A generalization of contact metric manifolds

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Abstract. In this paper, we give a characterization of a contact metric manifold as a special almost contact metric manifold and discuss an almost contact metric manifold which is a natural generalization of the contact metric manifolds introduced by Y. Tashiro.

M.S.C. 2010: 53B20, 53C20.
Key words: contact metric manifold; quasi contact metric manifold.

1 Introduction

A $(2n+1)$-dimensional smooth manifold M is called a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Then we call the 1-form η a contact form of M. It is well-known that given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X on M. A Riemannian metric g is said to be an *associated metric* to a contact form η if there exists a $(1,1)$-tensor field ϕ satisfying

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y) \tag{1.1}$$

for $X, Y \in \mathfrak{X}(M)$. A $(2n+1)$-dimensional smooth manifold equipped with a triple (ϕ, ξ, η) of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η on M satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad and \quad \eta(\xi) = 1 \tag{1.2}$$

for $X \in \mathfrak{X}(M)$ is called an *almost contact manifold* with the almost contact structure (ϕ, ξ, η). Further, an almost contact manifold $M = (M, \phi, \xi, \eta)$ equipped with a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(\xi, X) \tag{1.3}$$

for $X, Y \in \mathfrak{X}(M)$ is called an *almost contact metric manifold* with the almost contact metric structure (ϕ, ξ, η, g). From (1.1) $\sim$ (1.3), we may regard a contact metric manifold as a special almost contact metric manifold.

D. Chinea and C. Gonzalez [2] obtained a classification of the $(2n+1)$-dimensional almost contact metric manifold based on $U(n) \times I$ representation theory, which is

Balkan Journal of Geometry and Its Applications, Vol.19, No.2, 2014, pp. 94-105.

an analogy of the classification of the $2n$-dimensional almost Hermitian manifolds established by A. Gray and H. M. Hervella [3].

Now, let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $\bar{M} = M \times \mathbb{R}$ be the product manifold of M and a real line $\mathbb{R}$ equipped with the following almost Hermitian structure $(\bar{J}, \bar{g})$ defined by

$$\begin{aligned} &\bar{J}X = \phi X - \eta(X)\frac{\partial}{\partial t}, \quad \bar{J}\frac{\partial}{\partial t} = \xi, \\ &\bar{g}(X,Y) = e^{-2t}g(X,Y), \quad \bar{g}(X, \frac{\partial}{\partial t}) = 0, \quad \bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = e^{-2t} \end{aligned} \tag{1.4}$$

for $X, Y \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$. In the case where $\bar{J}$ is integrable, the corresponding almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is said to be *normal*. Especially, a normal contact metric manifold is called a Sasakian manifold. Y. Tashiro [5] discussed the relation ship between the classes of almost Hermitian manifolds and the corresponding ones of almost contact metric manifolds and showed the following:

Fact 1. $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is a Kähler manifold if and only if $M = (M, \phi, \xi, \eta, g)$ is a Sasakian manifold.

Fact 2. $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is an almost Kähler manifold if and only if $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold.

On the other hand, it is easily observed that any orientable hypersurface of an almost Hermitian manifold becomes an almost contact metric manifold in natural way. So, from the above observation, it seems natural to consider the almost contact metric manifold in connection with almost Hermitian geometry, for example, to discuss the classification of almost Hermitian manifolds. We denote by $\mathcal{K}$, $\mathcal{AH}$, $\mathcal{NK}$, $\mathcal{QK}$ and $\mathcal{H}$ the classes of Kähler manifolds, almost Kähler manifolds, nearly Kähler manifolds, quasi Kähler manifolds and Hermitian manifolds, respectively thus, their inclusion relations are as follows [3]:

$$\mathcal{K}\begin{smallmatrix}\subset \mathcal{AK} \subset \\ \subset \mathcal{NK} \subset\end{smallmatrix}\mathcal{QK}, \qquad \mathcal{AK} \cap \mathcal{NK} = \mathcal{K}, \qquad \mathcal{QK} \cap \mathcal{H} = \mathcal{K}. \tag{1.5}$$

In the next section, we shall reprove these facts and introduce a class of almost contact metric manifolds as the class of almost contact metric manifolds corresponding to the class of quasi Kähler manifolds, which is regarded as a generalization of the class of contact metric manifolds by taking account of (1.5).

In the sequel, we shall call such an almost contact metric manifold quasi contact metric manifold. In §4, we shall discuss the quasi contact metric manifolds from the view point of a generalization of contact metric manifolds.

2 Preliminaries

In this section, we shall prepare some fundamental formulas which we need in the forthcoming discussions in the present paper. Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $\bar{M} = M \times \mathbb{R}$ be the direct product manifold of M and a real line equipped with the almost Hermitian structure $(\bar{J}, \bar{g})$ defined by (1.4). Now, we denote by $[\phi, \phi]$ the (1,2)-tensor field defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] - [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \eta([X, Y])\xi \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$. Further, we denote by $\bar{N}$ the Nijenhuis tensor of the almost complex structure $\bar{J}$. Then, from (1.4), we have

$$\bar{N}(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi - \Big((L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X)\Big)\frac{\partial}{\partial t}, \tag{2.2}$$

$$\bar{N}(X, \frac{\partial}{\partial t}) = -(L_\xi \phi)X + (L_\xi \eta)(X)\frac{\partial}{\partial t} \tag{2.3}$$

for $X, Y \in \mathfrak{X}(M)$. We denote by $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ the following tensor fields on M defined respectively by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi, \tag{2.4}$$

$$N^{(2)}(X, Y) = (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X), \tag{2.5}$$

$$N^{(3)}(X) = -(L_\xi \phi)X, \tag{2.6}$$

$$N^{(4)}(X) = (L_\xi \eta)(X) \tag{2.7}$$

for $X, Y \in \mathfrak{X}(M)$. Then, from (2.2) $\sim$ (2.7), we have

$$\begin{aligned} \bar{N}(X, Y) &= N^{(1)}(X, Y) - N^{(2)}(X, Y)\frac{\partial}{\partial t}, \\ \bar{N}(X, \frac{\partial}{\partial t}) &= N^{(3)}(X) + N^{(4)}(X)\frac{\partial}{\partial t} \end{aligned} \tag{2.8}$$

for $X, Y \in \mathfrak{X}(M)$.

Proposition 2.1. *[1] For an almost contact manifold $M = (M, \phi, \xi, \eta)$ the vanishing of the tensor field $N^{(1)}$ implies the vanishing of the tensor fields $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$.*

Proposition 2.2. *[1] For a contact metric manifold $M = (M, \phi, \xi, \eta, g)$, $N^{(2)}$ and $N^{(4)}$ vanish. Moreover, $N^{(3)}$ vanishes if and only if ξ is a Killing vector field (namely, M is a K-contact manifold).*

Remark 2.1. From Proposition 2.2, taking account of (2.8), we see that an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is normal if and only if $N^{(1)}$ vanishes everywhere on M [1, p.71].

We here note that the following equality

$$\begin{aligned} N^{(2)}(X, Y) &= (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X) \\ &= \phi X(\eta(Y)) - \eta([\phi X, Y]) - \phi Y(\eta(X)) + \eta([\phi Y, X]) \\ &= (\nabla_{\phi X}\eta)(Y) + \eta(\nabla_{\phi X}Y) - \eta(\nabla_{\phi X}Y - \nabla_Y(\phi X)) \\ &\quad - (\nabla_{\phi Y}\eta)(X) - \eta(\nabla_{\phi Y}X) + \eta(\nabla_{\phi Y}X - \nabla_X(\phi Y)) \\ &= (\nabla_{\phi X}\eta)(Y) + \eta(\nabla_Y(\phi X)) - (\nabla_{\phi Y}\eta)(X) - \eta(\nabla_X(\phi Y)) \\ &= (\nabla_{\phi X}\eta)(Y) - (\nabla_Y\eta)(\phi X) - (\nabla_{\phi Y}\eta)(X) + (\nabla_X\eta)(\phi Y) \end{aligned} \tag{2.9}$$

for $X, Y \in \mathfrak{X}(M)$. We here define a (1,1)-tensor field h on M by

$$h = \frac{1}{2} L_\xi \phi. \tag{2.10}$$

The tensor field h plays an important role in the geometry of almost contact metric manifolds. From (2.10), we have easily the following equalities

$$hX = \frac{1}{2}\Big((\nabla_\xi \phi)X - \nabla_{\phi X}\xi + \phi \nabla_X \xi \Big), \tag{2.11}$$

and hence

$$h\xi = 0, \tag{2.12}$$

$$trh = 0. \tag{2.13}$$

Proposition 2.3. *Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold satisfying $\nabla_\xi \phi = 0$. Then h is symmetric with respect to the metric g if and only if $N^{(2)}$ vanishes everywhere on M.*

Proof. By the hypothesis from (2.9) and (2.11), we have

$$\begin{aligned} g(hX, Y) - g(X, hY) &= \frac{1}{2}\Big(-(\nabla_{\phi X}\eta)(Y) - (\nabla_X \eta)(\phi Y) + (\nabla_{\phi Y}\eta)(X) + (\nabla_Y \eta)(\phi X) \Big) \\ &= -\frac{1}{2} N^{(2)}(X, Y) \end{aligned} \tag{2.14}$$

for $X, Y \in \mathfrak{X}(M)$. Proposition 2.3 follows immediately from (2.14). □

The following is well-known.

Proposition 2.4. *An almost contact metric manifold $M = (M, \phi, \xi, \eta, \xi, \eta)$ is Sasakian if and only if $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$ holds for any $X, Y \in \mathfrak{X}(M)$.*

Now, we denote by $\bar{\nabla}$ the covariant derivative with respect to the metric $\bar{g}$ on $\bar{M}$. Then, from (1.4) by direct calculation, we have

$$\begin{aligned} &\bar{\nabla}_X Y = \nabla_X Y + g(X, Y)\frac{\partial}{\partial t}, \quad \bar{\nabla}_X \frac{\partial}{\partial t} = -X, \\ &\bar{\nabla}_{\frac{\partial}{\partial t}} X = -X, \quad \bar{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = -\frac{\partial}{\partial t} \end{aligned} \tag{2.15}$$

for $X, Y \in \mathfrak{X}(M)$. Thus, from (1.4) and (2.15), we have further

$$(\bar{\nabla}_X \bar{J})Y = (\nabla_X \phi)Y - g(X, Y)\xi + \eta(Y)X - \Big(g(\phi X, Y) + (\nabla_X \eta)(Y) \Big)\frac{\partial}{\partial t}, \tag{2.16}$$

$$(\bar{\nabla}_X \bar{J})\frac{\partial}{\partial t} = \nabla_X \xi + \phi X, \tag{2.17}$$

$$(\bar{\nabla}_{\frac{\partial}{\partial t}}\bar{J})X = 0, \quad (\bar{\nabla}_{\frac{\partial}{\partial t}}\bar{J})\frac{\partial}{\partial t} = 0 \tag{2.18}$$

for $X, Y \in \mathfrak{X}(M)$. We here show the Facts 1 and 2, from (2.16) $\sim$ (2.18), we see that $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is Kähler if and only if

$$\begin{aligned}&(\nabla_X\phi)Y - g(X,Y)\xi + \eta(Y)X = 0,\\ &\nabla_X\xi + \phi X = 0\end{aligned} \tag{2.19}$$

for $X, Y \in \mathfrak{X}(M)$. Thus, from (2.19), taking account of Proposition 2.4, it follows immediately that $M = (M, \phi, \xi, \eta, g)$ is Sasakian.

Similarly, from (2.16) $\sim$ (2.18), taking account of (1.4), we see that $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is almost Kähler if and only if

$$\begin{aligned}0 &= \bar{g}((\bar{\nabla}_X\bar{J})Y, Z) + \bar{g}((\bar{\nabla}_Y\bar{J})Z, X) + \bar{g}((\bar{\nabla}_Z\bar{J})X, Y)\\ &= e^{-2t}\Big(g((\nabla_X\phi)Y, Z) + g((\nabla_Y\phi)Z, X) + g((\nabla_Z\phi)X, Y)\Big)\\ &= -3e^{-2t}d\Phi(X, Y, Z).\end{aligned} \tag{2.20}$$

$$\begin{aligned}0 &= \bar{g}((\bar{\nabla}_X\bar{J})\frac{\partial}{\partial t}, Z) + \bar{g}((\bar{\nabla}_{\frac{\partial}{\partial t}}\bar{J})Z, X) + \bar{g}((\bar{\nabla}_Z\bar{J})X, \frac{\partial}{\partial t})\\ &= e^{-2t}\Big((\nabla_X\eta)(Z) - (\nabla_Z\eta)(X) - 2\Phi(X, Z)\Big)\end{aligned} \tag{2.21}$$

for $X, Y \in \mathfrak{X}(M)$, where $\Phi(X, Y) = g(X, \phi Y)$. Thus, from (2.20) and (2.21), it follows that

$$d\eta(X, Y) = \Phi(X, Y) \tag{2.22}$$

for $X, Y \in \mathfrak{X}(M)$, and hence $d\Phi = 0$. Therefore, we see that $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold.

Definition 2.2. An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called *a quasi contact metric manifold* if the corresponding almost Hermitian manifold $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ defined by (1.4) is a quasi Kähler manifold.

We note that a quasi contact metric manifold was primary introduced as a contact O^*-manifold by Tashiro [5].

Now, we shall derive the condition for an almost contact metric manifold to be a quasi contact metric manifold. Again, from (2.16) $\sim$ (2.18), we see that $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is quasi Kähler if and only if

$$\begin{aligned}0 =&(\bar{\nabla}_X\bar{J})Y + (\bar{\nabla}_{\bar{J}X}\bar{J})\bar{J}Y\\ =&(\nabla_X\phi)Y - g(X,Y)\xi + \eta(Y)X\\ &+ (\nabla_{\phi X}\phi)\phi Y - g(X,Y)\xi - \eta(Y)\nabla_{\phi X}\xi + \eta(Y)X\\ &- \Big((\nabla_X\eta)(Y) + (\nabla_{\phi X}\eta)(\phi Y) + 2g(\phi X, Y)\Big)\frac{\partial}{\partial t}\\ =&(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y - 2g(X,Y)\xi + 2\eta(Y)X - \eta(Y)\nabla_{\phi X}\xi\\ &- \Big((\nabla_X\eta)(Y) + (\nabla_{\phi X}\eta)(\phi Y) + 2g(\phi X, Y)\Big)\frac{\partial}{\partial t},\end{aligned} \tag{2.23}$$

$$0 = (\bar{\nabla}_X \bar{J})\frac{\partial}{\partial t} + (\bar{\nabla}_{\bar{J}X} \bar{J})\bar{J}\frac{\partial}{\partial t} = \nabla_X \xi - \phi \nabla_{\phi X} \xi + 2\phi X, \tag{2.24}$$

$$0 = (\bar{\nabla}_{\frac{\partial}{\partial t}} \bar{J})Y + (\bar{\nabla}_{J\frac{\partial}{\partial t}} \bar{J})\bar{J}Y = (\nabla_\xi \phi)(\phi Y) - \eta(Y)\nabla_\xi \xi - (\nabla_\xi \eta)(\phi Y)\frac{\partial}{\partial t} \tag{2.25}$$

for $X, Y \in \mathfrak{X}(M)$. Thus, from (2.23) $\sim$ (2.25) it follows that $M = (M, \phi, \xi, \eta, g)$ is a quasi contact metric manifold if and only if the following equalities

$$(\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X, Y)\xi - 2\eta(Y)X + \eta(Y)\nabla_{\phi X} \xi, \tag{2.26}$$

$$(\nabla_X \eta)(Y) + (\nabla_{\phi X} \eta)(\phi Y) + 2g(\phi X, Y) = 0, \tag{2.27}$$

$$\nabla_X \xi - \phi \nabla_{\phi X} \xi + 2\phi X = 0, \tag{2.28}$$

$$(\nabla_\xi \phi)\phi Y - \eta(Y)\nabla_\xi \xi = 0, \tag{2.29}$$

$$(\nabla_\xi \eta)(\phi Y) = 0. \tag{2.30}$$

From (2.29), we get set $Y = \xi$, then we get

$$\nabla_\xi \xi = 0 \tag{2.31}$$

hold for any $X, Y \in \mathfrak{X}(M)$. Thus, from (2.29) and (2.31), we get

$$(\nabla_\xi \phi)\phi Y = 0. \tag{2.32}$$

From (2.32), we get further

$$(\nabla_\xi \phi)\phi^2 Y = 0,$$

and hence

$$-(\nabla_\xi \phi)Y + \eta(Y)(\nabla_\xi \phi)\xi = 0,$$

and hence

$$\nabla_\xi \phi = 0. \tag{2.33}$$

Further, from (2.28), we have

$$(\nabla_{\phi X} \eta)(\phi Y) + 2g(\phi X, Y) + (\nabla_X \eta)(Y) = 0 \tag{2.34}$$

for any $X, Y \in \mathfrak{X}(M)$, which is nothing but (2.27). Nearly the equality (2.28) is equivalent to the equality (2.27). Summing up the above arguments, we have the following:

Proposition 2.5. *An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is a quasi contact metric manifold if and only if the equalities (2.26), (2.27), (2.31) and (2.33) hold everywhere on M.*

Proposition 2.6. *[1] Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold satisfying the following condition:*

$$(C_1) \quad (\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX$$

for any $X, Y \in \mathfrak{X}(M)$. Then, the following equalities $(C_2) \sim (C_4)$ are derived from the equality (C_1):

$$\begin{aligned} &(C_2) \quad (\nabla_X \eta)Y + (\nabla_{\phi X}\eta)(\phi Y) + 2g(\phi X, Y) = 0,\\ &(C_3) \quad \nabla_\xi \phi = 0,\\ &(C_4) \quad \nabla_\xi \xi = 0 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. We change X and Y for ϕX and ϕY in (C_1), respectively, we get

$$(\nabla_{\phi X}\phi)\phi Y + (\nabla_{(-X+\eta(X)\xi)}\phi)(-Y + \eta(Y)\xi) = 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi,$$

and hence

$$\begin{aligned} &(\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y - \eta(Y)(\nabla_X \phi)\xi - \eta(X)(\nabla_\xi \phi)Y + \eta(X)\eta(Y)(\nabla_\xi \phi)\xi\\ &= 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi, \end{aligned}$$

namely

$$(2.35) \qquad \begin{aligned} (\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)(\nabla_X \phi)\xi\\ + \eta(X)(\nabla_\xi \phi)Y - \eta(X)\eta(Y)(\nabla_\xi \phi)\xi \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, from (C_1) and (2.35), we have

$$(2.36) \qquad \begin{aligned} &\eta(Y)(\nabla_X \phi)\xi + \eta(X)(\nabla_\xi \phi)Y - \eta(X)\eta(Y)(\nabla_\xi \phi)\xi\\ &- \eta(X)\eta(Y)\xi + \eta(Y)X + \eta(Y)hX = 0 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, setting $X = Y = \xi$ in (2.36) and taking account of (2.12), we have

$$(2.37) \qquad (\nabla_\xi \phi)\xi = 0.$$

Further, setting $X = \xi$ and choosing Y perpendicular to ξ arbitrary in (2.36), and taking account of (2.37), we have

$$(2.38) \qquad (\nabla_\xi \phi)Y = 0.$$

Thus, from (2.37) and (2.38), we have (C_3). The equality (C_4) follows immediately from (C_3). Thus, from (2.10) and (C_3), we have

$$(2.39) \qquad hX = \frac{1}{2}(-\nabla_{\phi X}\xi + \phi \nabla_X \xi)$$

for $X \in \mathfrak{X}(M)$. Thus from (2.36), taking account of (C_3) and (2.39), we obtain

$$-\eta(Y)\phi\nabla_X \xi - \eta(X)\eta(Y)\xi + \eta(Y)X + \frac{1}{2}\eta(Y)(-\nabla_{\phi X}\xi + \phi \nabla_X \xi) = 0,$$

and hence

$$
\begin{aligned}
0 &= \eta(Y)\Big(-\phi\nabla_X\xi + X - \eta(X)\xi + \frac{1}{2}(-\nabla_{\phi X}\xi + \phi\nabla_X\xi)\Big) \\
&= \eta(Y)\Big(-\frac{1}{2}(\nabla_{\phi X}\xi + \phi\nabla_X\xi) + X - \eta(X)\xi\Big)
\end{aligned}
\tag{2.40}
$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, from (2.40), we have

$$
\nabla_{\phi X}\xi + \phi\nabla_X\xi = 2X - 2\eta(X)\xi \tag{2.41}
$$

for $X \in \mathfrak{X}(M)$. From (2.41), we have also

$$
\phi\nabla_{\phi X}\xi + \phi^2\nabla_X\xi = 2\phi X,
$$

and hence

$$
-\nabla_X\xi + \phi\nabla_{\phi X}\xi = 2\phi X. \tag{2.42}
$$

From (2.42), we have further

$$
(\nabla_X\eta)(Y) + (\nabla_{\phi X}\eta)(\phi Y) + 2g(\phi X, Y) = 0
$$

for any $X, Y \in \mathfrak{X}(M)$. Namely, we have (C_2). □

In the next section §3, we shall give a characterization for an almost contact metric manifold to be a contact metric manifold, and further, a characterization for an almost contact metric manifold to be a quasi contact metric manifold. Through similar arguments as the proof of Proposition 2.6, we have the following:

Proposition 2.7. *Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold satisfying the following condition:*

$$
(C_1') \quad (\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X, Y)\xi - 2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi
$$

for any $X, Y \in \mathfrak{X}(M)$. Then, the equalities $(C_2) \sim (C_4)$ in Proposition 2.6 are derived from (C_1').

Proof. Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold satisfying the condition (C_1'). By changing X and Y for ϕX and ϕY in (C_1'), respectively, we get

$$
(\nabla_{\phi X}\phi)\phi Y + (\nabla_{-X+\eta(X)\xi}\phi)(-Y + \eta(Y)\xi) = 2g(X, Y)\xi - \eta(X)\eta(Y)\xi,
$$

and hence

$$
\begin{aligned}
(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y =& 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(X)(\nabla_\xi\phi)Y \\
&+ \eta(Y)(\nabla_X\phi)\xi - \eta(X)\eta(Y)(\nabla_\xi\phi)\xi
\end{aligned}
\tag{2.43}
$$

for any $X, Y \in \mathfrak{X}(M)$. Thus,(C_1') and (2.43), we have

$$
\begin{aligned}
&-2\eta(X)\eta(Y)\xi + \eta(Y)(\nabla_X\phi)\xi + \eta(X)(\nabla_\xi\phi)Y - \eta(X)\eta(Y)(\nabla_\xi\phi)\xi \\
&= -2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi
\end{aligned}
\tag{2.44}
$$

for any $X, Y \in \mathfrak{X}(M)$. Setting $X = Y = \xi$ in (2.44), we have

$$(\nabla_\xi \phi)\xi = 0. \tag{2.45}$$

Thus, setting $X = \xi$, $Y \perp \xi$ in (2.44), taking account of (2.45), we have

$$(\nabla_\xi \phi)Y = 0. \tag{2.46}$$

Thus, from (2.45) and (2.46), we have (C_3), thus, we see that (2.43) reduces to

$$(\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi - \eta(Y)\phi\nabla_X \xi. \tag{2.47}$$

Thus, from (C_1') and (2.47), we have

$$-2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi = -2\eta(X)\eta(Y)\xi - \eta(Y)\phi\nabla_X \xi,$$

and hence

$$\eta(Y)\Big(\nabla_{\phi X}\xi + \phi\nabla_X \xi + 2\eta(X)\xi - 2X\Big) = 0 \tag{2.48}$$

for any $X, Y \in \mathfrak{X}(M)$. From (2.48), we have further

$$\nabla_{\phi X}\xi + \phi\nabla_X \xi + 2\eta(X)\xi - 2X = 0. \tag{2.49}$$

Thus, from (2.49), we have also

$$\phi\nabla_{\phi X}\xi + \phi^2\nabla_X \xi - 2\phi X = 0,$$

and hence

$$\phi\nabla_{\phi X}\xi - \nabla_X \xi - 2\phi X = 0. \tag{2.50}$$

We may easily check that (2.50) is equivalent to (C_2), and hence

$$(\nabla_{\phi X}\eta)(\phi Y) + (\nabla_X \eta)(Y) + 2g(\phi X, Y) = 0$$

for any $X, Y \in \mathfrak{X}(M)$. □

3 A characterization of contact metric manifolds

First of all, we shall show the following.

Lemma 3.1. *Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold satisfying the equality (C_3) in Proposition 2.6. Then, the tensor field h anti-commutes with ϕ and the following equality*

$$g(hX, Y) - g(hY, X) = -\frac{1}{2}N^{(2)}(X, Y) \tag{3.1}$$

holds for any $X, Y \in \mathfrak{X}(M)$.

Proof. From the hypotheses, it follows immediately that M satisfies the equality (C_4). Thus, taking account of (2.11), we have

$$\begin{aligned}(\phi h + h\phi)X &= \frac{1}{2}(-\phi\nabla_{\phi X}\xi + \phi^2\nabla_X\xi - \nabla_{\phi^2 X}\xi + \phi\nabla_{\phi X}\xi)\\ &= \frac{1}{2}(\eta(\nabla_X\xi)\xi - \eta(X)(\nabla_\xi\xi)) = 0\end{aligned} \tag{3.2}$$

for any $X \in \mathfrak{X}(M)$. and hence h anti-commutes with ϕ. Further, from (2.11) with (C_3) and (2.9), we have

$$\begin{aligned}g(hX,Y) - g(hY,X) &= \frac{1}{2}\Big(- \nabla_{\phi X}\eta(Y) - (\nabla_X\eta)(\phi Y) + (\nabla_{\phi Y}\eta)(X) + (\nabla_Y\eta)(\phi X)\Big)\\ &= -\frac{1}{2}N^{(2)}(X,Y)\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$.

Now, let $M = (M, \phi, \xi, \eta, g)$ be a contact metric manifold. Then, it is well-known that the tensor field h is symmetric with respect to the metric g and anti-commutes with ϕ and M satisfies the following conditions

(C_0) $\nabla_X\xi = -\phi X - \phi hX,$

(C_1) $(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX$

for any $X, Y \in \mathfrak{X}(M)$. Thus, we see that the equalities $(C_2) \sim (C_4)$ in Proposition 2.6 hold on M and (C_0) is equivalent to (2.28)(and hence (2.27)) by virtue of Proposition 2.6 together with its proof. □

Thus, from the above arguments and Lemma 3.1, we have the following theorem.

Theorem 3.2. *A contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the following conditions*

(C) *h is symmetric*

(C_1) $(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. First, from Proposition 2.6, it follows that M satisfies the conditions $(C_2) \sim (C_4)$. From (C_2), we have

$$(\nabla_X\eta)(Y) + (\nabla_{\phi X}\eta)(\phi Y) = -2g(\phi X, Y),$$

and hence

$$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) + (\nabla_{\phi X}\eta)(\phi Y) - (\nabla_{\phi Y}\eta)(\phi X) = -4g(\phi X, Y) \tag{3.3}$$

for any $X, Y \in \mathfrak{X}(M)$. Further, from the condition (C), taking account of (2.9) and (2.14), we have

$$(\nabla_{\phi X}\eta)(Y) - (\nabla_Y\eta)(\phi X) - (\nabla_{\phi Y}\eta)(X) + (\nabla_X\eta)(\phi Y) = 0 \tag{3.4}$$

for any $X, Y \in \mathfrak{X}(M)$. From (3.4), we have also,

$$(\nabla_{\phi^2 X}\eta)(Y) - (\nabla_Y \eta)(\phi^2 X) - (\nabla_{\phi Y}\eta)(\phi X) + (\nabla_{\phi X}\eta)(\phi Y) = 0,$$

and hence

$$\begin{aligned} & -(\nabla_X \eta)(Y) + \eta(X)(\nabla_\xi \eta)(Y) + (\nabla_Y \eta)(X) \\ & - \eta(X)(\nabla_Y \eta)(\xi) - (\nabla_{\phi Y}\eta)(\phi X) + (\nabla_{\phi X}\eta)(\phi Y) = 0 \end{aligned} \tag{3.5}$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, from (3.3) and (3.5), we have

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = -2g(\phi X, Y) = 2\Phi(X, Y), \tag{3.6}$$

and hence

$$d\eta(X, Y) = \Phi(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Therefore, M is a contact metric manifold. The converse is evident. This completes the proof of Theorem 3.2. □

4 Quasi contact metric manifolds

First of all, we shall show the following

Lemma 4.1. *Let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Then the conditions (C_1) and (C_1') are equivalent to each other.*

Proof. We assume that M satisfies the condition (C_1). Then, it follows from Proposition 2.6 that M satisfies the condition (C_2), and hence, we have

$$\nabla_X \xi - \phi \nabla_{\phi X}\xi + 2\phi X = 0 \tag{4.1}$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, from (4.1) we get

$$\phi \nabla_X \xi = -\nabla_{\phi X}\xi + 2X - 2\eta(X)\xi. \tag{4.2}$$

Since M satisfies the condition (C_3) by virtue of Proposition 2.6, from (2.11) and (4.2), we have

$$hX = \frac{1}{2}(-\nabla_{\phi X}\xi + \phi\nabla_X \xi) = -\nabla_{\phi X}\xi + X - \eta(X)\xi. \tag{4.3}$$

Thus, from (4.3), we see that the equality (C_1) reduces to

$$(\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X, Y)\xi - 2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi \tag{4.4}$$

for any $X, Y \in \mathfrak{X}(M)$. The equality (4.4) is nothing but the equality (C_1').

Conversely, we assume that M satisfies the condition (C_1'). Then, it follows from Proposition 2.7 that M also satisfies the condition (C_2), and hence we have (4.1) and hence, we have (4.2) and (4.3). Thus, finally we see that the equality (C_1') reduces (C_1). □

Therefore, from Proposition 2.5 and Lemma 4.1, we have the following Theorem.

Theorem 4.2. *A quasi contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the following condition (C_1):*

$$(C_1) \quad (\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX$$

for any $X, Y \in \mathfrak{X}(M)$.

Remark 4.1. It is well-known that a 4-dimensional quasi Kähler manifold is necessarily an almost Kähler manifold. Thus, a 3-dimensional quasi contact metric manifold is necessarily a contact metric manifold. Some classes of 3-dimensional contact metric manifolds have been discussed in [4]. From our discussion in this paper, the following question will naturally arise.

Question. Does there exist a $(2n+1)(n \geq 2)$-dimensional quasi contact metric manifold which is not a contact metric manifold?

Acknowledgement. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2011-0012987).

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