

A GENERALIZATION OF CONVEX FUNCTIONS VIA SUPPORT PROPERTIES*

AHARON BEN-TAL and ADI BEN-ISRAEL

(Received 29 October 1974; revised 16 May 1975)

Abstract

With respect to a given family of functions \mathcal{F} , a function is said to be \mathcal{F} -convex, if it is supported, at each point, by some member of \mathcal{F} . For particular choices of \mathcal{F} one obtains the convex functions and the generalized convex functions in the sense of Beckenbach. \mathcal{F} -convex functions are characterized and studied, retaining some essential results of classical convexity.

Introduction

Let \mathcal{F} be a family of functions: $R^n \rightarrow R$, depending on $(n + 1)$ parameters. A function $f: R^n \rightarrow R$ is called \mathcal{F} -convex if its graph is supported at each point by some member of \mathcal{F} . For particular choices of \mathcal{F} , the \mathcal{F} -convex functions reduce to the ordinary proper convex functions and the sub \mathcal{F} -functions of Beckenbach.

In this paper we study the basic properties of \mathcal{F} -convex functions.

Sections 2 and 3 contain definitions and examples.

Section 4 gives first order conditions (so called because they involve only first derivatives and “generalized gradients”) for \mathcal{F} -convexity. For families \mathcal{F} which possess certain uniqueness property, \mathcal{F} -convexity is characterized by an analog of the gradient inequality. The remaining results in Section 4 are conditions for \mathcal{F} -convexity or strict \mathcal{F} -convexity, in terms of the injectivity properties of the “generalized gradients” (termed here “ \mathcal{F} -gradients”).

* This paper is dedicated to Professor Olga Taussky Todd. It was presented at the Gatlinburg VI meeting in Munich, December 15–22, 1974 (sponsored by the Stifterverband für die Deutsche Wissenschaft through the German Research Council).

This research was partly supported by the Center for Cybernetic Studies, The University of Texas. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Second order conditions for \mathcal{F} -convexity and strict \mathcal{F} -convexity are given in Section 5. These conditions involve a “generalized Hessian matrix” $H(x)$, depending on f and \mathcal{F} , which in the classical case reduces to the Hessian Matrix $\partial^2 f / \partial x_i \partial x_j$.

The main results here are Theorems 5.1 and 5.5. An analog of the differential inequality of Peixoto (1949), characterizing sub- \mathcal{F} functions, is obtained as a special case.

Section 6 deals with the monotonicity properties of the \mathcal{F} -gradient x^* of an \mathcal{F} -convex function, where \mathcal{F} belongs to a certain class of functions. The derivative of x^* is computed and the result is used, for the “separable” families to establish that x^* is a P_0 -function [P -function] if f is \mathcal{F} -convex [strictly \mathcal{F} -convex].

In a sequel paper we study the corresponding generalizations of conjugacy and duality in the sense of Fenchel (see Rockafellar (1970)). These results involve a conjugate family \mathcal{F}^* , and are hidden in the classical case by the fact that there $\mathcal{F} = \mathcal{F}^*$. In future papers the authors intend to apply the theory developed here to derive duality results for nonconvex programs and to furnish a unified theory for various recent results on generalized Lagrangians.

2. \mathcal{F} -convex functions: definitions and examples

2.1 DEFINITIONS. Let \mathcal{F} be a family of functions: $X \rightarrow R$ where $X \subset R^n$ with range

$$(2.1) \quad \Xi \triangleq \bigcup \{\text{range } F : F \in \mathcal{F}\}.$$

Let f be a function: $R^n \rightarrow R$ with domain

$$(2.2) \quad \text{dom } f \subset X$$

and let S be an open subset of $\text{dom } f$. Then f is called \mathcal{F} -convex in S if for every $x \in S$, there exists an $F \in \mathcal{F}$ such that

$$(2.3) \quad f(x) = F(x) \text{ and } f(z) \geq F(z) \text{ for all } x \neq z \in S,$$

in which case F is called a *support of $f: S$ at x* . The function f is called *strictly \mathcal{F} -convex in S* if strict inequality holds in (2.4) for all $x \neq z \in S$.

The name \mathcal{F} -convex function was used recently (Roberts and Varberg (1973), p. 241) to denote the sub- \mathcal{F} functions, see Example 2.2.

If there is no need to specify S , for example if $S = \text{dom } f$, the above names are abbreviated by omitting S , e.g., *\mathcal{F} -convex*, *support of f at x* , etc.

2.2. EXAMPLE. Let \mathcal{F} be the family of affine functions: $R^n \rightarrow R$, i.e.,

$$(2.4) \quad \mathcal{F} = \{F(\cdot) = \langle x^*, \cdot \rangle - \xi^* : x^* \in R^n, \xi^* \in R\}.$$

Then a function $f: R^n \rightarrow R$ is \mathcal{F} -convex if and only if it is a proper convex function, i.e., a convex function whose epigraph is a non empty set containing no vertical lines, ([Rockafellar (1970), §4].

2.3 EXAMPLE. Let \mathcal{F} be a family of continuous functions: $X \rightarrow R$ where X is the open interval (a, b) , and such that

(B) For any two distinct points in X , say,

$$a < x_1 < x_2 < b$$

and any two real numbers $\{y_1, y_2\}$, there is a unique $F \in \mathcal{F}$ satisfying

$$F(x_i) = y_i, \quad (i = 1, 2).$$

We call such an \mathcal{F} a *Beckenbach family in (a, b)* . Beckenbach (1937) called a function $f: (a, b) \rightarrow R$ a *sub- \mathcal{F} function* if for any two points

$$a < x_1 < x_2 < b$$

the member of \mathcal{F} , F_{12} , defined by

$$(2.5) \quad F_{12}(x_i) = f(x_i), \quad (i = 1, 2),$$

satisfies

$$(2.6) \quad f(x) \leq F_{12}(x), \quad x_1 < x < x_2.$$

Peixoto (1948, 1949, Theorem 1) showed that if f is a sub- \mathcal{F} function and $a < x_0 < b$ then there exist two functions

$$F_i \in \mathcal{F}, \quad (i = 1, 2),$$

such that

$$F_i(x_0) = f(x_0), \quad (i = 1, 2),$$

$$F_2(x) \leq F_1(x) \leq f(x), \quad (a < x < x_0),$$

and

$$F_1(x) \leq F_2(x) \leq f(x), \quad x_0 < x < b.$$

(Furthermore, if the derivatives $f'(x_0)$, $F_1'(x_0)$ and $F_2'(x_0)$ exist, they are equal). Thus both F_1 and F_2 support f at x_0 .

Therefore every sub- \mathcal{F} function is \mathcal{F} -convex. We will now prove the converse for Beckenbach families \mathcal{F} .

2.4 PROPOSITION. *Let \mathcal{F} be a Beckenbach family in (a, b) . Then a function $f: (a, b) \rightarrow R$ is \mathcal{F} -convex in (a, b) if and only if f is a sub- \mathcal{F} function.*

PROOF. The proof of the "if" part was cited above.

To prove “only if” suppose f is not a sub- \mathcal{F} function, i.e., there are three points

$$a < x_1 < x_0 < x_2 < b$$

such that the function $F_{12} \in \mathcal{F}$, defined by (2.5), satisfies

$$(2.7) \quad F_{12}(x_0) < f(x_0).$$

Suppose that $F_0 \in \mathcal{F}$ is a support of f at x_0 , i.e.,

$$(2.8) \quad f(x_0) = F_0(x_0) \text{ and } f(x) \geq F_0(x), \quad a < x < b.$$

From (2.5), (2.7) and (2.8) it follows that F_{12} and F_0 intersect twice over the interval (a, b) , contradicting (B). Therefore f is not \mathcal{F} -convex. \square

2.5 EXAMPLE. Let $G(x, y, z)$ be a continuous function: $(a, b) \times R \times R \rightarrow R$, such that

(P1) For each $\{x_0, y_0, y'_0\} \in (a, b) \times R \times R$, the differential equation

$$(2.9) \quad y'' = G(x, y, y'), \quad (a < x < b),$$

has a unique solution $y = y(x)$ satisfying

$$(2.10) \quad y(x_0) = y_0, \quad y'(x_0) = y'_0.$$

(P2) The solution of (2.9) is continuous with respect to the initial values y_0, y'_0 .

(P3) For any two points $\{x_i, y_i\} \in (a, b) \times R$ ($i = 1, 2$) with $x_1 \neq x_2$, there is a unique solution of (2.9) satisfying

$$(2.11) \quad y(x_i) = y_i, \quad i = 1, 2.$$

Let \mathcal{F} be the Beckenbach family of solutions of (2.9). Peixoto (1949, Theorem 2) showed that a function $f \in C^2(a, b)$ is a sub- \mathcal{F} function if and only if

$$(2.12) \quad f'' \geq G(x, f, f'), \quad a < x < b.$$

2.6 EXAMPLE. While sub- \mathcal{F} functions are continuous (Beckenbach (1937), Roberts and Varberg (1973), p. 242), an \mathcal{F} -convex function need not be continuous in its domain, even if each $F \in \mathcal{F}$ is continuous:

Let \mathcal{F} be the family of functions: $R \rightarrow R$

$$F(x) = \xi^* \cos(x^* x)$$

depending on the two parameters

$$x^* \in R \quad \xi^* \in R.$$

Then the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

is \mathcal{F} -convex. Indeed, for every $x \neq 0$, the function $F = F(x^*, \xi^*; \cdot)$ defined by

$$x^* = \frac{\pi}{x} \quad \xi^* = -1$$

supports f at x . Also every $F \in \mathcal{F}$ supports f at 0.

We show now that an \mathcal{F} -convex function inherits from \mathcal{F} its lower semi continuity.

2.7 PROPOSITION. *Let \mathcal{F} be a family of l.s.c. (= lower semi continuous) functions, and let f be \mathcal{F} -convex in $\text{dom } f$. Then f is l.s.c. in $\text{dom } f$.*

PROOF. Suppose f is not l.s.c.. Then there exists an $x \in \text{dom } f$ such that

$$(2.13) \quad f(x) > \liminf_{y \rightarrow x} f(y).$$

Let $F \in \mathcal{F}$ support f at x . Then

$$F(x) = f(x) > \liminf_{y \rightarrow x} f(y) \geq \liminf_{y \rightarrow x} F(y),$$

by (2.3) and (2.13), contradicting the lower semi continuity of F . \square

2.8 NOTES. For further generalizations of convexity see the surveys in Beckenbach and Bellman (1965), Chapter 4, and Roberts and Varberg (1973), Chapter VIII.

For functions of several variables, the analogs of the sub \mathcal{F} -functions are the subfunctions in particular the subharmonic functions; see Beckenbach and Bellman (1965), p. 146, Beckenbach and Jackson (1953) and Jackson (1968), where applications to second order differential inequalities are surveyed.

3. Requirements on \mathcal{F}

3.1 GENERAL. With Examples 2.2, 2.3 and 2.5 as our motivation, we consider from now on only families \mathcal{F} of functions $F: R^n \rightarrow R$ depending continuously on $n + 1$ parameters

$$\{x^*, \xi^*\} \in X^* \times \Xi^*$$

where the sets of parameters X^* and Ξ^* are given subsets of R^n and R respectively. The general member of \mathcal{F} is thus denoted by

$$(3.1) \quad F(\cdot) = F(x^*, \xi^*; \cdot), \quad (x^* \in X^*, \xi^* \in \Xi^*),$$

with function values

$$(3.2) \quad F(x) = F(x^*, \xi^*; x), \quad x \in X.$$

We assume that the mapping: $\{x^*, \xi^*\} \rightarrow F(x^*, \xi^*; \cdot)$ is one to one on $X^* \times \Theta^*$.

3.2 THE CLASS A. Let $D^k(X)$ denote the functions: $R^n \rightarrow R$ which are k times differentiable in X . If $\mathcal{F} \subset D(X) \stackrel{\Delta}{=} D^1(x)$ we define the set

$$(3.3) \quad Z \stackrel{\Delta}{=} \cup \left\{ \text{range} \begin{bmatrix} F \\ F_x \end{bmatrix} : F \in \mathcal{F} \right\} \subset R \times R^n$$

where F_x is the gradient of F with respect to x .

A family \mathcal{F} of differentiable functions is said to be in *class A*, denoted by $\mathcal{F} \in A$, if for every $x \in X$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in Z$, the system

$$(3.4) \quad \xi = F(x^*, \xi^*; x)$$

$$(3.5) \quad y = F_x(x^*, \xi^*; x)$$

has a unique solution $\{x^*, \xi^*\} \in X^* \times \Xi^*$.

If $\mathcal{F} \subset D(X)$ and if f and S are a function: $R^n \rightarrow R$ and an open subset of $\text{dom } f$ respectively, we denote by

$$(3.6) \quad f \stackrel{S}{\approx} \mathcal{F}$$

the facts

$$(D1) \quad S \subset \text{dom } f \subset X$$

$$(D2) \quad f \in D(S)$$

$$(D3) \quad \text{range} \left\{ \begin{bmatrix} f(x) \\ f_x(x) \end{bmatrix} : x \in S \right\} \subset Z.$$

We abbreviate $f \stackrel{\text{dom } f}{\approx} \mathcal{F}$ by $f \approx \mathcal{F}$.

If $\mathcal{F} \in A$, $f \approx \mathcal{F}$ and $x \in \text{dom } f$ we denote by

$$(3.7) \quad (x^*_f(x), \xi^*_f(x))$$

the unique solution of

$$(3.8) \quad f(x) = F(x^*, \xi^*; x)$$

$$(3.9) \quad f_x(x) = F_x(x^*, \xi^*; x).$$

3.3 THE CLASS C . A family \mathcal{F} is said to be in class C , denoted by $\mathcal{F} \in C$, if for every $\{x^*, x\} \in X^* \times X$ the function $F(x^*, \cdot; x)$ is a strictly decreasing function of ξ^* , $\xi^* \in \Xi^*$. In this case, we denote by $F^I(x, \cdot; x^*)$ the *inverse function* of $F(x^*, \cdot; x)$. It satisfies the identity

$$(3.10) \quad \xi = F(x^*, F^I(x, \xi; x^*); x), \quad \xi \in \Xi.$$

If $\mathcal{F} \in A \cap C$, $f \approx \mathcal{F}$ and $x \in \text{dom } f$, then (3.8) gives

$$(3.11) \quad \xi^* = F^I(x, f(x); x^*)$$

which, substituted in (3.9), gives

$$(3.12) \quad f_x(x) = F_x(x^*, F^I(x, f(x); x^*); x).$$

The unique solution of (3.12) is then called the \mathcal{F} -gradient of f at x , and is denoted by $x^*_f(x)$.

3.4 EXAMPLE. Let \mathcal{F} be the family (2.4) of affine functions: $R^n \rightarrow R$. Then

(a) $\mathcal{F} \subset D(R^n)$, $F_x(x^*, \xi^*; x) = x^*$ for every $F \in \mathcal{F}$ and $x \in R^n$, and (3.3) gives $Z = R \times R^n$.

(b) $\mathcal{F} \in A$. For every $x \in R^n$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in R \times R^n$, the unique solution of (3.4)–(3.5) is

$$x^* = y, \quad \xi^* = \langle y, x \rangle - \xi.$$

(c) $\mathcal{F} \in C$.

(d) $f \approx \mathcal{F}$ means that $f \in D(\text{dom } f)$.

(e) If $f \approx \mathcal{F}$ then for every $x \in \text{dom } f$

$$(3.13) \quad x^*_f(x) = f_x(x), \quad \xi^*_f(x) = \langle f_x(x), x \rangle - f(x).$$

Thus the \mathcal{F} -gradient of f , x^*_f , coincides here with its ordinary gradient f_x .

3.5 EXAMPLE. Let ϕ be a given function: $X^* \times X \rightarrow R$ and let the family \mathcal{F} consist of the functions $F(x^*, \xi^*; \cdot)$, $\{x^*, \xi^*\} \in X^* \times \Xi^*$, with values

$$(3.14) \quad F(x^*, \xi^*; x) = \phi(x^*, x) - \xi^*, \quad x \in X.$$

Then: $\mathcal{F} \in C$ and

(a) $\mathcal{F} \in A$ if and only if the following two conditions hold:

(a1) $\phi(x^*, \cdot) \in D(X)$ for every $x^* \in X^*$.

(a2) For every $x \in X$, $y \in \cup_{x^*} \text{range } \phi_x(x^*, \cdot)$, the system $y = \phi_x(x^*, x)$ has a unique solution x^* .

The \mathcal{F} -gradient of f at x , $x^*_f(x)$, is the unique solution x^* of

$$(3.15) \quad f_x(x) = \phi_x(x^*, x).$$

A concrete example is the following family \mathcal{F} given by

$$(3.16) \quad F(x^*, \xi^*; x) = \frac{\langle x^*, x \rangle}{\langle a, x \rangle + \beta} - \xi^*$$

with domain $X = \{x: \langle a, x \rangle + \beta > 0\}$, and $X^* = \mathbb{R}^n$. Here a is a given vector in \mathbb{R}^n and β is a given nonzero scalar. Note that, for the special case $a = 0, \beta = 1$, the family (3.16) reduces to the family (2.4) of affine function. Here condition (a1) clearly holds, while condition (a2) holds if and only if the linear system of equations

$$y_i = \sum_{j \neq i} \left\{ \frac{-x_j a_i}{(\langle a, x \rangle + \beta)^2} x^*_j \right\} + \frac{\langle a, x \rangle + \beta - a_i x_i}{(\langle a, x \rangle + \beta)^2} x^*_i, \quad i = 1, 2, \dots, n$$

has a unique solution. This occurs of course if and only if $\det G \neq 0$ where G is the $n \times n$ matrix

$$G = \begin{bmatrix} \langle a, x \rangle + \beta - a_1 x_1, & -a_2 x_2, & \dots, & -a_n x_n \\ -a_1 x_2, & \langle a, x \rangle + \beta - a_2 x_2, & & \\ -a_1 x_n, & -a_2 x_n, & \dots, & \langle a, x \rangle + \beta - a_n x_n \end{bmatrix}$$

Using Sylvester's identity (Gantmacher (1959), section II.3) it can be shown, by induction, that $\det G = [\langle a, x \rangle + \beta]^{n-1} \beta$, and since $\beta \neq 0$ it follows that $\det G \neq 0$ and hence $\mathcal{F} \in \mathcal{A}$. For a \mathcal{F} -convex function f , the \mathcal{F} -gradient is given by

$$x^*_f(x) = G^{-1}(\nabla f(x)).$$

4. First order conditions for \mathcal{F} -convexity

In this section we give first order conditions (so-called because they involve only first derivatives and the "gradients" $\{x^*_f, \xi^*_f\}$ of f , see (3.7)) for \mathcal{F} -convexity, for families \mathcal{F} in class \mathcal{A} . These conditions use the extremal property of the supports implied by the inequality (2.3). First we require

4.1 LEMMA. Let $\mathcal{F} \in \mathcal{A}$, $f: \mathbb{R}^n \rightarrow \mathbb{R}$, and let $f \stackrel{s}{\approx} \mathcal{F}$. If $f: S$ is supported (by some $F \in \mathcal{F}$) at a point $x \in S$, then

$$(4.1) \quad F(x^*_f(x), \xi^*_f(x); \cdot)$$

is the unique support of f at x .

PROOF. Let $F(x_0^*, \xi_0^*; \cdot) \in \mathcal{F}$ support $f: S$ at x , i.e.,

$$(4.2) \quad h(z) \stackrel{\Delta}{=} f(z) - F(x_0^*, \xi_0^*; z) \geq 0, \quad \forall z \in S,$$

and

$$(4.3) \quad h(x) = f(x) - F(x_0^*, \xi_0^*; x) = 0.$$

Therefore $h(z)$ is minimized, in S , by $z = x$. Since S is open, this implies that x is a critical point of h , i.e.,

$$(4.4) \quad h_z(x) = f_x(x) - F_x(x_0^*, \xi_0^*; x) = 0.$$

Since $\mathcal{F} \in \mathbf{A}$, a comparison of (4.3)–(4.4) and (3.8)–(3.9) shows that

$$\{x_0^*, \xi_0^*\} = \{x_f^*(x), \xi_f^*(x)\}$$

proving that (4.1) is the unique support at x . \square

4.2 THEOREM. Let $\mathcal{F} \in \mathbf{A}$, $f: R^n \rightarrow R$, and $f \stackrel{s}{\approx} \mathcal{F}$. Then f is \mathcal{F} -convex in S if and only if for every $x \in S$

$$(4.5) \quad f(z) \geq F(x_f^*(x), \xi_f^*(x); z), \quad \forall x \neq z \in S.$$

Furthermore, f is strictly \mathcal{F} -convex in S if and only if for every $x \in S$

$$(4.6) \quad f(z) > F(x_f^*(x), \xi_f^*(x); z), \quad \forall x \neq z \in S.$$

PROOF. *If.* From (4.5) and (3.8) it follows, for any $x \in S$, that the function (4.1) supports $f: S$ at x . Moreover, x is the unique point of support if (4.6) holds.

Only if. Let f be \mathcal{F} -convex in S . Then, by Lemma 4.1, for any $x \in S$, the function (4.1) is the unique support of $f: S$ at x . The inequality (4.5) then follows from (2.3). Similarly (4.6) follows from the strict \mathcal{F} -convexity of f . \square

4.3 EXAMPLE. Let \mathcal{F} be the family (2.4) of affine functions: $R^n \rightarrow R$,

$$\mathcal{F} = \{F(x^*, \xi^*; \cdot) = \langle x^*, \cdot \rangle - \xi^*: x^* \in R^n, \xi^* \in R\}.$$

Then, using (3.13), the inequality (4.5) reduces to

$$f(z) \geq \langle f_x(x), z - x \rangle + f(x), \quad \forall x \neq z \in S,$$

the classical gradient inequality.

4.4 COROLLARY. (a) Let $\mathcal{F} \in \mathbf{A}$, and let $f: R^n \rightarrow R$, $f \stackrel{s}{\approx} \mathcal{F}$, be \mathcal{F} -convex in

S. Then f is strictly \mathcal{F} -convex in S if and only if the mapping

$$(4.7) \quad x \rightarrow \{x_{\mathcal{F}}^*(x), \xi_{\mathcal{F}}^*(x)\}$$

is one to one on S .

(b) Let, in addition, $\mathcal{F} \in \mathcal{C}$. Then f is strictly \mathcal{F} -convex in S if and only if the mapping

$$(4.8) \quad x \rightarrow x_{\mathcal{F}}^*(x)$$

is one to one on S .

PROOF. From Lemma 4.1 it follows, for every $x \in S$, that the function (4.1) is the unique support of $f: S$ at x . By definition, f is strictly \mathcal{F} -convex in S if, and only if, every support of $f: S$ supports f at exactly one point of S . This is equivalent to the mapping (4.7) being one to one on S .

To prove the last part, note that the additional hypothesis $\mathcal{F} \in \mathcal{C}$ implies

$$(4.9) \quad \left[x \xrightarrow{1:1} x_{\mathcal{F}}^*(x) \text{ on } S \right] \Leftrightarrow \left[x \xrightarrow{1:1} \{x_{\mathcal{F}}^*(x), \xi_{\mathcal{F}}^*(x)\} \text{ on } S \right].$$

Indeed, the implication \Rightarrow is always true. Conversely, suppose that $x_{\mathcal{F}}^*$ is not one to one on S , i.e., there exist $x_1, x_2 \in S, x_1 \neq x_2$, such that

$$(4.10) \quad x_{\mathcal{F}}^*(x_1) = x_{\mathcal{F}}^*(x_2) \stackrel{\Delta}{=} x_0^*.$$

Let $\xi_1^* \stackrel{\Delta}{=} \xi_{\mathcal{F}}^*(x_1) = F^1(x_1, f(x_1); x_0^*)$ and let

$$F^i(\cdot) \stackrel{\Delta}{=} F(x_0^*, \xi_i^*; \cdot), \quad (i = 1, 2).$$

Then

$$F^i(x_i) = f(x_i), \quad F_x^i(x_i) = f_x(x_i), \quad i = 1, 2.$$

Hence by Theorem 4.2, F^i supports f at x_i . If $\xi_1^* = \xi_2^*$, then this and (4.10) contradicts the fact that $(x_{\mathcal{F}}^*, \xi_{\mathcal{F}}^*)$ is 1:1, established earlier. Thus suppose that $\xi_1^* > \xi_2^*$. This implies, since $\mathcal{F} \in \mathcal{C}$, that $F^1(z) < F^2(z) \forall z \in S$. In particular

$$F^2(x_1) > F^1(x_1) = f(x_1)$$

contradicting the fact that F^2 is a support. □

4.5 THEOREM. *Let $\mathcal{F} \in \mathcal{A} \cap \mathcal{C}$, $f: R^n \rightarrow R$, and $f \stackrel{S}{\approx} \mathcal{F}$. Then f is strictly \mathcal{F} -convex in S if the following two conditions hold.*

(a) *The mapping $x_{\mathcal{F}}^*$ is one to one on S .*

(b) *For every $x \in S$ and for every sequence $\{z_k\} \subset S$ which either converges*

to a point $y \in \text{bdry } S$ or $\|z_k\| \rightarrow \infty$ there exists an $\hat{x} \in S$ such that

$$(4.11) \quad \limsup_{k \rightarrow \infty} \{F^I(z_k, f(z_k), x^*_j(x)) - F^I(\hat{x}, f(\hat{x}), x^*_j(x))\} \leq 0$$

where F^I is defined in §3.3.

PROOF. For any $x \in S$ consider the function

$$(4.12) \quad T(z) \stackrel{\Delta}{=} F^I(z, f(z); x^*_j(x)).$$

We show first that $z = x$ is a critical point of T . Differentiating the identity

$$(4.13) \quad F(x^*, F^I(y, f(y); x^*); y) - f(y) = 0$$

with respect to y we get

$$(4.14) \quad F_x(\cdot, \cdot; \cdot) + F_{\xi^*}(\cdot, \cdot; \cdot)[F^I_x(y, f(y); x^*) + F^I_{\xi}(y, f(y); x^*)f_x(y)] - f_x(y) = 0$$

where

$$(\cdot, \cdot; \cdot) = (x^*, F^I(y, f(y); x^*); y).$$

Now $F_{\xi^*} \neq 0$, since $\mathcal{F} \in \mathcal{C}$. Therefore, for $y = x$ and $x^* = x^*_j(x)$, it follows from (4.14) and (3.12) that

$$(4.15) \quad F^I_x(x, f(x); x^*_j(x)) + F^I_{\xi}(x, f(x); x^*_j(x))f_x(x) = 0$$

which, by (4.12), is the same as $T_z(x) = 0$, proving that $z = x$ is critical.

Moreover, $z = x$ is the unique critical point of T in S . For suppose that $x \neq x' \in S$ is another critical point of T , i.e.

$$T_z(x') = F^I_x(x', f(x'); x^*_j(x)) + F^I_{\xi}(x', f(x'); x^*_j(x))f_x(x') = 0$$

implying that for $y = x'$ and $x^* = x^*_j(x)$, (4.14) reduces to

$$F_x(x^*_j(x), F^I(x', f(x'); x^*_j(x)); x') - f_x(x') = 0$$

which, together with (3.12), implies that

$$x^*_j(x') = x^*_j(x),$$

contradicting (a).

We show next that

$$(4.16) \quad \sup \{T(z); z \in S\} = T(x).$$

Indeed if this supremum occurs at some $z = y \in \text{bdry } S$ or if a supremizing sequence $\{z_k\}$ is such that $\|z_k\| \rightarrow \infty$ then the supremum is also attained at $\hat{x} \in S$, by (4.11). Therefore $z = \hat{x}$ is a critical point of T , proving that $\hat{x} = x$, since the latter is the unique critical point in S , and therefore (4.16) becomes

$$F^I(x, f(x); x^*_j(x)) > F^I(z, f(z); x^*_j(x)), \quad \forall x \neq z \in S,$$

which is the same as

$$f(z) > F(x_j^*(x), F^1(x, f(x); x_j^*(x)); z), \quad \forall x \neq z \in S,$$

proving that f is strictly \mathcal{F} -convex in S , by theorem 4.2. □

4.6 EXAMPLE. Consider the family

$$\mathcal{F} = \{\phi(x^*, \cdot) - \xi^* : x^* \in X^*, \xi^* \in \Xi^*\},$$

of Example 3.5 and let $\bar{\mathcal{F}} \in \mathcal{A}$, $f: R^n \rightarrow R$, and $f \overset{S}{\approx} \bar{\mathcal{F}}$. Then condition (b) of Theorem 4.5 follows from

(b1) For every $x^* \in \text{range} \{x_j^*(x) : x \in S\}$ and every sequence $\{z_k\}$ as in Theorem 4.5(b),

$$(4.17) \quad \liminf_{k \rightarrow \infty} \{f(z_k) - \phi(x^*, z_k)\} = +\infty. \quad \square$$

In particular, if

$$S = \text{dom } f = X = R^n$$

and

$$(4.18) \quad \limsup_{\|x\| \rightarrow \infty} \frac{\phi(x^*, x)}{\|x\|} < \infty, \quad \forall x^* \in \text{range } x_j^*.$$

then condition (b) of Theorem 4.5 is satisfied if

$$\lim_{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|} = \infty.$$

Note that (4.18) is trivially satisfied by the family \mathcal{F} of affine functions. Hence, a differentiable function $f: R^n \rightarrow R$ is strictly convex if the following two conditions hold.

(a) The mapping

$$x \rightarrow f_x(x)$$

is one to one on R^n .

$$(b) \quad \lim_{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|} = \infty.$$

As a concrete example of condition (b1) let \mathcal{F} be the family of functions: $R^2 \rightarrow R$ given by

$$(4.19) \quad F(x^*, \xi^*; x) = x_1^* e^{-x_1} + x_2^* x_2 e^{-x_1} - \xi^*$$

with $X = X^* = R^2$, $\Xi^* = R$.

Consider the function $f: R^2 \rightarrow R$

$$(4.20) \quad f(x) = \frac{1}{2}e^{2x_1} + \frac{1}{2}x_2^2 e^{-x_1}$$

with $\text{dom } f = R^2$. Then f is \mathcal{F} -convex in R^2 since:

(a) The \mathcal{F} -gradient

$$x^*_f(x) = \begin{bmatrix} -e^{3x_1} - \frac{1}{2}x_2^2 \\ x_2 \end{bmatrix}$$

is one-to-one, and

$$(4.21) \quad \text{range } x^*_f = \left\{ (x^*_1, x^*_2) \in R^2: x^*_1 + \frac{1}{2}x^{*2}_2 < 0 \right\}.$$

$$(b) \quad \begin{aligned} f(z) - \phi(x^*, z) &= \left(\frac{1}{2}e^{2z_1} + \frac{1}{2}z_2^2 e^{-z_1} \right) - (x^*_1 e^{-z_1} + x^*_2 z_2 e^{-z_1}) \\ &= \frac{1}{2}e^{2z_1} - \left(x^*_1 + \frac{1}{2}x^{*2}_2 \right) e^{-z_1} + \frac{1}{2}(z_2 - x^*_2)^2 e^{-z_1} \end{aligned}$$

by (4.21) the coefficients of all exponentials are positive and hence

$$\lim_{|z^*_1| \rightarrow \infty} [f(z) - \phi(x^*, z)] = \infty \quad \forall x^* \in \text{range } x^*_f.$$

5. Second order conditions for \mathcal{F} -convexity

In this section we collect second order conditions (involving second derivatives) for \mathcal{F} -convexity.

5.1 THEOREM. Let $\mathcal{F} \in \mathcal{A} \cap D^2(X)$, $f: R^n \rightarrow R$, $f \stackrel{S}{\approx} \mathcal{F}$ and $f \in D^2(S)$.

Then:

(a) f is \mathcal{F} -convex in S only if, for every $x \in S$, the matrix

$$(5.1) \quad H(x) \stackrel{\Delta}{=} f_{xx}(x) - F_{xx}(x^*_f(x), \xi^*_f(x); x)$$

is positive semi definite. (A matrix $H \in R^{n \times n}$ is called here positive semi definite if

$$\langle Hz, z \rangle \geq 0, \quad \forall z \in R^n.$$

We do not mean by this that H is symmetric.)

(b) Let S be convex and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S . Then f is \mathcal{F} -convex in S if

$$(5.2) \quad \left\langle y, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x^*_f(x), \xi^*_f(x); x+sy)) y ds \right\rangle \geq 0,$$

for every $x \in S$ and $y \in S - x$.

If strict inequality holds in (5.2), F is strictly \mathcal{F} -convex in S .

PROOF. (a) Let f be \mathcal{F} -convex in S . Then, for any $x \in S$, the function

$$(5.3) \quad h(z) \stackrel{\Delta}{=} f(z) - F(x^*_f(x), \xi^*_f(x); z)$$

satisfies

$$(5.4) \quad h(x) = 0, \quad h_z(x) = 0, \quad \text{by (3.8) - (3.9),}$$

and

$$h(z) \geq 0, \quad \forall z \in S, \quad \text{by Theorem 4.2.}$$

Therefore $z = x$ minimizes h in S . Since S is an open set, it follows that

$$h_{zz}(x) = H(x)$$

is positive semi-definite.

(b) The function h of (5.3) satisfies

$$\begin{aligned} h(z) &= h(z) - h(x) - \langle h_z(x), z - x \rangle, \quad \text{by (5.4),} \\ &= \langle h_z(x + t(z - x)) - h_z(x), z - x \rangle, \quad \text{for some } 0 < t < 1, \\ &\quad \text{by a mean value theorem (Ortega and Rheinboldt} \\ &\quad \text{(1970), Theorem 3.2.2).} \\ &= \left\langle z - x, \left(\int_0^1 h_{zz}(x + st(z - x)) ds \right) t(z - x) \right\rangle, \\ &\quad \text{by a mean value theorem (Ortega and Rheinboldt} \\ &\quad \text{(1970), Theorem 3.2.7),} \\ &= \frac{1}{t} \left\langle y, \int_0^1 (f_{xx}(x + sy) - F_{xx}(x^*_f(x), \xi^*_f(x); x + sy)) y ds \right\rangle, \\ &\quad \text{where } y = t(z - x). \end{aligned}$$

Thus, (5.2) implies that

$$(5.5) \quad h(z) \geq 0, \quad \forall z \in S,$$

proving that f is \mathcal{F} -convex in S , by Theorem 4.2.

Similarly, strict inequality in (5.2) implies strict inequality in (5.5), hence strict \mathcal{F} -convexity. \square

5.2 EXAMPLE. Let \mathcal{F} be the family (2.4) of affine functions: $R^n \rightarrow R$. Then the matrix $H(x)$ of (5.1) reduces to the Hessian of f

$$H(x) = f_{xx}(x)$$

and Theorem 5.1 gives the classical conditions for convexity in terms of the Hessian.

5.3 EXAMPLE. Let \mathcal{F} be the Beckenbach family of solutions of the second order differential equation

$$(2.9) \quad y'' = G(x, y, y'), \quad (a < x < b),$$

discussed in Example 2.5. Then (5.1) becomes

$$H = f'' - G(x, f, f').$$

Now, suppose that $\mathcal{F} \subset C^2(X)$, $f \in C^2(S)$, then $H(x) > 0$ implies $H(x + sy) > 0$ for $0 < s < 1$ and y sufficiently close to x . Thus (5.2) is a strict inequality in some neighborhood of x , and we conclude that f is, locally, strictly \mathcal{F} -convex. By Proposition 2.4 this implies that f is locally strictly sub- \mathcal{F} , which by (Beckenbach (1937), Theorem 7) implies that f is sub- \mathcal{F} globally in (a, b) . This result is the analog of (Peixoto (1949), Theorem 3). To get the analogous result of (Peixoto (1949), Theorem 1), we need the implication $H(x) \geq 0 \Rightarrow H(x + sy) \geq 0$, for $0 < s < 1$ and y sufficiently close to x , for which Peixoto's additional requirement, (P2) of Example 2.5, is needed (see the proof of Lemma 1 in Peixoto (1949)).

5.4 DEFINITION. A mapping $T: R^n \rightarrow R^n$ is called an extended one to one mapping on R^n if

- (a) $x, y \in R^n$, $x \neq y \Rightarrow T(x) \neq T(y)$.
- (b) The inverse images $T^{-1}(B)$ of bounded sets $B \subset R^n$ are bounded.

5.5 THEOREM. Let $\mathcal{F} \in \mathcal{A} \cap \mathcal{C} \cap C^2(R^n)$, $f: R^n \rightarrow R$, $f \in C^2(R^n)$ and $f \approx \mathcal{F}$. Then f is strictly \mathcal{F} -convex in R^n if the following two conditions hold

- (a) x^* is an extended one to one mapping on R^n .
- (b) For every $x \in R^n$, the matrix

$$(5.6) \quad H(x) = f_{xx}(x) - F_{xx}(x^*(x), F^1(x, f(x); x^*(x))); x)$$

is positive definite. Conversely, if f is strictly \mathcal{F} -convex in R^n then x^* is one to one and the matrix $H(x)$ is positive semi-definite for every $x \in R^n$.

PROOF. First we note, by (3.11), that (5.6) and (5.1) are the same. For any $x \in R^n$ consider now the function (4.12)

$$T(z) = F^I(z, f(z); x^*(x)).$$

As in the proof of Theorem 4.5 it follows from (a) that $z = x$ is the unique critical point of T in R^n .

Differentiating the identity (4.13) twice with respect to y we get, by using (4.15) and (3.12),

$$(5.7) \quad T_{zz}(x) = \frac{1}{F_{\xi^*}} H(x)$$

(where F_{ξ^*} is evaluated at $\{x^*(x), F^I(x, f(x); x)\}$). From (5.7), (b) and $F \in C$ it follows that $T_{zz}(x)$ is negative definite. Therefore $z = x$ is an isolated local maximizer of T , and its unique critical point in R^n .

Thus, by Leighton's Theorem (Leighton (1966), see also Szegö (1968)), $z = x$ is the global maximizer of T , i.e.,

$$F^I(x, f(x); x^*(x)) > F^I(z, f(z); x^*(x)), \quad \forall x \neq z \in R^n,$$

which is the same as

$$f(z) > F(x^*(x), F^I(x, f(z); x^*(x)); z), \quad \forall x \neq z \in R^n,$$

proving that f is strictly \mathcal{F} -convex in R^n by Theorem 4.2.

If f is strictly \mathcal{F} -convex in R^n then (a) and (b) follow from Corollary 4.4 and Theorem 5.1(a) respectively. □

5.6 EXAMPLE. Let \mathcal{F} and f be given by (4.19) and (4.20) respectively. Then the matrix (5.1) is positive definite

$$H(x) = \begin{bmatrix} 3e^{2x_1} & 0 \\ 0 & e^{-x_1} \end{bmatrix}$$

and f is strictly \mathcal{F} -convex in R^2 , by Theorem 5.5.

6. Monotonicity of \mathcal{F} -gradients

In this section we prove monotonicity results for the \mathcal{F} -gradient x^* of an \mathcal{F} -convex function. We recall that a mapping $g: R^n \rightarrow R^n$ is a P -function [P_0 -function] if for every $x, y \in \text{dom } g$, $x \neq y$, there is an index $k = k(x, y) \in \{1, 2, \dots, n\}$ such that

$$(x_k - y_k)(g_k(x) - g_k(y)) > 0 \quad [(x_k - y_k)(g_k(x) - g_k(y)) \geq 0 \text{ and } x_k \neq y_k],$$

see Moré and Rheinboldt (1973). In particular, a mapping $g: R^n \rightarrow R^n$ is *monotone (strictly monotone)* if for every $x, y \in \text{dom } g, x \neq y$, we have

$$\langle x - y, g(x) - g(y) \rangle \geq 0 \quad [\langle x - y, g(x) - g(y) \rangle > 0].$$

We also require the following

6.1 DEFINITIONS. A family \mathcal{F} is said to be in class A_1 , denoted by $\mathcal{F} \in A_1$, if $\mathcal{F} \in A$ and for every $\{x^*, \xi^*; x\} \in X^* \times \Xi^* \times X$ the derivatives in (6.1) are continuous and the matrix

$$(6.1) \quad J(x^*, \xi^*; x) = \begin{bmatrix} F_{\xi^*}(x^*, \xi^*; x) & F_{x^*}^T(x^*, \xi^*; x) \\ F_{\xi^*x}(x^*, \xi^*; x) & F_{x^*x}(x^*, \xi^*; x) \end{bmatrix}$$

is nonsingular, say

$$(6.2) \quad \det J(x^*, \xi^*; x) < 0.$$

This matrix is the Jacobian matrix of the function

$$\begin{bmatrix} f(\cdot, \cdot; x) \\ F_x(\cdot, \cdot; x) \end{bmatrix},$$

see (3.4) – (3.5).

A family \mathcal{F} is said to be in class A_2 , denoted by $\mathcal{F} \in A_2$, if $\mathcal{F} \in A_1$ and for every $x \in X$ the matrix

$$(6.3) \quad J_0(x) \stackrel{\Delta}{=} \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^*x} - F_{\xi^*x} F_{x^*}^T],$$

where all derivatives are evaluated at $\{x_j^*(x), \xi_j^*(x); x\}$, is positive definite.

6.2 LEMMA. Let $\mathcal{F} \in A_1 \cap C, f: R^n \rightarrow R, f \stackrel{S}{\approx} \mathcal{F}$ and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S . Then, for every $x \in S$,

$$(6.4) \quad D_x x_j^*(x) = J_0(x)^{-1} H(x)$$

where $D_x x_j^*(x)$ denotes the derivative of x_j^* at x and J_0 and H are given by (6.3) and (5.1) respectively.

PROOF. For any $x \in S$ consider the system

$$(3.8) \quad F(x^*, \xi^*; x) - f(x) = 0$$

$$(3.9) \quad F_x(x^*, \xi^*; x) - f_x(x) = 0$$

which, since $\mathcal{F} \in A$, has a unique solution $\{x_j^*(x), \xi_j^*(x)\}$. The implicit function

theorem, applicable since $\mathcal{F} \in A_1$, then gives

$$(6.5) \quad \begin{bmatrix} D_x \xi_j^*(x) \\ D_x x_j^*(x) \end{bmatrix} = \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^*x} & F_{x^*x} \end{bmatrix}^{-1} \begin{bmatrix} f_x(x) - F_x(x_j^*(x), \xi_j^*(x); x) \\ f_{xx}(x) - F_{xx}(x_j^*(x), \xi_j^*(x); x) \end{bmatrix}$$

where the derivatives

$$\begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^*x} & F_{x^*x} \end{bmatrix}$$

are evaluated at $\{x_j^*(x), \xi_j^*(x); x\}$.

Using (3.9) and (5.1), we rewrite (6.5) as

$$(6.6) \quad F_{\xi^*} D_x \xi_j^*(x) + F_{x^*}^T D_x x_j^*(x) = 0$$

$$(6.7) \quad F_{\xi^*x} D_x \xi_j^*(x) + F_{x^*x} D_x x_j^*(x) = H(x).$$

Now $F_{\xi^*} \neq 0$ since $\mathcal{F} \in C$. Eliminating $D_x \xi_j^*(x)$ from (6.6) and substituting in (6.7) gives

$$H(x) = \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^*x} - F_{\xi^*x} F_{x^*}^T] D_x x_j^*(x).$$

The proof is completed by showing that the matrix

$$[F_{\xi^*} F_{x^*x} - F_{\xi^*x} F_{x^*}^T]$$

is nonsingular, which follows since

$$(6.8) \quad \det [F_{\xi^*} F_{x^*x} - F_{\xi^*x} F_{x^*}^T] = F_{\xi^*}^{n-1} \det \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^*x} & F_{x^*x} \end{bmatrix},$$

by Sylvester's identity (Gantmacher (1959), Section II.3), $\neq 0$, since $\mathcal{F} \in C \cap A_1$.

□

6.3 EXAMPLE. Let \mathcal{F} be the family (2.4) of affine functions: $R^n \rightarrow R$. Then

$$x_j^*(x) = f_x(x), \text{ by Example 3.4,}$$

$$J_0(x) = I \text{ by (6.3) since } F_{x^*x} = I, \quad F_{\xi^*x} = 0$$

and (6.4) reduces to the obvious

$$(6.9) \quad D_x f_x(x) = f_{xx}(x).$$

If f is a convex [strictly convex] differentiable function, then its gradient f_x is monotone [strictly monotone] in $\text{dom } f$. This is an immediate consequence of the gradient inequality (Example 4.3), and Theorem 4.2. Alternatively and less directly, the monotonicity of f_x can be shown to follow from (6.9) and the fact that f_{xx} is positive semi definite, see, e.g. Ortega and Rheinboldt (1970),

Theorem 5.4.3. Two other cases in which the factorization (6.4) is used to establish a monotonicity property of the \mathcal{F} -gradient $x_{\mathcal{F}}^*$, will now be given.

6.4 THEOREM. Let $\mathcal{F} \in \mathbf{A}_2 \cap C^2(X)$ where $X = I_1 \times I_2 \times \dots \times I_n$ is the product of open intervals $I_i \subset \mathbb{R}$, ($i = 1, \dots, n$). Let each $F \in \mathcal{F}$ be of the form

$$(6.10) \quad F(x^*, \xi^*; x) = \sum_{i=1}^n F^i(x^*, x_i) - \xi^*$$

where $F^i(x^*, \cdot): I_i \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, n$). Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be \mathcal{F} -convex [strictly \mathcal{F} -convex] with $\text{dom} f \supset X$ and $f \in C^2(X)$. Then $x_{\mathcal{F}}^*$ is a P_{σ} -function [P -function] in X .

PROOF. From (6.10), (6.3) and $\mathcal{F} \in \mathbf{A}_2$ it follows that

$$J_0(x) = F_{x^*x}$$

is a diagonal, positive definite matrix. From (6.4) and Theorem 5.1(a) it therefore follows, for an \mathcal{F} -convex function f , that $D_x x_{\mathcal{F}}^*(x)$ is a P_{σ} -matrix, (see Fiedler and Pták (1962, 1966), proving that $x_{\mathcal{F}}^*$ is a P_{σ} -function, by Moré and Rheinboldt (1973), Corollary 5.3.

If f is strictly \mathcal{F} -convex, then, by Corollary 4.4(b) (applicable since $\mathcal{F} \in \mathbf{C}$), it follows for any $x, y \in X$, $x \neq y$, that there is a $k = k(x, y) \in \{1, 2, \dots, n\}$ such that

$$x_k \neq y_k \quad \text{and} \quad x_{\mathcal{F}}^*(x)_k \neq x_{\mathcal{F}}^*(y)_k,$$

proving that $x_{\mathcal{F}}^*$ is a P -function. □

A special case of Theorem 6.4 is the following, one dimensional result:

6.5 COROLLARY. Let $\mathcal{F} \in \mathbf{A}_1 \cap \mathbf{C}$ be a family of functions: $\mathbb{R} \rightarrow \mathbb{R}$, let $f: \mathbb{R} \rightarrow \mathbb{R}$, S an open subset of $\text{dom} f$, and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S . If f is \mathcal{F} -convex in S then $x_{\mathcal{F}}^*$ is a nondecreasing function in S .

PROOF. Using (6.3), (6.8) and (6.1) we write

$$J_0(x) = \frac{1}{F_{\xi^*}} \det J(x_{\mathcal{F}}^*(x), \xi_{\mathcal{F}}^*(x); x)$$

> 0 , by (6.2) and $\mathcal{F} \in \mathbf{C}$.

Therefore

$$\frac{d}{dx} x_{\mathcal{F}}^*(x) \geq 0, \quad \text{by (6.4) and Theorem 5.1(a).} \quad \square$$

6.6 COROLLARY. Let \mathcal{F} , f and S be as in Corollary 6.5, where S is an interval (a, b) . If

$$f''(x) > F_{xx}(x^*(x), \xi^*(x), x), \quad x \in S,$$

then f is strictly \mathcal{F} -convex.

PROOF. From (6.4) and (6.11) we infer that x^* is 1:1 on (a, b) . As in the proof of Theorem 5.6 this implies that $z = x$ is a local minimizer of $h(z) \stackrel{\Delta}{=} f(z) - F(x^*(x), \xi^*(x); z)$ and that no other critical point exists in (a, b) . Hence $z = x$ is the unique global minimizer of $h(z)$, which was previously shown to be equivalent to the strict \mathcal{F} -convexity of f . \square

6.7 COROLLARY. Let \mathcal{F} be as in Theorem 6.4, with $X = \mathbb{R}^n$. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with $\text{dom } f = \mathbb{R}^n$, $f \in C^2(\mathbb{R}^n)$, $f \approx \mathcal{F}$ is strictly \mathcal{F} -convex, if the matrix $H(x)$ is positive definite.

PROOF. Follows from (6.4) and Theorem 5.5. \square

Acknowledgement

The authors wish to thank the anonymous referee for his valuable comments and in particular for suggesting Example 2.6.

References

- E. F. Beckenbach, (1937), 'Generalized convex functions', *Bull. Amer. Math. Soc.*, **43**, 363–371.
- E. F. Beckenbach and R. Bellman, (1965), *Inequalities*, (Springer-Verlag New York Inc., 2nd revised printing, 1965).
- E. F. Beckenbach and L. K. Jackson, (1953), 'Subfunctions of several variables', *Pacific J. Math.*, **3**, 291–313.
- L. Danzer, B. Grunbaum and V. L. Klee, (1963), 'Helly's Theorem and Its Relatives', in *Convexity*, V. L. Klee (ed.), Proceedings of Symposia in Pure Mathematics, Vol. VII, American Mathematical Society, 101–180.
- M. Fiedler and V. Pták, (1962), 'On matrices with non-positive off-diagonal elements and positive principal minors', *Czech. Math. J.*, **12**, 382–400.
- M. Fiedler and V. Pták, (1966), 'Some generalizations of positive definiteness and monotonicity', *Numer. Math.*, **9**, 163–172.
- F. R. Gantmacher, (1959), *The Theory of Matrices*, Vol. I, (Chelsea Publishing Co., New York, 1959).
- L. K. Jackson, (1968), 'Subfunctions and second-order ordinary differential inequalities', *Advances in Mathematics*, **2**, 307–363.
- W. Leighton, (1966), 'On Liapunov functions with a single critical point', *Pacific J. Math.*, **19**, 467–472.
- J. Moré and W. Rheinboldt, (1973), 'On P - and S -functions and related classes of n -dimensional nonlinear mappings', *Linear Algebra and Its Applications*, **6**, 45–68.

- T. S. Motzkin, (1949), 'Approximation by curves of a unisolvent family', *Bull. Amer. Math. Soc.*, **55**, 789–793.
- J. M. Ortega and W. C. Rheinboldt, (1970), '*Iterative Solution of Nonlinear Equations in Several Variables*', (Academic Press, New York, 1970).
- M. M. Peixoto, (1948), 'On the existence of a derivative of generalized convex functions', *Summa Brasiliensis Math.*, **2**, No. 3.
- M. M. Peixoto, (1949), 'Generalized convex functions and second order differential inequalities', *Bull. Amer. Math. Soc.*, **55**, 563–572.
- A. W. Roberts and D. W. Varberg, (1973), *Convex Functions*, (Academic Press, New York, 1973).
- R. T. Rockafellar, (1970), *Convex Analysis*, (Princeton University Press, Princeton, N. J., 1970).
- G. P. Szegő, (1968), 'A theorem of Rolle's type in E^n for functions of the class C^1 ', *Pacific J. Math.*, **27**, 193–195.

Center for Cybernetic Studies
The University of Texas
Austin, Texas 78712
U.S.A.