A GENERALIZATION OF EPSTEIN'S ZETA FUNCTION

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§ 0. Introduction.

Koecher defined in [3] the following zeta function associated with the matrix $S^{(n)}$ of a positive quadratic form and one complex variable ρ

$$(1) \qquad Z_{n_1}(S, \rho) = \sum_A |{}^t ASA|^{-\rho}.$$

Here $n \geq n_1$, and the sum is over a complete set of representatives for the n by n_1 integral rank n_1 matrices $A^{(n, n_1)}$ with respect to the equivalence relation $A \sim B$ if $A = BU$ for some unimodular matrix U. The unimodular group $\mathfrak{U}_{n_1}$ is defined by $\mathfrak{U}_{n_1} = \{U^{(n_1)} : U \text{ integral}, \ n_1 \text{ by } n_1 \text{ with determinant} \ |U| = \pm 1\}$. We use the notation $|S|$ = determinant of S and $S[A] = {}^t ASA$ throughout. Superscripts in parentheses on matrices denote the number of rows and columns. Thus $A^{(n, n_1)}$ has n rows and n_1 columns.

Koecher shows in [3] that $Z_{n_1}(S, \rho)$ converges for $Re\rho > \dfrac{n}{2}$. But his proof of the analytic continuation and the functional equation,

$$(2) \qquad R_{n_1}(S, \rho) = |S|^{\frac{-n_1}{2}} R_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right)$$

where $R_{n_1}(S, \rho) = \pi^{n_1\left(\frac{n_1-1}{4} - \rho\right)} \prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right) Z_{n_1}(S, \rho)$,

has a gap. This is remedied neatly using an idea of Selberg. One can annihilate the trouble-making terms of the theta function with an appropriate differential operator. We outline these results in § 1 because they do not appear in the literature.

Selberg has defined in [6] a zeta function associated with a positive matrix S and $n-1$ complex variables $\rho = (\rho_1, \rho_2, \cdots, \rho_{n-1})$. This function can be seen to be essentially the same as

$$(3) \qquad \zeta_{(n)}(S, \rho) = \sum \prod_{i=1}^{n-1} |S[U_i]|^{-\rho_i},$$

where the sum is over a complete set of representatives for unimodular $U^{(n)} = (U_i{}^{(n,i)}*)$, with respect to the equivalence relation $U \sim V$ if $U = VP$ for P unimodular and upper triangular. Selberg states that this series converges for $Re\rho_i > 1$ and that $\zeta_{(n)}(S, \rho)$ satisfies $n!$ functional equations and can be continued to a meromorphic function in C^{n-1}. There are various proofs of this fact. See [4] and [9]. Selberg used both approaches. We consider the method used in [9] briefly in §2 because we generalize this method to obtain our main results. And we make use of the $n-1$ functional equations which generate the rest:

For $i = 2, 3, \cdots, n-2,$

$$(4) \qquad \pi^{-\rho_i}\Gamma(\rho_i)\zeta(2\rho_i)\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_i)}\Gamma(1 - \rho_i)\zeta(2(1 - \rho_i))\zeta_{(n)}(S, \rho')$$

where $\rho_i' = 1 - \rho_i$, $\rho_{i\pm1}' = \rho_{i\pm1} + \rho_i - \dfrac{1}{2}$, $\rho_j' = \rho_j$ for $j \neq i$, $i \pm 1$.

For $i = 1,$

$$\pi^{-\rho_1}\Gamma(\rho_1)\zeta(2\rho_1)\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_1)}\Gamma(1-\rho_1)\zeta(2(1-\rho_1))\zeta_{(n)}\Big(S, 1 - \rho_1, \rho_1 + \rho_2 - \frac{1}{2}, \rho_3, \cdots\Big)$$

For $i = n - 1,$

$$\pi^{-\rho_{n-1}}\Gamma(\rho_{n-1})\zeta(2\rho_{n-1})\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_{n-1})}\Gamma(1 - \rho_{n-1})\zeta(2(1 - \rho_{n-1})) \times$$
$$\times \zeta_{(n)}\Big(S, \rho_1, \cdots, \rho_{n-3}, \rho_{n-1} + \rho_{n-2} - \frac{1}{2}, 1 - \rho_{n-1}\Big).$$

It should be noted that an easy "proof" of the $n!$ functional equations derives from an integral formula of Harish-Chandra for spherical functions (see [1], Proposition 6.8, page 428). Maass has shown in [4] that a further functional equation is trivial, namely:

$$(5) \qquad \zeta_{(n)}(S^{-1}, \rho) = |S|^{\sum_{i=1}^{n-1} \rho_i} \zeta_{(n)}(S, \tilde{\rho}),$$

where $\tilde{\rho} = (\rho_{n-1}, \rho_{n-2}, \cdots, \rho_2, \rho_1)$.

We consider here a generalization of both (1) and (3):

$$(6) \qquad \zeta_{n_1, \cdots, n_r}(S, \rho_1, \cdots, \rho_{r-1}) = \sum \prod_{i=1}^{r-1} |S[U_i]|^{-\rho_i}.$$

Here $n = \sum\limits_{j=1}^{r} n_j$, with positive integers n_j, and $N_i = \sum\limits_{j=1}^{i} n_j$, for $i = 1, 2, \cdots, r$. In the sum, $U^{(n)} = (U_i^{(n, N_i)}*)$ runs over a complete set of representatives for unimodular matrices, with respect to the equivalence relation $U \sim V$ if $U = VP$, with P unimodular and having block form

$$
P = \begin{pmatrix} P_1^{(n_1)} & & & * \\ & P_2^{(n_2)} & & \\ 0 & & \ddots & \\ & & & P_r^{(n_r)} \end{pmatrix}.
$$

We show in §3 that (6) converges when $Re\, \rho_i > \dfrac{n}{2}$, $i = 1, 2, \cdots, r-1$.

It comes out of the proof of (4) in §2 that

$$(7) \qquad \zeta_{(n)}(S, \rho)\big|_{\rho_i = 0} = \zeta_{1,\ldots,1,\overset{i}{2},1,\ldots,1}(S, \rho_1, \cdots, \overset{\overset{i}{\vee}}{}, \rho_{n-1}), \text{ where } ``\vee" \text{ denotes omis-}$$

sion of the variable ρ_i. In §3 we generalize this relation to

$$(8) \qquad \zeta_{n_1,\ldots,n_r}(S, \rho_{N_1}, \cdots, \rho_{N_r-1})\big|_{\rho_{N_i} = 0,\, l \neq i_j} = \zeta_{m_1,\ldots,m_\lambda}(S, \rho_{M_1}, \cdots, \rho_{M_\lambda-1}),$$

where $N_{i_j} = \sum\limits_{k=1}^{i_j} n_k = \sum\limits_{k=1}^{j} m_k = M_j$, for $j = 1, \cdots, \lambda$. The proof is by induction using results of §1. Thus Koecher's functions are essentially specializations of Selberg's arrived at by setting all but one variable equal to zero. Similarly the functions (6) are specializations of Selberg's function (3).

In §4 we use the functional equations in §2 for Selberg's function and the results of §3 to obtain relations between Koecher's functions $\zeta_{i,n-i}$ and $\zeta_{n-i,i}$. We have two ways of doing this ((4) and (5)).

§1. Koecher's Zeta Function

We first note the relation between Koecher's function (1) and the case $r = 2$ of (6).

LEMMA 1.1. $\qquad Z_{n_1}(S, \rho) = \zeta_{n_1, n-n_1}(S, \rho) \prod\limits_{i=0}^{n_1-1} \zeta\left(2\left(\rho - \dfrac{i}{2}\right)\right).$

Proof. Consider the map h defined by $h(A^{(n, n_1)}) = (U^{(n, n_1)}, B^{(n_1)})$, where B is the greatest right divisor of A and $A = UB$ with U primitive. For the definitions of greatest right divisor and primitive, see Siegel [8], volume I, pages 331 and 332. The map h gives a one-to-one map from integral

rank n_1 non right equivalent A to primitive non right equivalent U and integral rank n_1 non right equivalent B. One uses a result of Koecher-formula (1.10) of [3], page 7 — to complete the proof.

The main results of this section are the functional equation and analytic continuation of $Z_{n_1}(S, \rho)$. The proof requires a series of lemmas, which we shall simply state. For more details, see [9].

Define the theta function by

$$(9) \qquad \theta(S^{(n)}, X^{(n_1)}) = \sum' e^{-\pi\sigma(S[A]X)} = \theta^S(X),$$

where $\sigma(S) = $ trace of S, and the sum is over all integral $A^{(n, n_1)}$. If we restrict summation to A of rank r, we denote the result $\theta_r(S, X) = \theta_r^S(X)$. Koecher proves (using the Poisson summation formula, in [3], page 8) that $\theta(S, X)$ converges for $Re\, S > 0$, $X > 0$, $n \geq n_1$ and that it satisfies the transformation formula

$$(10) \qquad \theta(S, X) = |S|^{\frac{-n_1}{2}} |X|^{\frac{-n}{2}} \theta(S^{-1}, X^{-1}).$$

Selberg defines differential operators D_a on the Riemannian symmetric space P_n of positive $Y^{(n)}$ as follows:

$$(11) \qquad D_a = |Y|^a D_Y |Y|^{1-a},$$

where $D_Y = \left| \frac{1}{2}(1 + \delta_{ij}) \dfrac{\partial}{\partial y_{ij}} \right|$, δ_{ij} being the Kronecker delta. It is easy to see that the D_a are invariant differential operators with respect to the diffeomorphisms τ_A of the space P_n defined by $\tau_A(Y) = Y[A]$, for $Y \in P_n$, $A \in GL(n, \mathbf{R})$. The general linear group $GL(n, \mathbf{R})$ consists of all non-singular n by n real matrices.

Further define L^* as the formal adjoint of the differential operator L on P_n, with respect to the τ_A-invariant measure $d\mu = |Y|^{-\frac{n+1}{2}} \prod\limits_{1 \leq i \leq j \leq n} dy_{ij}$. That is, $\int_{P_n} (Lf)g \, d\mu = \int_{P_n} f \, L^*g d\mu$ for f and g infinitely differentiable on P_n with compact support (i.e., in $C_0^\infty(P_n)$). Let $\alpha(Y) = Y^{-1}$. Then one can show that

$$(12) \qquad L^* = L^\alpha,$$

where $L^\alpha(f) = [L(f \circ \alpha)] \circ \alpha^{-1}$ for $f \in C^\infty(P_n)$.

Moreover the algebra of all L which are invariant with respect to the τ_A (i.e. $L^{\tau_A} = L$), for $A \in GL(n, \mathbf{R})$, is commutative. For a proof, see Selberg [5], page 51.

One easily sees that

(13) $$D_Y e^{-\sigma(H\,Y)} = |W|(-1)^n e^{-\sigma(WY)}$$

Also

(14) $$(D_a)^\alpha = (D_a)^* = (-1)^n D_{1+\frac{n+1}{2}-a}$$

Now define

(15) $$f_r^\rho(X) = \prod_{i=1}^{r-1} |X_i|^{\rho_i}.$$

if $X = \begin{pmatrix} X_i^{(N_i)} & * \\ * & * \end{pmatrix}$, $N_i = \sum_{j=1}^{i} n_j$, $i = 1, 2, \cdots, r$, and $n = \sum_{j=1}^{r} n_j = N_r$. Note that $\zeta_{n_1, \cdots, n_r}(S, \rho) = \sum f_r^{-\rho}(S[U])$. The f_r^ρ are eigenfunctions for the invariant differential operators L on the space P_n of positive symmetric matrices . We compute the eigenvalues for D_a in the case $r = 2$ later in this section. Writing $X = {}^t TT$ for T upper triangular with positive diagonal entries we have, if $n = r$, i.e., $n_1 = n_2 = \cdots = n_r = 1$,

(16) $$f_n^\rho({}^t TT) = \prod_{i=1}^{n-1} t_i^{r_i} = \chi^r(T), \quad \text{where} \quad T = \begin{pmatrix} t_1 & & & \\ & t_2 & & * \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix},$$

and $r_i = 2\sum_{j=i}^{n-1} \rho_j$. A similar result holds for arbitrary r. χ is a character on the triangular group and can be extended to $GL(n, \boldsymbol{R})$ by making it constant on cosets $O_n T$, where T is an upper triangular matrix and O_n is the orthogonal group. The result is a right spherical function.

We need to define the following gamma factor

(17) $$G_{n_1}(\rho) = \prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right).$$

Now we can prove the functional equation and obtain the analytic continuation of $Z_{n_1}(S, \rho)$ to the whole complex plane as a meromorphic function.

THEOREM 1.2. *Let L be the invariant differential operator on P_{n_1}*

$$L = D_a D_1,$$

for $a = \dfrac{1}{2}(n_1 - n + 1)$. *Let* F_{n_1} *be a fundamental domain for* P_{n_1} *with respect to* τ_U *for* $U \in \mathfrak{U}_{n_1}$ *(recall that* $\tau_U(Y) = {}^t U\, Y U = Y[U]$*). Then we have the following integral representation:*

$$(18) \qquad 2 Z_{n_1}(S^{(n)}, \rho) g(\rho) G_{n_1}(\rho) \pi^{\frac{1}{4}(n_1^2 - n_1) - n_1 \rho} = \int_{F_{n_1}} f_2^\rho (L\theta_{n_1}^S)\, d\mu \equiv J(S, \rho)$$

$$(f_2^\rho(X) = |X|^\rho, \quad X \in P_{n_1}).$$

The integral can be analytically continued for all ρ *and*

$$J(S, \rho) = |S|^{\frac{-n_1}{2}} J\left(S^{-1}, \frac{n}{2} - \rho\right).$$

Further $g(\rho) = \displaystyle\prod_{i=1}^{n_1} \left(\rho + \dfrac{n_1 - i - n}{2}\right)\left(\rho - \dfrac{i-1}{2}\right) = g\left(\dfrac{n}{2} - \rho\right).$

Proof. By changing to triangular matrix variables T through $Y = {}^t TT$, it is easy to see that

$$J(S, \rho) = g(\rho) \int_{F_{n_1}} f_2^\rho \theta_{n_1}^S \, d\mu = 2 g(\rho) Z_{n_1}(S, \rho) \pi^{\frac{1}{\pi}(n_1^2 - n_1) - n_1 \rho} G_{n_1}(\rho),$$

where $L^* f_2^\rho = g(\rho) f_2^\rho$, since $f_2^\rho(X) = |X|^\rho$ is an eigen function for L. We compute $g(\rho)$ later.

Now

$$J(S, \rho) = \int_{\substack{F_{n_1} \\ |X| \geq 1}} + \int_{\substack{F_{n_1} \\ |X| \leq 1}}$$

$$= \int_{\substack{F_{n_1} \\ |X| \geq 1}} \{f_2^\rho L\theta_{n_1}^S + (f_2^\rho \circ \alpha)((L\theta_{n_1}^S) \circ \alpha)\}\, d\mu$$

$$= \int_{\substack{F_{n_1} \\ |X| \geq 1}} \{f_2^\rho L\theta_{n_1}^S + f_2^{-\rho} L^\alpha (|S|^{\frac{-n_1}{2}} f_2^{\frac{n}{2}} \theta_{n_1}^{S^{-1}})\}\, d\mu$$

$$+ \sum_{r=0}^{n_1 - 1} \int_{\substack{F_{n_1} \\ |X| \geq 1}} f_2^{-\rho} \{|S|^{\frac{-n_1}{2}} L^\alpha (f_2^{\frac{n}{2}} \theta_r^{S^{-1}}) - (L\theta_r^S) \circ \alpha\}\, d\mu$$

using the transformation formula (10). We have chosen L so that all the terms of the last integral are zero. For recall that $L = D_a D_1 = D_1 D_a$. And (13) implies $D_Y(\theta_r^S) = 0$, for $0 \leq r < n_1$. And so $D_1(\theta_r^S) = f_2^1 D_Y(\theta_s^S) = 0$, $0 \leq r < n_1$.

Now $(-1)^{n_1}D_a^* = D_{-\frac{n}{2}+1} = f_2^{\frac{n+2}{2}}D_Y f_2^{\frac{-n}{2}}$ by (14). So $(-1)^{n_1}D_a^*(f_2^{\frac{n}{2}}\theta_r^{S-1}) =$

$f^{\frac{n+2}{2}}D_Y(\theta_r^{S-1}) = 0, \ 0 \leqslant r < n_1.$ Therefore

$$J(S,\rho) = \int_{F_{n_1} \atop |X|\geq 1}\left\{f_2\theta_{n_1}^S + f_2^{-\rho}|S|^{-\frac{1}{2}n_1}L^\alpha(f_2^{\frac{1}{2}n}\theta_{n_1}^{S-1})\right\}d\mu.$$

The functional equation $J(S,\rho) = |S|^{-\frac{1}{2}n_1}J\left(S^{-1}, \frac{1}{2}n - \rho\right)$ will follow immediately from the convergence of the above integral for all ρ, since $L\theta_{n_1}^S = f^{-\frac{1}{2}n}L^\alpha(\frac{1}{2}^n\theta_{n_1}^S).$

The convergence of $J(S,\rho)$ is easy. For we may assume $Re\,\sigma$ is arbitratily large in the function $f_2^\sigma(X) = |X|^\sigma$, since this only increases the function when $|X| \geq 1$. Here we use also the fact that $Z_{n_1}(S,\rho)$ converges for $Re\,\rho > \frac{n}{2}$, to complate the proof that $J(S,\rho)$ converges for all ρ. In fact we obtain the inequality (which we shall use in the proof of Theorem 3.3)

$$J(S,\rho) \leqslant g(|\rho|+n)\int_{F_{n_1}}f_2^{|\rho|+n}\theta_{n_1}^S d\mu + |S|^{-\frac{1}{2}n_1}g'(-(|\rho|+n))\int_{F_{n_1}}f_2^{|\rho|+n}\theta_{n_1}^{S-1}d\mu,$$

were $Lf_2^{-\rho} = g'(\rho)f_2^{-\rho}$. Actually $g(\rho) = g'(\rho)$, as we shall see.

Next we compute $g(\rho)$ and $g'(\rho)$. Suppose $D_1^*|Y|^\rho = h(\rho)|Y|^\rho(-1)^{n_1}$. Then

$$(-1)^{n_1}h(\rho)\int_{P_{n_1}}e^{-\sigma(Y)}|Y|^\rho d\mu = \int_{P_{n_1}}e^{-\sigma(Y)}D_1^*|Y|^\rho d\mu$$

$$= \int_{P_{n_1}}(D_1e^{-\sigma(Y)})|Y|^\rho d\mu = (-1)^{n_1}\int_{P_{n_1}}|Y|e^{-\sigma(Y)}|Y|^\rho d\mu.$$

It is clear that

$$h(\rho) = \frac{(-1)^{n_1}\int_{P_{n_1}}e^{\sigma-(Y)}|Y|^{1+\rho}d\mu}{(-1)^{n_1}\int_{P_{n_1}}e^{-\sigma(Y)}|Y|^\rho d\mu} = \frac{\prod_{i=0}^{n_1-1}\Gamma\left(1+\rho-\frac{i}{2}\right)}{\prod_{i=0}^{n_1-1}\Gamma\left(\rho-\frac{i}{2}\right)} = \prod_{i=0}^{n_1-1}\left(\rho-\frac{i}{2}\right).$$

So $|Y|^{\frac{n_1+1}{2}}D_Y|Y|^{1-\frac{n_1+1}{2}+\rho} = h(\rho)|Y|^\rho$ and $D_Y|Y|^\rho = h\left(\rho+\frac{n_1-1}{2}\right)|Y|^{\rho-1} =$

$k(\rho)|Y|^{\rho-1}$. It follows that $k(\rho) = \prod_{i=1}^{n_1}\left(\rho + \frac{n_1-i}{2}\right)$ and $g(\rho) = k\left(\rho - \frac{n}{2}\right) \times$

$k\left(\rho - \frac{n_1-1}{2}\right), \ g'(\rho) = k\left(-\rho + \frac{1}{2}(n-n_1+1)\right)k(-\rho) = g\left(\frac{n}{2}-\rho\right) = g(\rho).$

This completes the proof.

It will be necessary later to compute the residue of Koecher's zeta function at $\frac{n}{2}$.

THEOREM 1.3.
$$\operatorname*{Res}_{\rho=\frac{n}{2}} Z_{n_1}(S^{(n)}, \rho) = \frac{1}{2}\pi^{\frac{1}{2}(1+nn_1-n_1{}^2)}|S|^{\frac{-n_1}{2}} \times$$
$$\prod_{k=2}^{n_1}\zeta(k)\Gamma\left(\frac{k}{2}\right)\prod_{j=0}^{n_1-1}\Gamma\left(\frac{n-j}{2}\right)^{-1}$$

Proof. The method of Siegel [8], volume III, pages 328 to 333, can be modified to show this result. One needs also formula (3.18) of Koecher [3], page 14. Siegel's proof is for $n = n_1$, so some work is required to obtain the result. For this, see [9], pages 82–91.

§ 2. Selberg's Zeta Function.

We are considering $\zeta_{(n)}(S^{(n)}, \rho)$ defined by (3). This is the case $r = n$ of (6). Since the results of this section have been stated by many authors (Selberg [6], Maass [4], and Godement in a 1962 lecture at Johns Hopkins University), we shall be brief. We must note some details of the proofs for later use.

Define $\mathfrak{P}(n_1, \cdots, n_r) = \left\{ U : U \in \mathfrak{U}_n, \ U = \begin{pmatrix} U_1^{(n_1)} & & * \\ & \ddots & \\ 0 & & U_r^{(n_r)} \end{pmatrix} \right\}$.

LEMMA 2.1. (*A Decomposition for* $\zeta_{(n)}$ *with respect to* $\mathfrak{P}_i^* = \mathfrak{P}(1, \cdots, 1, \underset{i}{2}, 1, \cdots, 1)$)

$$\zeta_{(n)}(S, \rho) = \sum_{V \in \mathfrak{U}_{n/\mathfrak{P}_i^*}} \prod_{\substack{j=1 \\ j \neq i}}^{n-1} |S[V_j]|^{-\rho_j} \zeta_{(2)}(T, \rho_i).$$

Here $V^{(n)} = (V_j^{(n,j)}*)$, $V_{i+1} = (V_{i-1}V_{i-1}')$, *and* $T = |S[V_{i-1}]|\{S[V_{i-1}'] - (S[V_{i-1}])^{-1}[{}^tV_{i-1}SV_{i-1}']\}$. *Also*

$$|T| = |S[V_{i-1}]|\ |S[V_{i+1}]|\ \textit{for}\ i \neq 1,\ n-1,$$
$$|T| = |S[V_2]|\ \textit{if}\ i = 1,\ \textit{and}$$
$$|T| = |S|\ |S[V_{n-1}]|\ \textit{if}\ i = n-1.$$

Proof. Note that $\mathfrak{U}_n \supset \mathfrak{P}_i^* \supset \mathfrak{P}_{(n)} = \mathfrak{P}(1, \cdots, 1)$. Therefore any $U \in \mathfrak{U}_n/\mathfrak{P}_{(n)}$ may be written uniquely as $U = VW$, $V \in \mathfrak{U}_n/\mathfrak{P}_i^*$, $W \in \mathfrak{P}_i^*/\mathfrak{P}_{(n)}$. And we can take

$$W = \begin{pmatrix} I^{(i-1)} & 0 & 0 \\ 0 & W^* & 0 \\ 0 & 0 & I^{(n-1-i)} \end{pmatrix} \text{ with } W^* \in \mathfrak{U}_2/\mathfrak{P}_{(2)}, \quad \mathfrak{P}_{(2)} = \mathfrak{P}(1,1).$$

Now we have $|S[V W_j]| = |S[V_j]|$ for $j \neq i$. And $|S[V W_i]| = |T[W_1^*]|$. This proves the lemma.

The analytic continuation and functional equation of $\zeta_{(2)}(T, \rho_i)$ (an Epstein zeta function) can be used to derive the same for $\zeta_{(n)}$. One obtains the functional equations (4) and formula (7) immediately (after an argument on the convergence of the series, which we omit here, as we must generalize it in §3).

In order to complete the analytic continuation with this approach it is convenient (and not surprising, considering the integral formula of Harish-Chandra, stated in [1], proposition 6.8, page 428) to introduce new variables z, with $\rho_i = z_{i+1} - z_i + \frac{1}{2}$. One uses (4) to show that for $\zeta'_{(n)}(z) = \zeta_{(n)}(\rho(z))$ we have

(19)
$$\Xi(z) = |S|^{z_n - \frac{1}{2}} \pi^{-2 \sum_{j=1}^{n-1} j z_j} \prod_{1 \leq i < j \leq n} \Gamma\left(z_j - z_i + \frac{1}{2}\right) \zeta\left(2\left(z_j - z_i + \frac{1}{2}\right)\right) \zeta'_{(n)}(z)$$

is invariant under all permutations of $z_1, z_2, \cdots, z_n$.

One obtains the analyticity of $\prod_{1 \leq i < j \leq n}\left((z_j - z_i - \frac{1}{2}\right)\zeta'_{(n)}(z)$ in $\Delta^* = \{z \in C^n :$ there is a permutation σ such that $Re\,(z_{\sigma(j+1)} - z_{\sigma(j)}) > n, \; j = 2, 3, \cdots, n\}$. The region Δ^* is a connected tube and its convex hull is C^n because it is fairly easy to see that it contains n independent lines. Applying Theorem 2.5.10 of Hörmander [2] one obtains the result that $\zeta_{(n)}$ can be continued to C^{n-1}. For more details of the above arguments see §3 of [9].

§3. The General Decomposition of the Zeta Function $\zeta_{n_1,\ldots,n_r}$ Corresponding to $\mathfrak{P}=\mathfrak{P}(n_1, \cdots, n_r)$ with respect to $\mathfrak{P}^*=\mathfrak{P}(m_1, \cdots, m_\lambda) \supset \mathfrak{P}$.

We first show the convergence of $\zeta_{n_1,\ldots,n_r}(S^{(n)}, \rho_1, \cdots, \rho_{r-1}) = \zeta(S, \rho)$ for $Re\,\rho_i > \frac{n}{2}$, $i = 1, 2, \cdots, r-1$. This results from the following theorem, since $Z_m(S^{(n)}, \rho)$ converges whenever $Re\,\rho > \frac{n}{2}$, as Koecher proves in [3], page 7. (Koecher bounds $Z_m(S^{(n)}, \rho)$ by $c^\rho(Z_1(S, \rho))^m$, c being a positive constant. Now the Epstein zeta function $Z_1(S, \rho)$ converges for $Re\,\rho > \frac{n}{2}$ using methods like those of Hecke for the Dedekind zeta function).

THEOREM 3.1. *(Convergence of the Zeta Function for Arbitrary r).* *For real ρ_i,*

$$\zeta_{n_1,\ldots,n_r}(S^{(n)},\ \rho_1,\ \cdots,\rho_{r-1}) \le \prod_{i=1}^{r-1} \zeta_{N_i,n-N_i}(S,\rho_i).$$

Here $n = \sum_{j=1}^{r} n_j$ *and* $N_i = \sum_{j=1}^{i} n_j.$

Proof. Define a map $f:\mathfrak{U}_n/\mathfrak{P}\ (n_1,\ \cdots,n_r) \longrightarrow \prod_{i=1}^{r-1}\mathfrak{A}_{n,N_i}/\mathfrak{A}_{N_i}$, where $\mathfrak{U}_{n,N_i}=\{V^{(n,N_i)}: V \text{ primitive}\}.$ The map f is defined as follows. Let $U \in \mathfrak{U}_n/\mathfrak{P}$ $(n_1,\ \cdots,n_r)$ and $U = (U_i^{(n,N_i)}*)$. Suppose $U_i = V_iB_i$, with B_i the greatest right divisor of U_i and V_i primitive. (The notion of greatest right divisor and primitive is defined in Siegel [8], volume I, pp. 331,332). Define $f(U) = (V_1,\ \cdots,V_{r-1})$. The map is well-defined.

The map is shown to be one-to-one by an induction process. Suppose $f(U) = f(U')$, where $U_i = V_iB_i$ and $U_i' = V_iB_i'$ as above. Then $U_1B_1^{-1}B_1'=U_1'$. Let $R_1 = B_1^{-1}B_1'\in\mathfrak{U}_{N_1}$. In general let $R_i = B_i^{-1}B_i'$. We assume as the induction hypothesis that $R_i\in\mathfrak{P}\ (n_1,\ \cdots,n_i)$. Then $U_{i+1}\begin{pmatrix} R_i & 0 \\ 0 & I^{(n_{i+1})} \end{pmatrix}$ and U_{i+1}' have the same first N_i columns. So $B_{i+1}\begin{pmatrix} R_i & 0 \\ 0 & I^{(n_{i+1})} \end{pmatrix}$ and B_{i+1}' have the same first N_i columns. (Here take $B_r = U$, $B_r' = U'$). Therefore $R_{i+1} = (B_{i+1})^{-1}B_{i+1}'\in\mathfrak{P}\ (n_1,\ \cdots,n_{i+1})$. The result follows that $U^{-1}U'\in\mathfrak{P}\ (n_1,\ \cdots,n_r)$, $U = U'$, and f is one-to-one.

Now that we have the convergence of the zeta function for arbitrary r, we generalize Lemma 2.1.

THEOREM 3.2. *Suppose* $\mathfrak{P} = \mathfrak{P}\ (n_1,\ \cdots,\ n_r) \subset \mathfrak{P}^* = \mathfrak{P}\ (m_1,\ \cdots,m_\lambda).$ *Let* $N_{i_j} = \sum_{k=1}^{i_j} n_k = M_j = \sum_{k=1}^{j} m_k.$ *Then we have the following representation:*

$$\zeta_{n_1,\ldots,n_r}(S, \rho_{N_1},\ \cdots,\rho_{N_{r-1}})$$

$$=\sum_{V\in\mathfrak{U}_n/\mathfrak{P}^*} \prod_{j=1}^{\lambda} |S[V_{j-1}]|^{-\sum_{k=i_{j-1}}^{i_j-1}\rho_{N_k}} \zeta_{n_{i_{j-1}+1},\ldots,n_{i_j}}(T_j, \rho_{N_{i_{j-1}+1}},\ \cdots,\rho_{N_{i_j-1}}).$$

Here $V = (V_j^{(n,M_j)}*)$, *and* $T_j = \{S - (S[V_{j-1}])^{-1}[{}^tV_{j-1}S]\}[Y_{j-1}]$, *if* $V_j = (V_{j-1}Y_{j-1}).$ *Note that* $|T_j| = |S[V_j]|/|S[V_{j-1}]|.$

Proof. Write $U \in \mathfrak{U}/\mathfrak{K}$ in the form $U = VW$, $V \in \mathfrak{U}/\mathfrak{P}^*$, $W \in \mathfrak{P}^*/\mathfrak{P}$. Then

$$W = \begin{pmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_\lambda \end{pmatrix}, \quad W_j \in \mathfrak{U}_{m_j}/\mathfrak{P} \, (n_{i_{j-1}+1}, \, \cdots, \, n_{i_j}).$$ Now if $i_{j-1} < k \le i_j$, we can

write as before:

$$|S[VW_{N_k}]| = \left| S\left[(V_{j-1}Y_{j-1}) \begin{pmatrix} Z_{j-1}^{(M_{j-1})} & 0 \\ 0 & Q_{j-1}^{(m_j, N_k - M_{j-1})} \end{pmatrix} \right] \right|$$

where $W = \begin{pmatrix} Z & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & * \end{pmatrix}$. It follows that

$$\begin{aligned}
|S[VW_{N_k}]| &= |S[(V_{j-1}Z_{j-1}, \ Y_{j-1}Q_{j-1})]| \\
&= |S[V]| \ |S[YQ] - (S[VZ])^{-1}[{}^t(VZ)SYQ]| \\
&= |S[V]| \ |\{S - (S[V])^{-1}[{}^tVS]\} [YQ]| \\
&= |S[V_{j-1}]| \ |T_j[Q_{j-1}]|.
\end{aligned}$$

This proves the theorem.

Next we generalize formula (7).

THEOREM 3.3. $\zeta_{n_1,\ldots,n_r}(S, \rho_{N_1}, \cdots, \rho_{N_{r-1}})|_{\rho_{N_k}=0, k \ne i_j} = \zeta_{m_1,\ldots,m_\lambda}(S, \rho_{M_1}, \cdots,$
$\rho_{M_{\lambda-1}})$, where $N_{i_j} = M_j$. (*Here we use the notation of Theorem 3.2*).

Proof. We proceed by induction on r. The case $r = 2$ is needed in the induction step, so we shall prove it in detail. That is, we shall show that $\zeta_{n_1,n_2}(T, 0) = 1$. This is equivalent (by Lemma 1.1) to the proof that $Z_{n_1}(T^{(n)}, 0) = \prod\limits_{k=0}^{n_1-1} \zeta(-k)$.

By Theorem 1.2,

$$\prod_{i=1}^{n_1} \Gamma\left(\rho - \frac{i-1}{2}\right) \pi^{-n_1\rho} Z_{n_1}(S, \rho) = |S|^{-\frac{n_1}{2}} \prod_{i=1}^{n_1} \Gamma\left(\frac{n+1-i}{2} - \rho\right) \pi^{-n_1\left(\frac{n}{2}-\rho\right)} Z_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right).$$

We take the residue at $\rho = 0$ on both sides, recalling the facts:

$$\operatorname*{Res}_{\rho=0} \Gamma(\rho) = 1 \text{ and } \operatorname*{Res}_{\rho=0} Z_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right) = - \operatorname*{Res}_{\rho=\frac{n}{2}} Z_{n_1}(S^{-1}, \rho)$$

$$= -\frac{1}{2} \pi^{\frac{1}{2}(-n_1^2 + n_1 n + 1)} |S|^{\frac{n_1}{2}} \prod_{k=2}^{n_1} \zeta(k) \Gamma\left(\frac{k}{2}\right) \prod_{i=1}^{n_1-1} \Gamma\left(\frac{n-i}{2}\right)^{-1}, \text{ by Theorem 1.3.}$$

It follows that

$$Z_{n_1}(S, 0) = \frac{-\dfrac{1}{2}\pi^{-\frac{1}{2}(n_1^2-1)} \prod_{i=2}^{n_1} \zeta(i)\Gamma\left(\dfrac{i}{2}\right)}{\prod_{i=1}^{n_1-1}\Gamma\left(-\dfrac{i}{2}\right)}.$$

Let $\phi(x) = \zeta(2x)\Gamma(x)$. Then the functional equation of the Riemann zeta function is $\Phi(1-x) = \pi^{\frac{3}{2}-2x}\phi\left(x-\dfrac{1}{2}\right)$. This means that $\zeta(-y)\Gamma\left(-\dfrac{y}{2}\right) = \pi^{-y-\frac{1}{2}}\zeta(y+1)\Gamma\left(\dfrac{y+1}{2}\right)$. Therefore

$$Z_{n_1}(S, 0) = \prod_{i=0}^{n_1-1} \zeta(-i).$$

This completes the case $\gamma = 2$.

Now we proceed to the induction step. We suppose for convenience that $i \neq 1$. By Theorem 3.2 for $\mathfrak{P}^* = \mathfrak{P}(n_1 + n_2, n_3, \cdots, n_\gamma)$ we have $M_j = N_{j+1}$, $j = 1, 2, \cdots, \gamma - 1$, and

$$\zeta_{n_1,\ldots,n_\gamma}(S, \rho_{N_1}, \cdots, \rho_{N_{\gamma-1}}) = \sum_{V \in \mathfrak{A}_n/\mathfrak{P}^*} \zeta_{n_1,n_2}(T_1, \rho_{N_1}) \prod_{j=1}^{\gamma-2} |S[V_j]|^{-\rho_{N_{j+1}}},$$

where $V = (V_j^{(n, M_j)*})$ and $T_1 = S[V_1]$.

Multiplying by the factor in Theorem 1.2 we obtain:

$$2\pi^{-\frac{1}{4}(n_1^2-n_1)-n_1\rho_{N_1}} \left\{ \prod_{i=1}^{n_1} \left(\rho_{N_1} + \frac{i-1-n}{2}\right)\left(\rho_{N_1} - \frac{i-1}{2}\right)\right\} \phi\left(\rho_{N_1} - \frac{i-1}{2}\right)\zeta(S, \rho)$$

$$= \sum_{V \in \mathfrak{U}_n/\mathfrak{P}^*} J(T_1, \rho_{N_1}) \prod_{j=1}^{\gamma-2} |S[V_j]|^{-\rho_{N_{j+1}}},$$

$$(\zeta(S, \rho) = \zeta_{n_1,\ldots,n_\gamma}(S, \rho_{N_1}, \cdots, \rho_{N_{\gamma-1}})).$$

The theorem will follow from the case $\gamma = 2$ provided that we can show the convergence of this representation in a domain like $n > Re\,\rho_{N_1} > -1$, $Re\,\rho_{N_j} > L$, $j = 2, \cdots, \gamma - 1$, for some sufficiently large L.

First one can show that there is a positive constant c, depending only on S, ρ, and n, such that

(20)
$$\sum_{V \in \mathfrak{U}/\mathfrak{P}^*} |J(S[V_1], \rho_{N_1}) \prod_{j=1}^{\gamma-2} |S[V_j]|^{-\rho_{N_{j+1}}}|$$
$$< c \sum_{V \in \mathfrak{U}/\mathfrak{P}^*} |J(I[V_1], \rho_{N_1}) \prod_{j=1}^{\gamma-2} |I[V_j]|^{-\rho_{N_{j+1}}}|.$$

To prove this inequality, use the following facts.

a) If $W^{(n)} = (w_{ij})$ is reduced and $W_0 = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}$, where $w_{ii} = w_i$, then there is a positive constant c, depending only on n, such that for all vectors x

$$\frac{1}{c}\, W_0[x] \leqslant W[x] \leqslant c\, W_0[x] \quad \text{and} \quad \frac{1}{c} W_0^{-1}[x] \leqslant W^{-1}[x] \leqslant c W_0^{-1}[x].$$

b) By the proof of Theorem 1.2, $J(S[V_1], \rho_{N_1})$ is a sum of 2 integrals. And $(S[V_1])^{\pm 1}$ comes into these integrals by means of the trace and a determinant.

c) Recall that $\sigma(W[Y]) = \sum\limits_{k=1}^{n} W[y_k]$, where $Y = (y_1 y_2 \cdots y_n)$. And $W = S[V]$, where $V = (v_1 v_2 \cdots v_n)$, implies that $w_i = S[v_i]$.

d) If W is reduced there exists a positive constant c depending only on n such that $|W| \geqslant c w_1 w_2 \cdots w_n$. And if W is positive then $|W| \leqslant w_1 w_2 \cdots w_n$.

e) To apply the previous, one assumes in the sums on the right hand side of (20) that $S[V_j]$, $j = 2, 3, \cdots, r-2$ are reduced and that $S[V_1]$ or $(S[V_1])^{-1}$ is reduced. In the sums on the left hand side one assumes that $I[V_1]$ or $(I[V_1])^{-1}$ is reduced.
These facts and a little computation suffice to prove (20).

To show convergence we may assume $S = I$, the identity. Then $T = T_1 = I[V_1]$ is integral. From the proof of Theorem 1.2 it follows that for $n > Re\, \rho > -1$,

$$|J(T, \rho)| \leqslant g(n) \int_{F_{n_1}} |X|^n \theta_{n_1}^T d\mu + |g(n)| \, |T|^{-\frac{1}{2}n_1} \int_{F_{n_1}} |X|^n \theta_{n_1}^{T^{-1}} d\mu.$$

If T is a matrix with integer entries, the first integral is less than a bound independent of T. For $\int_{F_{n_1}} |X|^n \theta_{n_1}^T d\mu \leq \xi(n) Z_{n_1}(T, n)$, where $\xi(n)$ is a product of Γ-functions, etc., and $\xi(n($ is independent of T. We may assume T to be Minkowski-reduced since $Z_{n_1}(T[U], n) = Z_{n_1}(T, n)$ for $U \in \mathfrak{U}_{n_1 + n_2}$. From Koecher [3], page 7, we have $Z_{n_1}(T, n) \leq c Z_{1^{n_1 + n_2 - 1}}(T, n)^{n_1}$. And $Z_1(T, n) = \frac{1}{2} \sum\limits_{m \neq 0} T[m]^{-n} \leq c^n \sum T_0[m]^{-n} \leq c^n t_1^{-n} Z_1(I, n) \leq c^n Z_1(I, n)$, a bound B independent of T, since $t_1 \geq 1$. (Here T_0 is the diagonal matrix with the same entries

as the diagonal entries of T and $T = \begin{pmatrix} t_1 & & & * \\ & t_2 & & \\ & & \ddots & \\ * & & & t_n \end{pmatrix}$.)

If T is integral, so is adjoint $T = \operatorname{adj} T$. Thus we are able to use the same estimate for the second integral in the formula for $J(T, \rho_{N_1})$. We have

$$\int_{F_{n_1}} |X|^n \theta_{n_1}^{T-1} d\mu = \int_{F_{n_1}} |X|^n \theta_{n_1}^{|T|^{-1}\operatorname{adj}T} d\mu$$

$$= |T|^n \int_{F_{n_1}} |X|^n \theta_{n_1}^{\operatorname{adj}T} d\mu \leq |T|^n B, \text{ where } B \text{ is a bound independent of } T.$$

Therefore when $n > Re\,\rho_{N_1} > -1$, there is a positive constant B independent of T such that

$$|J(T, \rho_{N_1})| \leq |T|^n B.$$

It follows that for $n > Re\,\rho_{N_1} > -1$:

$$\sum_{V \in \mathfrak{U}_n/\mathfrak{B}*} |J(T, \rho_{N_1})| \prod_{j=1}^{r-2} |I[V_j]|^{-\rho_{N_{j+1}}}$$

$$\leq B \sum_{V \in \mathfrak{U}_n/\mathfrak{B}*} |I[V_1]|^n \prod_{j=1}^{r-2} |I[V_j]|^{-Re\,\rho_{N_{j+1}}},$$

$$= B \zeta_{n_1+n_2, n_3, \ldots, n_{r-1}}(I, -n + Re\,\rho_{N_2}, Re\rho_{N_3}, \cdots, Re\,\rho_{N_{r-1}}),$$

which converges for $n > Re\,\rho_{N_1} > -1$ and $Re\,\rho_{N_j} > \dfrac{3n}{2}$, $j = 2, \cdots, r-1$. This finishes the proof.

§4. Relations Between Koecher's Zeta Functions.

There are two methods of obtaining relations between $\zeta_{i,n-i}$ and $\zeta_{n-i,i}$. First we use (4) and the case $r = n$, $\lambda = 2$ of (8). Then we use (5) and the same case of (8).

THEOREM 4.1. Let $F_n(\rho) = |S|^{-\rho_{n-1}+\frac{1}{2}} \pi^{2\sum\limits_{j=1}^{n-1} j\rho_j - \frac{(n-1)n}{2}} \times$

$$\prod_{i=1}^{n-1} \frac{\phi\left(1 - \rho_{n-1} - \cdots - \rho_i + \dfrac{n-1-i}{2}\right)}{\phi\left(\rho_{n-1} + \cdots + \rho_i - \dfrac{n-1-i}{2}\right)}. \quad \text{Then } \zeta_{i,n-i}(S, \rho) =$$

$$\prod_{j=0}^{n-i-1} F_n(\rho^j)|_{\rho_j=0, j\neq i} \zeta_{n-i,i}\Big(S, \frac{n}{2}-\rho_i\Big). \quad \text{Here} \quad \rho^\sigma = \Big(\frac{n}{2}-\sum_{i=1}^{n-1}\rho_i, \rho_1, \rho_2, \cdots, \rho_{n-2}\Big) \quad and$$

$$\phi(x) = \zeta(2x)\Gamma(x).$$

Proof. If we apply the functional equations (4) in the order $n-1$, $n-2, \cdots, 2, 1,$ or if we use the invariance of formula (19) in §2 under $\sigma = (n\ \ n-1)\cdots(32)(21) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}$ we obtain:

$$\pi^{-\sum_{j=1}^{n-1} j\rho_j + \frac{(n-1)(n-2)}{4}} \prod_{i=1}^{n-1} \phi\Big(\rho_{n-1}+\cdots+\rho_i - \frac{n-1-i}{2}\Big)\zeta_{(n)}(\rho)$$

$$= |S|^{\frac{1}{2}-\rho_{n-1}} \pi^{\sum_{j=1}^{n-1} j\rho_j - \frac{1}{4}(n+2)(n-1)} \prod_{i=1}^{n-1} \phi\Big(1-\rho_{n-1}-\cdots-\rho_i+\frac{n-1-i}{2}\Big)\times$$

$$\zeta_{(n)}\Big(\frac{n}{2}-\sum_{i=1}^{n-1}\rho_i, \rho_1, \cdots, \rho_{n-2}\Big).$$

Therefore $\zeta_{(n)}(\rho)=F_n(\rho)\zeta_{(n)}(\rho^\sigma)$. And $\zeta_{(n)}(\rho)=\prod_{j=0}^{i-1} F(\rho^{\sigma^j})\zeta_{(n)}(\rho^{\sigma^i})$. It is clear that $\rho^{\sigma^j} = (\rho_{n-(i-1)}, \cdots, \rho_{n-1}, \frac{n}{2}-\sum_{j=1}^{n-1}\rho_j, \rho_1, \rho_2, \cdots, \rho_{n-(i+1)})$, where ρ_{n-i} is omitted. The result follows easily using (8). A small computation convinces one that $\prod_{j=1}^{n-i-1} F(\rho^{\sigma^j})|_{\rho_k=0, k\neq i}$ makes sense.

THEOREM 4.2. $\zeta_{i,n-i}(S^{-1}, \rho) = |S|^\rho \zeta_{n-i,i}(S, \rho).$

Proof. Use (8) and (5).

Theorems 4.1 and 4.2 combine to give once again the functional equation of Koecher's zeta function.

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