots, k-1\Big\}. \end{aligned}$$

(3). It is a direct consequence of (2). □

Corollary 3.3 ([4], Theorem 2). *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map satisfying the following:*

1. *X is T-orbitally complete.*
2. *There exists $k \in \mathbb{N}$ and $q \in [0, 1)$ such that for all $x, y \in X$,*

$$d(T^k x, T^k y) \leq q.\max\Big\{d(x, y), d(x, T^k x), d(y, T^k y), d(x, T^k y), d(y, T^k x)\Big\}. \quad (10)$$

Then we have

1. *T has a unique fixed point x^* in X;*
2. *$d(T^n x, x^*) \leq \dfrac{q^m}{1-q}\max\Big\{d(T^i x, T^{i+k} x) : i = 0, 1, \ldots, k-1\Big\}$ for all $x \in X$ and $n \in \mathbb{N}$ where m is the greatest integer not exceeding $\dfrac{n}{k}$;*
3. *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*

Now, we denote the multi-valued mapping $F : X \longrightarrow BN(X)$ of generalized quasi-contraction by

$$\begin{aligned}\rho(Fx, Fy) \leq\ & q.\max\Big\{d(x, y), \rho(x, Fx), \rho(y, Fy), D(x, Fy), D(y, Fx),\\ & D(F^2 x, x), D(F^2 x, Fx), D(F^2 x, y), D(F^2 x, Fy)\Big\},\end{aligned} \quad (11)$$

for some $q \in [0, 1)$ and for all $x, y \in X$. The following theorem presents the fixed point theorem for multi-valued version of generalized quasi-contractive mapping.

Theorem 3.4. *Let (X, d) be a metric space and $F : X \longrightarrow BN(X)$ be a multi-valued map. Suppose that F is a generalized quasi-contraction and X is F-orbitally complete. Then we have*

1. *F has a unique fixed point x^* in X and $Fx^* = \{x^*\}$.*
2. *For each $x_0 \in X$, there exists an orbit $\{x_n\}_n$ of F at x_0 such that $\lim_{n\to\infty} x_n = x^*$ for all $x \in X$, and*
3. *$d(x_n, x^*) \leq \dfrac{(q^{1-a})^n}{1-q^{1-a}} d(x_0, x_1)$ for all $n \in \mathbb{N}$, where $a < 1$ is any fixed positive number.*

Proof. (1). Given $a \in (0, 1)$ and defined a single-valued mapping $T : X \longrightarrow X$ by the following statement:

$$for\ each\ x \in X,\ Tx \in Fx\ \ satisfies\ \ d(x, Tx) \geq q^a \rho(x, Fx).$$

By the Definition 2.3 and the condition of F, we have for every $x, y \in X$,

$$\begin{aligned}d(Tx, Ty) &\leq \rho(Fx, Fy)\\ &\leq q\max\Big\{d(x, y), \rho(x, Fx), \rho(y, Fy), D(x, Fy), D(y, Fx),\\ &\quad D(F^2 x, x), D(F^2 x, Fx), D(F^2 x, y), D(F^2 x, Fy)\Big\}\\ &= qq^{-a}\max\Big\{q^a d(x, y), q^a \rho(x, Fx), q^a \rho(y, Fy), q^a D(x, Fy), q^a D(y, Fx),\\ &\quad q^a D(F^2 x, x), q^a D(F^2 x, Fx), q^a D(F^2 x, y), q^a D(F^2 x, Fy)\Big\}\\ &\leq q^{1-a}\max\Big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),\\ &\quad d(T^2 x, x), d(T^2 x, Tx), d(T^2 x, y), d(T^2 x, Ty)\Big\}.\end{aligned}$$

By Theorem 3.1, we conclude that T has a unique fixed point x^*. Then $\rho(x^*, Fx^*) \leq q^a d(x^*, Tx^*) = 0$ implies that $\rho(x^*, Fx^*) = 0$. Then x^* is a fixed point of F and $Fx^* = \{x^*\}$. From the direct consequences of Theorem 3.1 where $x_n = T^n x$ for all $n \in \mathbb{N}$, we obtain that (2) and (3) hold. □

Corollary 3.5 ([4], Theorem 3). *Let (X, d) be a metric space and $F : X \longrightarrow BN(X)$ be a multi-valued map satisfying the following:*

1. *X is F-orbitally complete.*
2. *There exists $q \in [0, 1)$ such that for all $x, y \in X$,*

$$\rho(Fx, Fy) \quad \leq \quad q.\max\Big\{d(x, y), \rho(x, Fx), \rho(y, Fy), D(x, Fy), D(y, Fx)\Big\}. \tag{12}$$

Then we have

1. *F has a unique fixed point x^* in X and $Fx^* = \{x^*\}$.*
2. *For each $x_0 \in X$, there exists an orbit $\{x_n\}_n$ of F at x_0 such that $\lim_{n\to\infty} x_n = x^*$ for all $x \in X$, and*
3. *$d(x_n, x^*) \leq \dfrac{(q^{1-a})^n}{1 - q^{1-a}} d(x_0, x_1)$ for all $n \in \mathbb{N}$, where $a < 1$ is any fixed positive number.*

Example 3.6. *Let (X, d) and $T : X \longrightarrow X$ be defined by Example 2.5.*

It is easy to see that X is T-orbitally complete metric space. By the definition of the distance d and mapping T , we conclude that X and T satisfy all of the conditions in Theorem 3.1. Clearly, $x^* = 1$ is a unique fixed point of T.

Note that, if $x \in \{1, 2, 3\}$ then $T^n x = 1$ for $n = 1, 2, 3, ...$ and if $x \in \{4, 5\}$ then $T^n x = 1$ for $n = 2, 3, 4, ...$ That is $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.

Let $q \in \Big[0.5, 1\Big)$ be fixed by the generalized quasi-contraction of T which arises from Example 2.5. We see that the inequality $d(T^n x, x^*) \leq \dfrac{q^n}{1 - q} d(x, Tx)$ holds for all $x \in X$ and $n \in \mathbb{N}$.

Therefore, this example is presented to certify the results of Theorem 3.1. However, it is not applicable to Theorem 1.1.

Lemma 3.7. *Example 3.6 shows that our results are proper generalizations of Ćirić fixed point theorems in [4]. Then our results are exactly a new form of fixed point theorems in metric spaces. Moreover, we may generalized other fixed point theorems contained at most five mentioned values in the literature to that contain $d(T^2x, x)$, $d(T^2, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ in addition.*

Acknowledgements The first author was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU No.58000512). Moreover, the third author was funded by KingMongkut's University of Technology North Bangkok (Contract no. KMUTNB-GEN-58-08).

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math. **3** (1922) 133 – 181.
[2] V. Berinde, *Iterative approximation of fixed points*, Lecture Notes in Mathematics, Springer, 2006.
[3] V. Berinde, *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math. **24** (2008) no. 2, 10 – 19.
[4] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974) 267 – 273.
[5] L. B. Ćirić, *Non-self mappings satisfying non-linear contractive condition with applications*, Nonlinear Anal. **71** (2009) no. 7, 2927 – 2935.
[6] P. Collaço and J. C. E. Silva, *A complete comparison of 25 contraction conditions*, Nonlinear Anal. **30** (1997) no. 1, 471 – 476.
[7] P. Z. Daffer and H. Kaneko, *Fixed points of generalized contractive multi-valued mappings*, J. Math. Anal. Appl. **192** (1995) no. 2, 655 – 666.
[8] A. A. Eldred, J. Anuradha, and P. Veeramani, *On equivalence of generalized multi-valued contractions and Nadler's fixed point theorem*, J. Math. Anal. Appl. **336** (2005) no. 2, 751 – 757.
[9] A. A. Harandi, *Fixed point theory for set-valued quasi-contraction maps in metric spaces*, Appl. Math. Lett. **24** (2011), no. 11, 1791 – 1794.
[10] M. Jleli and B. Samet, *The Kannan's fixed point theorem in a cone rectangular metric space*, J. Nonlinear Sci. Appl. **2** (2009) no. 3, 161 – 167.
[11] R. Kannan, *Some results on fixed points*, Bull. Culcutta Math. Soc. **60** (1968) 71 – 76.
[12] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009) no. 12, 4341 – 4349.

[13] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969) no. 2, 475 – 488.
[14] B. E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977) 257 – 290.
[15] S. Rolewicz, *Metric linear spaces*, PWN-Polish Scientific Publishers, Warszawa, 2002.
[16] P. V. Semenov, *Fixed points of multivalued contractions*, Funct. Anal. Appl. **36** (2002) no. 2, 159 – 161.
[17] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd)(N.S.) **32** (1982) no. 46, 14 – 153.