Nonlinear Functional Analysis and Applications
Vol. 19, No. 3 (2014), pp. 359-377

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

A GENERALIZATION OF GERAGHTY'S THEOREM IN PARTIALLY ORDERED G-METRIC SPACES AND APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The purpose of this article is to present some coincidence and fixed point theorems for generalized contraction in partially ordered complete G-metric spaces. As an application, we give an existence and uniqueness for the solution of an initial-boundary-value problem. These results generalize and extend several well known results in the literature.

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity, see [15]-[19], [22, 23], [25]-[28]. The notion of D-metric space is a generalization of usual metric

[0]Received January 9, 2014. Revised June 8, 2014.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Fixed point, coincidence point, partially $G-$metric spaces, contraction, initial value.

spaces and it is introduced by Dhage [2, 3]. Recently, Mustafa and Sims [31]-[33] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [31], [34]-[36], they introduced a improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces and fixed point results, one can refer to the papers [1], [4]-[13], [20, 24, 29], [37]-[43] some of them have given some applications to matrix equations, ordinary differential equations, and integral equations.

Let S denotes the class of the functions β: $[0,+\infty) \to [0,1)$ which satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$. For example, functions
$\beta_1(x) = \begin{cases} \frac{\ln(1+x)}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$ $\beta_2(x) = \frac{1}{1+x}$, $\beta_3(x) = \begin{cases} \frac{\exp(x)-1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$
are in S.

2. Mathematical preliminaries

Definition 2.1. ([30]) Let X be a non-empty set, $G : X \times X \times X \to R_+$ be a function satisfying the following properties:

- $(G1)$ $G(x,y,z) = 0$ if $x = y = z$.
- $(G2)$ $0 < G(x,x,y)$ for all x, $y \in X$ with $x \neq y$.
- $(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$.
- $(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x)$ (symmetry in all three variables).
- $(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X,G) is called a $G-$metric space.

Definition 2.2. ([30]) Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is $G-$convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x;x_n,x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x;x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence x_n and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.3. ([30]) *Let (X,G) be a G-metric space. The following are equivalent:*

1. (x_n) *is G-convergent to x;*
2. $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$;
3. $G(x_n,x,x) \to 0$ *as* $n \to \infty$;
4. $G(x_n,x_m,x) \to 0$ *as* $n,m \to \infty$.

Definition 2.4. ([30]) Let (X,G) be a G-metric space. A sequence (x_n) is called a $G-$ Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.5. ([30]) *Let (X,G) be a G-metric space. Then the following are equivalent:*

(1) *The sequence (x_n) is G-Cauchy.*
(2) *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 2.6. ([30]) *Let (X,G) be a G-metric space. A mapping $f : X \to X$ is $G-$continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.*

Proposition 2.7. ([30]) *Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous all three of its variables.*

Definition 2.8. ([30]) A G-metric space (X,G) is called $G-$complete if every $G-$Cauchy sequence is $G-$convergent in (X,G).

Definition 2.9. (weakly compatible mappings ([30])) Two mappings $f, g : X \to X$ are weakly compatible if they commute at their coincidence points, that is $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Definition 2.10. ([30]) Let X be a non-empty set and S, T be self-mappings of X. A point $x \in X$ is called a coincidence point of S and T if $Sx = Tx$. A point $w \in X$ is said to be a point of coincidence of S and T if there exists $x \in X$ so that $w = Sx = Tx$.

Definition 2.11. ($g-$Nondecreasing Mapping ([30])) Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ are mappings. f is said to be $g-$Non decreasing if for $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$.

Now, we are ready to state and prove our main results.

Let Ψ denotes the class of the functions $\psi : [0, +\infty[\to [0, +\infty[$ which satisfies the following conditions:

(1) ψ is nondecreasing,
(2) ψ is sub-additive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$,
(3) ψ is continuous,

(4) $\psi(t) = 0 \iff t = 0.$

For example, functions $\varphi_1(t) = kt$, where $k > 0$, $\varphi_2(t) = \frac{t}{1+t}$, $\varphi_3(t) = \ln(1+t)$ and $\varphi_4(t) = \min\{1, t\}$ are in Ψ.

The following generalization of Banach's contraction principle is due to Geraghty [21].

Theorem 2.12. *Let (M,d) be a complete metric space and let $f : M \to M$ be a map. Suppose there exists $\beta \in S$ such that for each $x, y \in M$*

$$d(f(x), f(y)) \leq \beta\,(d(x,y))\, d(x,y).$$

Then f has a unique fixed point $z \in M$ and $\{f^n(x)\}$ converges to z, for each $x \in M$.

3. Main results

Now, we state our main results.

Lemma 3.1. *Let (X, G) be a G-metric space and (x_n) be a sequence in X such that $G(x_{n+1},\, x_{n+1},\, x_n)$ is decreasing and*

$$\lim_{n\to\infty} G(x_{n+1}, x_{n+1}, x_n) = 0. \tag{3.1}$$

If (x_{2n}) is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that the following four sequences tends to ε as $k \to \infty$,

$$\begin{array}{ll} G(x_{2m_k}, x_{2m_k}, x_{2n_k}), & G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}), \\ G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_k}), & G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_{k+1}}). \end{array} \tag{3.2}$$

Proof. If (x_{2n}) is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that

$$n_k > m_k > k; \quad G(x_{2m_k}, x_{2m_k}, x_{2n_k-2}) < \varepsilon, \;\; G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \geq \varepsilon$$

for all integer k. Then

$$\begin{array}{rcl} \varepsilon & \leq & G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \\ & \leq & G(x_{2m_k}, x_{2m_k}, x_{2n_k-2}) + G(x_{2n_{k-2}}, x_{2n_{k-2}}, x_{2n_k-1}) \\ & & +G(x_{2n_{k-1}}, x_{2m_{k-1}}, x_{2n_k}) \\ & < & \varepsilon + G(x_{2n_{k-2}}, x_{2n_{k-2}}, x_{2n_k-1}) + G(x_{2n_{k-1}}, x_{2n_{k-1}}, x_{2n_k}). \end{array}$$

From (3.1), we conclude that

$$\lim_{k\to\infty} G(x_{2m_k}, x_{2m_k}, x_{2n_k}) = \varepsilon. \tag{3.3}$$

Further,

$$G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \leq G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) + G(x_{2n_{k+1}}, x_{2n_{k+1}}, x_{2n_k})$$

and

$$G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) \leq G(x_{2m_k}, x_{2m_k}, x_{2n_k}) + G(x_{2n_k}, x_{2n_k}, x_{2n_{k+1}}).$$

Passing to the limit when $k \to \infty$ and using (3.1) and (3.3), we obtain

$$\lim_{k\to\infty} G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) = \varepsilon.$$

The remaining two sequences in (3.2) tend to ε can be proved in a similar way. □

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G- complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that*

$$\psi(G(fx, fy, fz)) \leq \beta\left(\psi(G(gx, gy, gz))\right)\psi(G(gx, gy, gz)) \tag{3.4}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. By the condition of the theorem there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, we can define $x_1 \in X$ such that $gx_1 = fx_0$, then $gx_0 \preceq fx_0 = gx_1$. Since f is $g-$nondecreasing, we have $fx_0 \preceq fx_1$. In this way we construct the sequence (x_n) recursively as

$$fx_n = gx_{n+1}, \quad \forall\, n \geq 1 \tag{3.5}$$

for which

$$\begin{aligned} gx_0 \;\preceq\; fx_0 = gx_1 \;\preceq\; fx_1 = gx_2 \;\preceq\; fx_2 \preceq \cdots \\ \preceq\; fx_{n-1} = gx_n \;\preceq\; fx_n = gx_{n+1} \;\preceq \cdots . \end{aligned} \tag{3.6}$$

First, we suppose that there exists $n_0 \in \mathbb{N}$ such that $\psi(G(fx_{n_0}, fx_{n_0}, fx_{n_0+1})) = 0$, then it follows from the properties of ψ, $G(fx_{n_0}, fx_{n_0}, fx_{n_0+1}) = 0$. So, $fx_{n_0} = fx_{n_0+1,}$ we have $gx_{n_0+1} = fx_{n_0+1.}$ Therefore x_{n_0+1} is a considance point of f and g. From now on we suppose $\psi(G(fx_n, fx_n, fx_{n+1})) \neq 0$ for all $n \geq 0$. The elements gx_n and gx_{n+1} are comparable, substituting $x = y = x_n$ and $z = x_{n+1}$ in (3.4), using (3.5) and (3.6), we have

$$\begin{aligned} \psi(G(fx_n, fx_n, fx_{n+1})) &\leq \beta\left(\psi(G(gx_n, gx_n, gx_{n+1}))\right)\psi(G(gx_n, gx_n, gx_{n+1})) \\ &\leq \psi(G(gx_n, gx_n, gx_{n+1})) \\ &= \psi(G(fx_{n-1}, fx_{n-1}, fx_n)). \end{aligned}$$

Thus it follows that $(\psi(G(fx_n, fx_n, fx_{n+1})))$ is a non increasing sequence and bounded below, so $\lim_{n\to\infty} \psi(G(fx_n, fx_n, fx_{n+1})) = r \geq 0$ exits. Assume that $r > 0$, then from (3.4), we have

$$\frac{\psi(G(fx_n, fx_n, fx_{n+1}))}{\psi(G(fx_{n-1}, fx_{n-1}, fx_n))} \leq \beta\left(\psi(G(gx_n, gx_n, gx_{n+1}))\right) \leq 1 \quad \text{for each } n \geq 1,$$

which yields that

$$\lim_{n\to\infty} \beta\left(\psi(G(gx_n, gx_n, gx_{n+1}))\right) = 1.$$

On the other hand, since $\beta \in S$, we have $\lim_{n\to\infty} \psi(G(fx_n, fx_n, fx_{n+1})) = 0$ and so $r = 0$. Now we show that (fx_n) is a Cauchy sequence. Suppose that (fx_n) is not a Cauchy sequence. Using Lemma 3.1, we know that there exist $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that the following four sequences tend to ε as k goes to infinity,

$$G(fx_{2m_k}, fx_{2m_k}, fx_{2n_k}), \quad G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}}),$$
$$G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}), \quad G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_{k+1}}).$$

Putting in the contractive condition $x = y = x_{2m_k}$ and $z = x_{2n_{k+1}}$, using (3.5) and (3.6), it follows that

$$\begin{aligned}&\psi(G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}}))\\ &\leq \beta\left(\psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))\right)\psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))\\ &\leq \psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k})).\end{aligned}$$

So

$$\frac{\psi(G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}}))}{\psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))} \leq \beta\left(\psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))\right) \leq 1$$

and $\lim_{k\to\infty} \beta\left(\psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))\right) = 1$. Since $\beta \in S$, it follows that

$$\lim_{k\to\infty} \psi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k})) = 0.$$

Since ψ is a continuous mapping, $\psi(\varepsilon) = 0$ and so $\varepsilon = 0$, which contradicts $\varepsilon > 0$. Therefore, (fx_n) is a Cauchy sequence in (X, G). Since (X, G) is a complete metric space, there exists $a \in X$ such that $\lim_{n\to\infty} fx_n = a = \lim_{n\to\infty} gx_{n+1}$. Since $g(X)$ is closed, then $a = gz$, and by (3.5) $fx_n = gx_{n+1}$ for all $n \geq 1$. We have

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = gz = a. \tag{3.7}$$

Now we prove that z is a coincidence point of f and g. By (3.6), we have (gx_n) is a non-decreasing sequence in X. By (3.7) and condition of our theorem

$$gx_n \preceq gz. \tag{3.8}$$

Putting $x = y = x_n$ in (3.4), by the virtue of (3.8), we get

$$\begin{aligned}&\psi(G(fx_n, fx_n, fz)\\ &\leq \beta\left(\psi(G(fx_{n-1}, fx_{n-1}, gz))\right)\psi(G(gx_n, gx_n, gz))\\ &\leq \psi(G(gx_n, gx_n, gz), \quad \text{for each } \ n \geq 1.\end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (3.7) and the continuity of ψ, we get

$$G(gz, gz, fz) = 0,$$

that is

$$fz = gz. \tag{3.9}$$

This complete the proof. □

Theorem 3.3. *If in Theorem* 3.2*, it is additionally assumed that*

$$gz \preceq ggz, \tag{3.10}$$

where z is as in the condition of theorem and f and g are weakly compatible, then f and g have a common fixed point in X.

Proof. Following the proof of the Theorem 3.2, we have (3.7), that is, a non-decreasing sequence (gx_n) converging to gz. Then by (3.10) we have $gz \preceq ggz$. Since f and g are weakly compatible, by (3.9), we have $fgz = gfz$. We set

$$w = gz = fz. \tag{3.11}$$

Therefore, we have

$$gz \preceq ggz = gw. \tag{3.12}$$

Also

$$fw = fgz = gfz = gw. \tag{3.13}$$

If $z = w$, then z is a common fixed point. If $z \neq w$, then necessarily $gz = gw$. We argue by contradiction, if $gz \neq gw$. By (3.4) and (3.8), we have

$$\frac{\psi(G(gx_n, gx_n, gw))}{\psi(G(gx_n, gx_n, gw))} \leq \beta\left(\psi(G(gx_n, gx_n, gw))\right) \leq 1.$$

By going to the limit as $n \to \infty$, by using the fact that $\beta \in S$ and the continuity of ψ, we get $\psi(G(gz, gz, gw)) = 0$, so $gz = gw$. This is a contradiction. Therefore, by (3.11) and (3.13), we have $w = gw = fw$. Hence w is a common fixed point. This completes the proof. □

Remark 3.4. Continuity of f is not required in Theorem 3.3. If we assumed f to be continuous, then (3.8) is not longer required for the theorem and can be omitted.

Theorem 3.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G- complete metric space. Let $f : X \to X$ be such that f is a nondecreasing. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that*

$$\psi(G(fx, fy, fz)) \leq \beta\left(\psi(G(x, y, z))\right) \psi(G(x, y, z)),$$

for all x, y , $z \in X$ with $x \preceq y \preceq z$. Assume that either f is continuous or X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the proof of the Theorem 3.2, we have (3.7), that is, a nondecreasing sequence (x_n) converging to z. Now we show, that z is a fixed of point of f. If f is continuous, then

$$z = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(z)$$

and hence $f(z) = z$. If the second condition of the theorem holds, then we have

$$G(f(z), f(z), z) \leq G(f(z), f(z), f((x_n)) + G(f(x_n), f(x_n), z).$$

On the other hand, since φ is nondecreasing and sub-additive, we have

$$\begin{aligned}
&\psi\left(G\left(f(z), f(z), z\right)\right) \\
&\leq \psi\left(G(f(z), f(z), f((x_n)))\right) + \psi\left(G(f(x_n), f(x_n), z)\right) \\
&\leq \beta\left(\psi(G(z, z, x_n))\right) \psi(G(z, z, x_n)) + \psi\left(G(x_{n+1}, x_{n+1}, z)\right) \\
&\leq \psi(G(z, z, x_n)) + \psi\left(G(x_{n+1}, x_{n+1}, z)\right).
\end{aligned}$$

Since $G(z, z, x_n) \to 0$, $G(x_{n+1}, x_{n+1}, z) \to 0$, $\psi\left(G(x_{n+1}, x_{n+1}, z)\right) \to 0$ and $\psi(G(z, z, x_n)) \to 0$ when n goes to infinity. Then

$$\psi\left(G\left(f(z), f(z), z\right)\right) = 0 \quad \Leftrightarrow \quad G\left(f(z), f(z), z\right) = 0.$$

Therefore, we get $f(z) = z$. This completes the proof. □

In the following, we give a sufficient condition for the uniqueness of the fixed point in Theorem 3.5. This condition is as follows.

(i) Every pair of elements in X has a lower bound or an upper bound.

In [12], it is proved that the condition (i) is equivalent to the following.

(ii) For every $x, y \in X$, there exists $z \in X$ which is comparable to x and y.

Theorem 3.6. *Adding the condition* (ii) *to the hypothesis of Theorem* 3.5*, The fixed point z is unique.*

Proof. Let y be another fixed point of f, from (ii), there exists $x \in X$ which is comparable to y and z. The monotonicity of f implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n \geq 0$. Moreover, we have

$$\begin{aligned}
& \psi(G(z, z, f^n(x)) \\
= \; & \psi(G(f^n(z), f^n(z), f^n(x)) \\
= \; & \psi(G(f(f^{n-1}(z)), f(f^{n-1}(z)), f(f^{n-1}(x))) \\
\leq \; & \beta\left(\psi(G(f^{n-1}(z), f^{n-1}(z), f^{n-1}(x)))\right) \psi(G(f^{n-1}(z), f^{n-1}(z), f^{n-1}(x)) \\
\leq \; & \psi(G(f^{n-1}(z), f^{n-1}(z), f^{n-1}(x)) \\
= \; & \psi(G(z, z, f^{n-1}(x)). \qquad (3.14)
\end{aligned}$$

Consequently, the sequence (γ_n) defined by $\gamma_n = \psi(G(z, z, f^{n-1}(x))$ is non-negative and non increasing and so

$$\lim_{n\to\infty} \psi(G(z, z, f^{n-1}(x)) = \gamma \geq 0.$$

Now, we show that $\gamma = 0$. Assume that $\gamma > 0$. By passing to the subsequences, if necessary, we may assume that $\lim_{n\to\infty} \beta(\gamma_n) = \delta$ exists. From (3.14), it follows that $\delta\gamma = \gamma$ and so $\delta = 1$. Since $\beta \in S$,

$$\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \psi(G(z, z, f^{n-1}(x)) = \gamma = 0.$$

This is a contradiction and so $\gamma = 0$. Similarly, we can prove that

$$\lim_{n\to\infty} \psi(G(y, y, f^{n-1}(x)) = 0.$$

Finally, from

$$G(z, z, y) \leq G(z, z, f^n(x)) + G(f^n(x), f^n(x), y),$$

and $G(x, x, y) \leq 2G(x, y, y)$ for any $x, y \in X$, we obtain

$$G(z, z, y) \leq G(z, z, f^n(x)) + 2G(f^n(x), y, y).$$

Since ψ is nondecreasing and sub-additive, it follows that

$$\begin{aligned}
\psi(G(z, z, y)) \; & \leq \; \psi(G(z, z, f^n(x))) + \psi(G(f^n(x), y, y)) \\
& \quad + \psi(G(f^n(x), y, y)) \\
& \leq \; \psi(G(z, z, f^n(x))) + 2\psi(G(f^n(x), y, y)).
\end{aligned}$$

Therefore, taking $n \to \infty$, we have

$$\psi(G(z, z, y)) = 0.$$

It follows that $G(z, z, y) = 0$ and so $z = y$. This completes the proof. □

Letting $\psi = id_X$, in Theorems 3.2 and 3.5, we can get the following results.

Corollary 3.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\beta \in S$ such that*

$$G(fx, fy, fz) \leq \beta\left(G(gx, gy, gz)\right) G(gx, gy, gz), \tag{3.15}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Corollary 3.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f : X \to X$ be such that f is a nondecreasing. Suppose that there exist $\beta \in S$ such that*

$$G(fx, fy, fz) \leq \beta\left(G(x, y, z)\right) G(x, y, z),$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$. Assume that either f is continuous or X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Example 3.9. Let $X = [0, 1]$. We define a partial ordered $\leq$ on X as $x \leq y$ if and only if $x \leq y$ for all $x, y \in X$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$. Then (X, G) is a complete $G-$metric space. Let f,g : $X \to X$ be two functions defined as, $f(x) = \frac{x}{6}$ and $g(x) = \frac{x}{2}$ for all $x \in X$. So, $f(X) \subset g(X) = \left[0, \frac{1}{2}\right]$. $g(X)$ is closed in X and f is $g-$nondecreasing. Let $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(x) = \ln(1 + x)$. ψ is continuous, sub-additive, nondecreasing and satisfies $\psi(x) = 0 \Longleftrightarrow x = 0$ and $\psi(x) < x$ for any $x > 0$. Let $\beta : [0, \infty) \to [0, 1)$ defined as $\beta(x) = \begin{cases} \frac{\ln(1+x)}{x} & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$

Without loss of generality, we assume that $x < y < z$ and satisfy the inequality (3.4) for all $x, y, z \in X$ with $x < y < z$. So

$$G(fx, fy, fz) = \frac{1}{3}(z - x) \quad \text{and} \quad G(gx, gy, gz) = (z - x).$$

Hence it is easy to see that $\frac{1}{3}x \leq \psi(x)$ for all $x \in X$. Therefore the inequality (3.4) is satisfied. Then we choose $x_0 = 0$ in $[0, 1]$, $f(0) \leq g(0)$. All conditions of Theorem 3.2 are satisfied. Here $x_0 = 0$ is a coincidence point of f and g.

Later, from the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application of Theorem 3.2 above. For this purpose, let

$$Y=\left\{\chi:\begin{array}{c}\chi:\mathbb{R}^{+}\to\mathbb{R}^{+}\text{ satisfies that }\chi\text{ is Lebesgue integrable,}\\ \text{summable on each compact of subset of }\mathbb{R}^{+},\text{ sub-additive}\\ \text{and }\int_0^{\epsilon}\chi(t)\,dt>0\text{ for each }\epsilon>0.\end{array}\right\}$$

Definition 3.10. The function $\chi:\mathbb{R}^{+}\to\mathbb{R}^{+}$is called sub-additive integrable function if for any $a,\, b\in\mathbb{R}^{+}$,

$$\int_0^{a+b}\chi(t)\,dt\le\int_0^{a}\chi(t)\,dt+\int_0^{b}\chi(t)\,dt.$$

Theorem 3.11. *Let $(X,\preceq)$ be a partially ordered set and suppose that (X,G) be a G-complete metric space. Let $f,g:X\to X$ be such that $f(X)\subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\beta\in S$ and $\psi\in\Psi$ such that for $\chi\in Y$,*

$$\begin{aligned}&\int_0^{\psi(G(fx,fy,fz))}\chi(t)\,dt\\ &\le\beta\left(\int_0^{\psi(G(gx,gy,gz))}\chi(t)\,dt\right)\int_0^{\psi(G(gx,gy,gz))}\chi(t)\,dt,\end{aligned}\tag{3.16}$$

for all $x,y,z\in X$ with $gx\preceq gy\preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n\preceq x$ for each $n\geq 0$. If there exists $x_0\in X$ such that $gx_0\preceq fx_0$, then f and g have a coincidence point.

Proof. For $\chi\in Y$, consider the function $\Lambda:\mathbb{R}^{+}\to\mathbb{R}^{+}$ defined by $\Lambda(x)=\int_0^{x}\chi(t)\,dt$. We note that $\Lambda\in\Psi$. Thus the inequality (3.16) becomes

$$\Lambda(\psi(G(fx,fy,fz)))\le\beta\left(\Lambda(\psi(G(gx,gy,gz)))\right)\Lambda(\psi(G(gx,gy,gz)))\,.\tag{3.17}$$

Setting $\Lambda\circ\psi=\psi_1$, $\psi_1\in\Psi$, so we obtain

$$\psi_1(G(fx,fy,fz))\le\beta\left(\psi_1(G(gx,gy,gz))\right)\psi_1(G(gx,gy,gz)).$$

Therefore by Theorem 3.2 above, f and g have a coincidence point. □

Corollary 3.12. *Let $(X,\preceq)$ be a partially ordered set and suppose that (X,G) be a G-complete metric space. Let $f:X\to X$ be a nondecreasing function. Suppose that there exist $\beta\in S$ and $\psi\in\Psi$ such that*

$$\begin{aligned} &\int_0^{\psi(G(fx,fy,fz))} \chi(t)\,dt \\ &\leq \beta\left(\int_0^{\psi(G(x,y,z))} \chi(t)\,dt\right)\int_0^{\psi(G(x,y,z))} \chi(t)\,dt, \quad \chi \in Y \end{aligned} \tag{3.18}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$. Assume that either f is continuous or X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.13. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\beta \in S$ such that for $\chi \in Y$,*

$$\int_0^{G(fx,fy,fz)} \chi(t)\,dt \leq \beta\left(\int_0^{G(gx,gy,gz)} \chi(t)\,dt\right)\int_0^{G(gx,gy,gz)} \chi(t)\,dt, \tag{3.19}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

4. Application

In this section, We show the existence of solution for the following initial-value problem by using Theorems 3.5 and 3.6.

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t,u,u_x), -\infty < x < \infty,\ 0 < t < T, \\ u(x,t) = \varphi(x), -\infty < x < \infty. \end{cases} \tag{4.1}$$

Where we assumed that φ is continuously differentiable and that φ and φ' are bounded and $F(x,t,u,u_x)$ is a continuous function.

Definition 4.1. We mean a solution of an initial-boundary-value problem for any $u_t(x,t) = u_{xx}(x,t) + F(x,t,u,u_x)$ in $\mathbb{R} \times I$, where $I = [0, T]$. A function $u = u(x, t)$ defined in $\mathbb{R} \times I$ such that

(a) $u \in C(\mathbb{R} \times I)$,
(b) $u_t,\ u_x,\ u_{xx} \in C(\mathbb{R} \times I)$,
(c) u_t and u_x are bounded in $\mathbb{R} \times I$,
(d) $u_t(x, t) = u_{xx}(x, t) + F(x, t, u(x,t), u_x(x, t)), \quad \forall\ (x,t) \in \mathbb{R} \times I.$

Now we consider the space $\Omega = \{v\left(x,\, t\right) : v, v_x \in C\left(\mathbb{R}\times I\right) \text{ and } \|v\| < \infty\}$, where

$$\|v\| = \sup_{x\in\mathbb{R},\ t\in I} |v\left(x,t\right)| + \sup_{x\in\mathbb{R},\ t\in I} |v_x\left(x,t\right)| .$$

The set Ω with the norm $\|\cdot\|$ is a Banach space. Obviously, the space with the $G-$metric given by

$$\begin{aligned} G(u,v,w) &= \sup_{x\in\mathbb{R},\ t\in I} |u\left(x,t\right)-v\left(x,t\right)| + \sup_{x\in\mathbb{R},\ t\in I} |u_x\left(x,t\right)-v_x\left(x,t\right)| \\ &\quad + \sup_{x\in\mathbb{R},\ t\in I} |v\left(x,t\right)-w\left(x,t\right)| + \sup_{x\in\mathbb{R},\ t\in I} |v_x\left(x,t\right)-w_x\left(x,t\right)| \\ &\quad + \sup_{x\in\mathbb{R},\ t\in I} |u\left(x,t\right)-w\left(x,t\right)| + \sup_{x\in\mathbb{R},\ t\in I} |u_x\left(x,t\right)-w_x\left(x,t\right)| \end{aligned}$$

is a complete $G-$metric space. The set Ω can also equipped with the a partial order given by

$$u,v\in\Omega, \quad u \preceq v \iff u\left(x,t\right) \le v\left(x,t\right), \quad u_x\left(x,t\right) \le v_x\left(x,t\right)$$

for any $x\in\mathbb{R}$ and $t\in I$. Obviously, $(\Omega,\preceq)$ satisfies the condition (ii), since for any $u,v\in\Omega$, $\max\{u,v\}$ and $\min\{u,v\}$ are the least and greatest lower bounds of u and v, respectively. Taking a monotone nondecreasing sequence $\{v_n\}\subseteq\Omega$ converging to v in Ω, for any $x\in\mathbb{R}$ and $t\in I$,

$$v_1\left(x,t\right) \le v_2\left(x,t\right) \le \cdots \le v_n\left(x,t\right) \le \cdots$$

and

$$v_{1x}\left(x,t\right) \le v_{2x}\left(x,t\right) \le \cdots \le v_{nx}\left(x,t\right) \le \cdots .$$

Further, since the sequences $\{v_n\left(x,t\right)\}$ and $\{v_{nx}\left(x,t\right)\}$ of real numbers converge to $v\left(x,t\right)$ and $v_x\left(x,t\right)$, respectively, it follows that, for all $x\in\mathbb{R}$, $t\in I$ and $n\ge 1$, $v_n\left(x,t\right)\le v\left(x,t\right)$ and $v_{nx}\left(x,t\right)\le v_x\left(x,t\right)$. Therefore, $v_n\le v$ for all $n\ge 1$ and so $(\Omega,\preceq)$ with the above mentioned metric satisfies the condition (I).

Definition 4.2. A lower solution of the initial-value problem (4.1) is a function $u\in\Omega$,

$$\begin{cases} u_t\left(x,t\right) = u_{xx}\left(x,t\right) + F\left(x,t,u,u_x\right), \ -\infty < x < \infty,\, 0 < t < T, \\ \\ u\left(x,t\right) = \varphi\left(x\right), -\infty < x < \infty, \end{cases}$$

where we assume that φ is continuously differentiable and that φ and φ' are bounded, the set Ω is defined in above and $F\left(x,t,u,u_x\right)$ is a continuous function. This section is inspired in [14, 20, 21].

Theorem 4.3. *Consider the problem* (4.1) *with* $F:\mathbb{R}\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *continuous and assume the following:*

(1) *for any* $c > 0$ *with* $|s| < c$ *and* $|p| < c$, *the function* $F(x,t,s,p)$ *is uniformly Holder continuous in* x *and* t *for each compact subset of* $\mathbb{R} \times I$;

(2) *there exists a constant* $c_F \leq \frac{1}{3}(T + 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}})^{-1}$ *such that*

$$0 \leq F(x,t,s_2,p_2) - F(x,t,s_1,p_1) \leq c_F \ln(s_2 - s_1 + p_2 - p_1 + 1)$$

for all (s_1,p_1) *and* (s_2,p_2) *in* $\mathbb{R} \times \mathbb{R}$ *with* $s_1 \leq s_2$ *and* $p_1 \leq p_2$;

(3) F *is bounded for bounded* s *and* p.

Then the existence of a lower solution for the initial-value problem (4.1) *provides the existence of the unique solution of the problem* (4.1).

Proof. The problem (4.1) is equivalent to the integral equation

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(x-\xi,t-\tau) F\left(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)\right) d\xi d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial-value (4.1) possesses a unique solution if and only if the above integral differential equation possesses a unique solution u such that u and u_x are continuous and bounded for all $x \in \mathbb{R}$ and $0 < t \leq T$. Define a mapping $f : \Omega \to \Omega$ by

$$\begin{aligned} (fu)(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau) F\left(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)\right) d\xi d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in I$. Note that, if $u \in \Omega$ is a fixed point of f, then u is a solution of the problem (4.1). Now, we show that the hypothesis in Theorems 3.5 and 3.6 are satisfied. The mapping f is nondecreasing since, by hypothesis, for $u \geq v$,

$$F\left(x,t,u(x,t),u_x(x,t)\right) \geq F\left(x,t,v(x,t),v_x(x,t)\right).$$

By using that $k(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,T]$, we conclude that

$$\begin{aligned}
(fu)(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))\,d\xi d\tau \\
&\geq \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))\,d\xi d\tau \\
&= (fv)(x,t)
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in I$. Besides, we have

$$\begin{aligned}
&|(fu)(x,t) - (fv)(x,t)| \\
&\leq \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)|F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)) \\
&\quad -F(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))|\,d\xi d\tau \\
&\leq \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)\cdot c_F \\
&\quad \times \ln(u(\xi,\tau)-v(\xi,\tau)+u_x(\xi,\tau)-v_x(\xi,\tau)+1)\,d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1) \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau) d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1)\,T. \qquad (4.2)
\end{aligned}$$

With the same way, we obtain

$$|(fv)(x,t) - (fw)(x,t)| \leq c_F \ln(G(u,v,w)+1)\,T \qquad (4.3)$$

and

$$|(fu)(x,t) - (fw)(x,t)| \leq c_F \ln(G(u,v,w)+1)\,T \qquad (4.4)$$

for all $u \geq v \geq w$. Similarly, we have

$$\begin{aligned}
&\left|\frac{\partial fu}{\partial x}(x,t) - \frac{\partial fu}{\partial x}(x,t)\right| \\
&\leq c_F \ln(G(u,v,w)+1) \int_0^t \int_{-\infty}^{+\infty} \left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right| d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1)\, 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}}, \qquad (4.5)
\end{aligned}$$

$$\begin{aligned}&\left|\frac{\partial fv}{\partial x}(x,t)-\frac{\partial fw}{\partial x}(x,t)\right|\\&\leq c_F\ln\left(G(u,v,w)+1\right)\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right|d\xi d\tau \qquad (4.6)\\&\leq c_F\ln\left(G(u,v,w)+1\right)2\pi^{\frac{-1}{2}}T^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\partial fu}{\partial x}(x,t)-\frac{\partial fw}{\partial x}(x,t)\right|\\&\leq c_F\ln\left(G(u,v,w)+1\right)\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right|d\xi d\tau \qquad (4.7)\\&\leq c_F\ln\left(G(u,v,w)+1\right)2\pi^{\frac{-1}{2}}T^{\frac{1}{2}}.\end{aligned}$$

Combining $(4.2)\,,(4.3)\,,(4.4)$ with $(4.5)\,,(4.6)$, (4.7), we obtain

$$G(fu,fv,fw)\leq 3c_F(T+2\pi^{\frac{-1}{2}}T^{\frac{1}{2}})\ln\left(G(u,v,w)+1\right)\leq\ln\left(G(u,v,w)+1\right)$$

which implies

$$\begin{aligned}\ln(G(fu,fv,fw)+1) &\leq \ln\left(\ln\left(G(u,v,w)+1\right)+1\right)\\ &= \frac{\ln\left(\ln\left(G(u,v,w)+1\right)+1\right)}{\ln\left(G(u,v,w)+1\right)}\ln\left(G(u,v,w)+1\right).\end{aligned}$$

Put $\psi\left(x\right)=\ln(x+1)$ and $\beta\left(x\right)=\frac{\psi(x)}{x}$. Obviously, $\psi:[0,\infty)\to[0,\infty)$ is continuous, sub-additive, nondecreasing and ψ is positive in $(0,\infty)$ with $\psi\left(0\right)=0$ and also $\psi\left(x\right)<x$ for any $x>0$ and $\beta\in S$. Finally, let $\alpha\left(x,t\right)$ be a lower solution for (4.1). Then we show that $\alpha\leq f\alpha$ integrating the following:

$$\begin{aligned}&\left(\alpha\left(\xi,\tau\right)k\left(x-\xi,t-\tau\right)\right)_\tau-\left(\alpha_\xi\left(\xi,\tau\right)k\left(x-\xi,t-\tau\right)\right)_\xi\\&\quad+\left(\alpha\left(\xi,\tau\right)k_\xi\left(x-\xi,t-\tau\right)\right)_\xi\\&\leq F\left(\xi,\tau,\alpha\left(\xi,\tau\right),\alpha_\xi\left(\xi,\tau\right)\right)k\left(x-\xi,t-\tau\right)\end{aligned}$$

for $-\infty<\xi<\infty$ and $0<\tau<t$. Then we obtain the following.

$$\begin{aligned}\alpha\left(x,t\right) &\leq \int_{-\infty}^{+\infty}k\left(x-\xi,t\right)\varphi\left(\xi\right)d\xi\\&\quad+\int_0^t\int_{-\infty}^{+\infty}k\left(x-\xi,t-\tau\right)F\left(\xi,\tau,\alpha\left(\xi,\tau\right),\alpha_\xi\left(\xi,\tau\right)\right)d\xi d\tau\\&= (f\alpha)(x,t)\end{aligned}$$

for all $x\in\mathbb{R}$ and $t\in(0,T]$. Therefore, by Theorems 3.5 and 3.6, f has a unique fixed point. This completes the proof. □

Acknowledgments: The authors acknowledge research support from the Qassim University, Grant 2630.

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