Progress of Theoretical Physics, Vol. 52, No. 2, August 1974

On the Extension of Inverse Scattering Method

Miki WADATI and Takashi KAMIJO

Institute for Optical Research, Kyoiku University
Hyakunintyo, Sinzyuku-ku, Tokyo

(Received February 14, 1974)

The inverse scattering method is extended into $n\times n$ matrix form so as to include wider classes of nonlinear differential equations. As a mathematical foundation for the extension, the inverse scattering problem for $n\times n$ Schrödinger equation is discussed and Gelfand-Levitan-Marchenko equation for the system is derived. Some of the applications of the theory are also discussed.

§ 1. Introduction

The inverse scattering method for solving the initial value problem of nonlinear differential equations was first proposed by Gardner, Greene, Kruskal and Miura.[1] In the past few years, this method was successfully applied to the Korteweg-de Vries equation,[2),3] nonlinear Schrödinger equation,[4] the Modified Korteweg-de Vries equation,[5] Sine-Gordon equation[6] and Reduced Maxwell Bloch equation for the system of two-level atoms.[7),8] As pointed out by Ablowitz, Kaup, Newell and Segur,[9] 2×2 matrix formalism which is based on the applicability of the inverse scattering problem for 2×2 Dirac equation[4] contains a wide class of nonlinear differential equations such as mentioned above.

The main aim of the present paper is to extend the idea of inverse scattering method into $n\times n$ matrix form so as to include wider classes of nonlinear differential equations. For the purpose, in § 2, we shall investigate the inverse scattering problem for the system of $n\times n$ Schrödinger equation on the entire axis $(-\infty<x<\infty)$, and obtain Gelfand-Levitan-Marchenko equation. For the case of semi-infinite interval $(0\leq x<\infty)$, this problem has been already studied by Newton and Jost,[10] and Agranovich and Marchenko.[11] In a sense, our work in § 2 is in parallel with Faddeev's version[12] of the original work by Gelfand and Levitan.[13] In § 3, we shall demonstrate some of the examples which may be solved by our extension of the inverse scattering method. The connection between our approach and the extension of Ablowitz et al.'s approach will be discussed in § 4.

Throughout the paper we shall use the following notations and definitions:

(1) The $n\times n$ matrix $U(x)$ whose (i,j) element is $u_{ij}(x)$ will be denoted by

$$U(x)=\|u_{ij}(x)\|_1^n\,.$$

(2) The absolute value of the matrix $B=\|b_{ij}\|$ will be defined as the non-negative

number

$$\max_i \sum_j |b_{ij}|$$

which we shall denote by $|B|$. The determinant of B will be denoted by $\det B$.

(3) A matrix is said to be continuous if all its elements are continuous functions. In the same sense, we shall refer to a matrix as being analytic, differentiable etc.

(4) I is the identity matrix, i.e.,

$$I=\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & \ddots 1\end{pmatrix}.$$

(5) B^* is the complex conjugate transpose of the matrix B. $\bar{k}$ is the complex conjugate of the quantity k.

(6) $W\{Y, Z\}$ is the Wronskian of the differentiable matrices $Y(x)$ and $Z(x)$, i.e.,

$$W\{Y, Z\} = Y(x)Z'(x) - Y'(x)Z(x),$$

where the differentiation with respect to x is denoted by a prime.

(7) The symbol $O\{\varphi(x)\}$ in matrix equations shall be used to denote the fact that all the elements of a matrix are $O\{\varphi(x)\}$.

§ 2. Inverse scattering problem

Consider a system of differential equations:

$$-y_i''(x, k) + \sum_{j=1}^{n} u_{ij}(x) y_j(x, k) = k^2 y_i, \quad -\infty<x<\infty. \quad (i=1, 2, \cdots, n) \tag{2·1}$$

We assume that the matrix of functions $u_{ij}(x)$,

$$U(x) = \|u_{ij}(x)\|_1^n \tag{2·2}$$

is Hermitian, i.e.,

$$U^*(x) = U(x). \tag{2·3}$$

Also we assume that the matrix $U(x)$ is continuous and satisfies the condition

$$\int_{-\infty}^{\infty} (1+|x|)|U(x)|dx < +\infty. \tag{2·4}$$

The system (2·1) is a $n\times n$ Schrödinger equation and we shall occasionally refer parameter k^2 and the matrix $U(x)$ as energy and the potential, respectively.

Every set of n solutions of the system (2·1) can be represented as a square matrix of n-th order $Y(x, k)$ satisfying the equation

$$-Y''(x, k) + U(x)Y(x, k) = k^2 Y(x, k), \quad -\infty<x<\infty. \tag{2·5}$$

The columns of every matrix solution of Eq. (2·5) are vector solutions of the system (2·1). Thus we shall study the matrix differential equation (2·5) instead of the system (2·1), and this proves to be more convenient.

(2-1) *Boundary conditions at infinity and Jost functions*

For real k, we introduce Jost functions $F_1(x,k)$ and $F_2(x,k)$ bound by the following boundary conditions at infinity:

$$\left.\begin{aligned} &\lim_{x\to\infty} F_1(x,k)e^{-ikx}=I\,, \\ &\lim_{x\to-\infty} F_2(x,k)e^{ikx}=I\,. \end{aligned}\right\} \tag{2·6}$$

Equation (2·5) with the boundary conditions (2·6) is equivalent to the integral equations:

$$F_1(x,k)=e^{ikx}I-\int_x^\infty \frac{\sin k(x-t)}{k}U(t)F_1(t,k)dt\,, \tag{2·7}$$

$$F_2(x,k)=e^{-ikx}I+\int_{-\infty}^x \frac{\sin k(x-t)}{k}U(t)F_2(t,k)dt\,. \tag{2·8}$$

These are equations of the Volterra type. Their successive approximation shows that we can extend the region of the parameter k into the upper half plane $\operatorname{Im} k\geq 0$ if the potential $U(x)$ satisfies the condition (2·4). Then the solutions $F_1(x,k)$ and $F_2(x,k)$ are analytic in the upper half plane $\operatorname{Im} k\geq 0$.

Let $Y(x)$ and $Z(x)$ be any two solutions of Eq. (2·5). Since the potential $U(x)$ is Hermitian, we have

$$-Y^{*\prime\prime}(x)+Y^*(x)U(x)=k^2Y^*(x)\,,$$
$$-Z''(x)+U(x)Z(x)=k^2Z(x)\,,$$

for $\operatorname{Im} k^2=0$. Hence,

$$Y^*(x)Z''(x)-Y^{*\prime\prime}(x)Z(x)=0$$

which implies that the Wronskian

$$W\{Y^*(x),Z(x)\}=Y^*(x)Z'(x)-Y^{*\prime}(x)Z(x)$$

is independent of x. Using this result and the boundary conditions (2·6), we obtain

$$\left.\begin{aligned} &W\{F_1^*(x,\bar{k}),\ F_1(x,k)\}=2ikI\,, \\ &W\{F_2^*(x,\bar{k}),\ F_2(x,k)\}=-2ikI\,, \\ &W\{F_1^*(x,\bar{k}),\ F_1(x,-k)\}=0\,, \\ &W\{F_2^*(x,\bar{k}),\ F_2(x,-k)\}=0\,, \end{aligned}\right\} \tag{2·9}$$

for $\operatorname{Im} k^2=0$.

(2-2) *A fundamental system of solutions*

For real $k(k\neq0)$, the pairs $F_1(x,k)$, $F_1(x,-k)$ and $F_2(x,k)$, $F_2(x,-k)$ are the fundamental systems of solutions of Eq. (2·5). Any solution of Eq. (2·5) can be represented as a linear combination of the solutions $F_1(x,k)$ and $F_1(x,-k)$ or $F_2(x,k)$ and $F_2(x,-k)$. Specifically, we have

$$F_2(x,k)=F_1(x,k)C_{11}(k)+F_1(x,-k)C_{12}(k), \qquad (2\cdot10)$$

$$F_1(x,k)=F_2(x,k)C_{22}(k)+F_2(x,-k)C_{21}(k). \qquad (2\cdot11)$$

Substituting expression (2·10) for $F_2(x,k)$ into (2·11) and carrying the same operation with $F_1(x,k)$, we find that compatibility of relations (2·10) and (2·11) requires fulfillment of the relations

$$\left.\begin{aligned} C_{11}(k)C_{22}(k)+C_{12}(-k)C_{21}(k)&=I,\\ C_{12}(k)C_{22}(k)+C_{11}(-k)C_{21}(k)&=0,\\ C_{22}(k)C_{11}(k)+C_{21}(-k)C_{12}(k)&=I,\\ C_{21}(k)C_{11}(k)+C_{22}(-k)C_{12}(k)&=0.\end{aligned}\right\} \qquad (2\cdot12)$$

The coefficient matrices $C_{ij}(k)$, $i,\ j=1,\ 2$ can be expressed in terms of the Wronskians of the solutions $F_1(x,k)$, $F_2(x,k)$ and their complex conjugate transposes. Using Eqs. (2·9), (2·10) and (2·11), we obtain

$$C_{11}(k)=\frac{1}{2ik}W\{F_1^*(x,k),\ F_2(x,k)\}, \qquad (2\cdot13\text{a})$$

$$C_{22}(k)=-\frac{1}{2ik}W\{F_2^*(x,k),\ F_1(x,k)\}, \qquad (2\cdot13\text{b})$$

$$C_{12}(k)=-\frac{1}{2ik}W\{F_1^*(x,-k),\ F_2(x,k)\}, \qquad (2\cdot13\text{c})$$

$$C_{21}(k)=\frac{1}{2ik}W\{F_2^*(x,-k),\ F_1(x,k)\}, \qquad (2\cdot13\text{d})$$

for real k. Comparing (2·13a) with (2·13b) and (2·13c) with (2·13d), we find that

$$\left.\begin{aligned} C_{11}(k)&=-C_{22}^*(k),\\ C_{12}(k)&=C_{21}^*(-k).\end{aligned}\right\} \qquad (2\cdot14)$$

Lemma 2·1: For real k, $\det C_{12}(k)\neq0$.
Proof: From Eqs. (2·12) and (2·14), we have

$$-C_{11}^*(k)C_{11}(k)+C_{12}^*(k)C_{12}(k)=I. \qquad (2\cdot15)$$

By multiplying (2·15) on the right by a vector $\boldsymbol{a}$, on the left by $\boldsymbol{a}^*$, we obtain

$$-\boldsymbol{a}^*C_{11}^*(k)C_{11}(k)\boldsymbol{a}+\boldsymbol{a}^*C_{12}^*(k)C_{12}(k)\boldsymbol{a}=\boldsymbol{a}^*\boldsymbol{a}. \qquad (2\cdot16)$$

If $C_{12}(k)\boldsymbol{a}=0$, the left-hand side and the right-hand side of Eq. (2·16) are non-positive and non-negative, respectively. Then $\boldsymbol{a}=0$, and lemma is proved.
(Q.E.D.)

For complex $k(\mathrm{Im}\,k\geq 0)$, we define $C_{ij}(k)$ as follows:

$$C_{11}(k)=\frac{1}{2ik}W\{F_1^*(x,\bar{k}),\ F_2(x,k)\}, \tag{2·17a}$$

$$C_{22}(k)=-\frac{1}{2ik}W\{F_2^*(x,\bar{k}),\ F_1(x,k)\}, \tag{2·17b}$$

$$C_{12}(k)=-\frac{1}{2ik}W\{F_1^*(x,-\bar{k}),\ F_2(x,k)\}, \tag{2·17c}$$

$$C_{21}(k)=\frac{1}{2ik}W\{F_2^*(x,-\bar{k}),\ F_1(x,k)\}, \tag{2·17d}$$

which reduce to Eqs. (2·13) on the real axis. Since $F_1(x,k)$, $F_2(x,k)$, $F_1^*(x,-\bar{k})$ and $F_2^*(x,-\bar{k})$ are analytic in the upper half plane, $C_{12}(k)$ and $C_{21}(k)$ are analytic in the upper half plane. Comparing (2·17a) with (2·17b) and (2·17c) with (2·17d), we find that

$$\left.\begin{aligned}C_{11}(k)&=-C_{22}^*(\bar{k}),\\ C_{12}(k)&=C_{21}^*(-\bar{k}),\end{aligned}\right\} \tag{2·18}$$

for complex $k(\mathrm{Im}\,k\geq 0)$.

Lemma 2·2: For $|k|\to\infty$

$$\left.\begin{aligned}&C_{11}(k)=O\left(\frac{1}{|k|}\right),\quad C_{22}(k)=O\left(\frac{1}{|k|}\right),\\ &C_{12}(k)=I+O\left(\frac{1}{|k|}\right),\\ &C_{21}(k)=I+O\left(\frac{1}{|k|}\right).\end{aligned}\right\} \tag{2·19}$$

Proof: Analysis of the successive approximations of Eqs. (2·7) and (2·8) leads to the estimates

$$\left.\begin{aligned}F_1(x,k)&=e^{ikx}I+O\left(\frac{1}{|k|}\right),\\ F_2(x,k)&=e^{-ikx}I+O\left(\frac{1}{|k|}\right),\end{aligned}\right\} \tag{2·20}$$

for $|k|\to\infty$ and a fixed x. Substitution of Eqs. (2·20) into Eqs. (2·17) gives the estimates (2·19).
(Q.E.D)

(2-3) *Bound state*

The bound states are determined from the condition:

$$\det C_{12}(k_j)=0\,, \tag{2·21}$$

as we shall see the reason in the following. If for a given k_j, the condition (2·21) is satisfied, then there exists a non-zero vector $\boldsymbol{a}$ so that $C_{12}(k_j)\boldsymbol{a}=0$. In the case, we can form a solution of basic system (2·1) such that

$$\left.\begin{aligned} \boldsymbol{y}(x,k_j)&=F_2(x,k_j)\boldsymbol{a}\\ &=F_1(x,k_j)C_{11}(k_j)\boldsymbol{a}\,. \end{aligned}\right\} \tag{2·22}$$

From the boundary conditions (2·6), we have

$$\boldsymbol{y}(x,k_j)\to e^{-ik_jx}\boldsymbol{a} \qquad \text{for } x\to-\infty\,, \tag{2·23}$$

$$\to e^{ik_jx}C_{11}(k_j)\boldsymbol{a} \qquad \text{for } x\to\infty\,. \tag{2·24}$$

Since a vector solution $\boldsymbol{y}(x,k_j)$ vanishes exponentially at $x=\pm\infty$ for $\mathrm{Im}\,k_j>0$, we conclude that for $k=k_j(\mathrm{Im}\,k_j>0)$, system (2·1) has a solution which is quadratically integrable on the entire axis. It is shown in the standard fashion that the energy eigenvalue k_j^2 is real, which implies

$$k_j=i\kappa_j,\quad \kappa_j;\ \text{real positive constant.}$$

The bound states therefore correspond exactly to the points in $\mathrm{Im}\,k>0$, $\mathrm{Re}\,k=0$, where $\det C_{12}(k)=0$ and where, therefore $C_{12}^{-1}(k)$ has a pole.

By virtue of Lemma 2·1, $\det C_{12}(k)$ does not vanish on the real axis. Further, asymptotic form (2·19) indicates that $\det C_{12}(k)$ differs from zero for sufficiently large $|k|$. Hence $\det C_{12}(k)$ can have only a finite number of zeros, which implies that a number of bound states are finite.

In order to investigate the property of the singular points of $C_{12}^{-1}(k)$, we shall need the following lemmas.

Lemma 2·3: Let $A(z)$ be a square matrix, which is analytic for $|z|<1$, such that $\det A(0)=0$ and $\det A(z)\neq 0$ for $0<|z|<1$. Then the inverse matrix $A^{-1}(z)$ has a simple pole at $z=0$ if and only if the relations

$$\left.\begin{aligned} &A(0)\boldsymbol{a}=0\,,\\ &A(0)\boldsymbol{b}+A'(0)\boldsymbol{a}=0\,, \end{aligned}\right\} \tag{2·25}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors, imply that $\boldsymbol{a}=0$.

Proof: See Newton and Jost,[10] or Agranovich and Marchenko.[11]

Lemma 2·4: For $k_j=i\kappa_j$ (κ_j; real positive constant),

$$C_{12}^*(k_j)C_{11}(k_j)-C_{11}^*(k_j)C_{12}(k_j)=0\,. \tag{2·26}$$

Proof: From the definition of $C_{ij}(k)$, Eq. (2·17), it can be shown that $C_{ij}(k_j)$

is independent of x for $k_j = i\kappa_j$ (κ_j; real positive constant). Therefore the left-hand side of Eq. (2·26) is independent of x. Using the asymptotic form of $F_1(x, k_j)$,

$$F_1(x, k_j) \to e^{ik_jx}I, \quad x \to \infty\,, \tag{2·27}$$

we obtain

$$\begin{aligned} &C_{12}^*(k_j)C_{11}(k_j) - C_{11}^*(k_j)C_{12}(k_j) \\ &= \frac{1}{4k_j^2}[\{F_2^{*\prime}(x, k_j) - ik_jF_2^*(x, k_j)\}\{F_2'(x, k_j) + ik_jF_2(x, k_j)\} \\ &\quad - \{F_2^{*\prime}(x, k_j) + ik_jF_2^*(x, k_j)\}\{F_2'(x, k_j) - ik_jF_2(x, k_j)\}] \\ &= \frac{1}{2ik_j}W\{F_2^*(x, k_j),\ F_2(x, k_j)\}, \quad x \to \infty, \end{aligned} \tag{2·28}$$

which reduces to zero because of the fourth equation of (2·9). (Q.E.D.)

Theorem 2·1: All the singularities of the matrix $C_{12}^{-1}(k)$ in the upper half plane $\mathrm{Im}\, k > 0$ are simple poles.

Proof: We begin by considering the differential equation

$$-F_2''(x, k) + U(x)F_2(x, k) = k^2F_2(x, k)\,. \tag{2·29}$$

Assuming that $\mathrm{Re}\, k = 0$ and $\mathrm{Im}\, k > 0$, we have

$$-\dot{F}_2''(x, k) + U(x)\dot{F}_2(x, k) = k^2\dot{F}_2(x, k) + 2kF_2(x, k)\,, \tag{2·30}$$

$$-F_2^{*\prime\prime}(x, k) + F_2^*(x, k)U(x) = k^2F_2^*(x, k)\,, \tag{2·31}$$

where dot indicates the differentiation with respect to k. From Eqs. (2·30) and (2·31), we obtain

$$\frac{d}{dx}W\{F_2^*(x, k),\ \dot{F}_2(x, k)\} = -2kF_2^*(x, k)F_2(x, k)\,. \tag{2·32}$$

Now let k coincide with one of the zeros of the det $C_{12}(k)$, which we again denote by k_j. We choose a non-zero vector such that

$$C_{12}(k_j)\boldsymbol{a} = 0\,. \tag{2·33}$$

Multiplying Eq. (2·32) on the left by $\boldsymbol{a}^*$ and on the right by $\boldsymbol{a}$, we obtain

$$\begin{aligned} &\frac{d}{dx}W\{\boldsymbol{a}^*F_2^*(x, k_j),\ \dot{F}_2(x, k_j)\boldsymbol{a}\} \\ &= -2k_j\boldsymbol{a}^*F_2^*(x, k_j)F_2(x, k_j)\boldsymbol{a}\,, \end{aligned} \tag{2·34}$$

and then

$$W\{\boldsymbol{a}^*F_2^*(x, k_j),\ \dot{F}_2(x, k_j)\boldsymbol{a}\}\Big|_{-\infty}^{\infty}$$

$$= -2k_j \int_{-\infty}^{\infty} [F_2(x, k_j)\boldsymbol{a}]^* [F_2(x, k_j)\boldsymbol{a}] dx . \tag{2·35}$$

Since the boundary conditions are as follows:

$$\left.\begin{aligned} F_2(x, k_j)\boldsymbol{a} &\to e^{-ik_jx}\boldsymbol{a} && \text{for } x\to -\infty , \\ &\to e^{ik_jx}C_{11}(k_j)\boldsymbol{a} && \text{for } x\to\infty , \end{aligned}\right\} \tag{2·36}$$

$$\left.\begin{aligned} \dot{F}_2(x, k_j)\boldsymbol{a} &\to -ixe^{-ik_jx}\boldsymbol{a} && \text{for } x\to -\infty , \\ &\to e^{-ik_jx}\dot{C}_{12}(k_j)\boldsymbol{a} && \text{for } x\to\infty , \end{aligned}\right\} \tag{2·37}$$

we finally arrive at

$$\boldsymbol{a}^* C_{11}^*(k_j)\dot{C}_{12}(k_j)\boldsymbol{a} = -i \int_{-\infty}^{\infty} [F_2(x, k_j)\boldsymbol{a}]^* [F_2(x, k_j)\boldsymbol{a}] dx$$
$$\neq 0 . \tag{2·38}$$

Let us now assume that the vector $\boldsymbol{a}$ satisfies not only the condition (2·33) but also the equation

$$C_{12}(k_j)\boldsymbol{b} + \dot{C}_{12}(k_j)\boldsymbol{a} = 0 , \tag{2·39}$$

where $\boldsymbol{b}$ is some other vector.

Multiplying Eq. (2·39) on the left by $[C_{11}(k_j)\boldsymbol{a}]^*$, we obtain

$$\boldsymbol{a}^* C_{11}^*(k_j) C_{12}(k_j)\boldsymbol{b} + \boldsymbol{a}^* C_{11}^*(k_j)\dot{C}_{12}(k_j)\boldsymbol{a} = 0 .$$

But by Lemma 2·4 and Eq. (2·33),

$$\boldsymbol{a}^* C_{11}^*(k_j) C_{12}(k_j)\boldsymbol{b}$$
$$= \boldsymbol{a}^* C_{12}^*(k_j) C_{11}(k_j)\boldsymbol{b} = 0 ,$$

and therefore

$$\boldsymbol{a}^* C_{11}^*(k_j)\dot{C}_{12}(k_j)\boldsymbol{a} = 0 .$$

This is a contradiction to Eq. (2·38).

This shows that no non-zero vector $\boldsymbol{a}$ exist satisfying conditions (2·33) and (2·39) simultaneously. Then by Lemma 2·3, $C_{12}^{-1}(k)$ has a simple pole at k_j.

(Q.E.D.)

(2-4) *Gelfand-Levitan-Marchenko equation*

We represent the Jost function $F_1(x, k)$ in the following form:

$$F_1(x, k) = e^{ikx}I + \int_x^{\infty} K_1(x, y) e^{iky} dy . \tag{2·40}$$

Substitution of the expression (2·40) for $F_1(x, k)$ into Eq. (2·7) yields

$$e^{ikx}I + \int_x^{\infty} K_1(x, y) e^{iky} dy$$

$$=e^{ikx}I+\int_x^\infty\Big(-\frac{\sin k(x-s)}{k}\Big)U(s)\Big[e^{iks}I+\int_s^\infty K_1(s,t)e^{ikt}dt\Big]ds\,. \quad (2\cdot41)$$

We now transform the integrals on the right-hand side of Eq. (2·41) first using the formulas

$$\frac{\sin k(s-x)}{k}e^{iks}=\frac{1}{2}\int_x^{2s-x}e^{iky}dy\,,$$

$$\frac{\sin k(s-x)}{k}e^{ikt}=\frac{1}{2}\int_{t+x-s}^{t+s-x}e^{iky}dy\,,$$

and then interchanging the orders of integration. This yields

$$\int_x^\infty K_1(x,y)e^{iky}dy$$

$$=\frac{1}{2}\int_x^\infty U(s)ds\int_x^{2s-x}e^{iky}dy+\frac{1}{2}\int_x^\infty U(s)ds\int_s^\infty K_1(s,t)dt\int_{t+x-s}^{t+s-x}e^{iky}dy$$

$$=\int_x^\infty e^{iky}\Big[\frac{1}{2}\int_{(x+y)/2}^\infty U(s)ds+\frac{1}{2}\int_{(x+y)/2}^\infty U(s)ds\int_s^{y+s-x}K_1(s,t)dt$$

$$+\frac{1}{2}\int_x^{(x+y)/2}U(s)ds\int_{x+y-s}^{y+s-x}K_1(s,t)dt\Big]dy\,. \quad (2\cdot42)$$

Making use of the uniqueness of the Fourier integral representation, we obtain the following integral equation for the matrix $K_1(x,y)$:

$$K_1(x,y)=\frac{1}{2}\int_{(x+y)/2}^\infty U(s)ds+\frac{1}{2}\int_x^{(x+y)/2}U(s)ds\int_{y+x-s}^{y+s-x}K_1(s,t)dt$$

$$+\frac{1}{2}\int_{(x+y)/2}^\infty U(s)ds\int_s^{y+s-x}K_1(s,t)dt\,. \quad (-\infty<x\leq y<\infty) \quad (2\cdot43)$$

Then,

$$K_1(x,x)=\frac{1}{2}\int_x^\infty U(s)ds\,, \quad (2\cdot44)$$

$$\frac{d}{dx}K_1(x,x)=-\frac{1}{2}U(x)\,. \quad (2\cdot45)$$

On the basis of Lemma 2·1, the expression for $F_2(x,k)$ can be written in the form

$$F_2(x,k)C_{12}^{-1}(k)=F_1(x,-k)+F_1(x,k)R(k)\,, \quad (2\cdot46)$$

where

$$R(k)=C_{11}(k)[C_{12}(k)]^{-1}\,. \quad (2\cdot47)$$

From Eqs. (2·12) and (2·14), it easily follows that

$$R(k)=R^*(-k). \tag{2·48}$$

We call the matrix $R(k)$ the reflection coefficient matrix.

In the following, it will be shown that the matrix $K_1(x, y)$ satisfies a linear integral equation whose kernel can be written explicitly in terms of the reflection coefficient matrix $R(k)$, the eigenvalue k_j^2 and the matrix R_j which is the residue of the reflection coefficient matrix $R(k)$ at the pole k_j.

To derive this equation, we begin with Eq. (2·46). We replace $F_1(x, k)$ in Eq. (2·46) by its expression (2·40). As a result,

$$F_2(x, k)C_{12}^{-1}(k)$$
$$=e^{-ikx}I+\int_x^\infty K_1(x, t)e^{-ikt}dt+e^{ikx}R(k)+\int_x^\infty K_1(x, t)e^{ikt}dtR(k),$$

or

$$F_2(x, k)(C_{12}^{-1}(k)-I)+F_2(x, k)-e^{-ikx}I$$
$$=e^{ikx}R(k)+\int_x^\infty K_1(x, t)e^{ikt}dtR(k)+\int_x^\infty K_1(x,t)e^{-ikt}dt. \tag{2·49}$$

As shown in Theorem 2·1, the matrix $C_{12}^{-1}(k)$ may have a finite number of simple poles at $k=k_j(j=1, 2, \cdots, N;\ k_j=i\kappa_j,\ \kappa_j$: real positive constant). Then, around each pole k_j, the matrix $C_{12}^{-1}(k)$ has the form

$$C_{12}^{-1}(k)=(k-k_j)^{-1}N_j+\cdots, \tag{2·50}$$

where the matrix $N_j\neq 0$.

Multiplying $e^{iky}I(y>x)$ on the both sides of Eq. (2·49) and integrating with respect to k from $-\infty$ to ∞, we have

$$I_1+I_2=I_3+I_4+I_5, \tag{2·51}$$

where

$$I_1=\int_{-\infty}^\infty F_2(x, k)[C_{12}^{-1}(k)-I]e^{iky}dk, \tag{2·52}$$

$$I_2=\int_{-\infty}^\infty [F_2(x, k)-e^{-ikx}I]e^{iky}dk, \tag{2·53}$$

$$I_3=\int_{-\infty}^\infty R(k)e^{ik(x+y)}dk, \tag{2·54}$$

$$I_4=\int_{-\infty}^\infty dk\int_x^\infty dtK_1(x, t)e^{ik(t+y)}R(k) \tag{2·55}$$

and

$$I_5=\int_{-\infty}^\infty dk\int_x^\infty K_1(x, t)e^{ik(y-t)}dt. \tag{2·56}$$

In evaluating I_1 and I_2 we can close integral contour in the upper half plane, since

$$F_2(x,k)\to e^{-ikx}I \qquad \text{as} \quad |k|\to\infty\,,$$

$$C_{12}^{-1}(k)\to I \qquad \text{as} \quad |k|\to\infty\,.$$

The integrand in I_2 is analytic in the upper half plane, then

$$I_2=0\,. \tag{2·57}$$

The integrand in I_1 is analytic in the upper half plane except for a finite number of poles at k_j corresponding to the simple poles of $C_{12}^{-1}(k)$. Then,

$$\begin{aligned} I_1&=2\pi i\sum_{j=1}^{N}F_2(x,k_j)N_je^{ik_jy}\\ &=2\pi i\sum_{j=1}^{N}F_1(x,k_j)C_{11}(k_j)N_je^{ik_jy}\\ &=2\pi i\sum_{j=1}^{N}e^{ik_j(x+y)}R_j+2\pi i\sum_{j=1}^{N}\int_x^\infty K_1(x,t)\,e^{ik_j(t+y)}dtR_j\,, \end{aligned} \tag{2·58}$$

where

$$R_j=C_{11}(k_j)N_j\,. \tag{2·59}$$

The matrix R_j is the residue of the reflection coefficient matrix $R(k)$ at the pole k_j.

While we keep the original expressions for I_3 and I_4, we evaluate I_5 as follows:

$$\begin{aligned} I_5&=\int_x^\infty dtK_1(x,t)\int_{-\infty}^\infty dke^{ik(y-t)}\\ &=2\pi\int_x^\infty dtK_1(x,t)\,\delta(y-t)\\ &=2\pi K_1(x,y) \qquad \text{for } x<y\,. \end{aligned} \tag{2·60}$$

Summing up the results, we arrive at the expression for Eq. (2·51):

$$K_1(x,y)+F(x+y)+\int_x^\infty dtK_1(x,t)F(y+t)\,dt=0\,, \quad x\leq y\,, \tag{2·61}$$

where

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^\infty dkR(k)e^{ikx}-i\sum_{j=1}^{N}e^{ik_jx}R_j\,. \tag{2·62}$$

Equation (2·60), which was derived on the assumption that $x<y$, also holds for $x=y$ by continuity. We refer to Eq. (2·61) with Eq. (2·62) as Gelfand-Levitan-Marchenko equation. Gelfand-Levitan-Marchenko equation indicates that, when we know the reflection coefficient matrix $R(k)$, the eigenvalue k_j^2 and the matrix R_j which is the residue of the matrix $R(k)$ at the pole x_j, we can evaluate the matrix $K_1(x,y)$, and then construct the potential $U(x)$ by the relation (2·45).

In other words, only the asymptotic forms of the eigenfunctions are necessary to reconstruct the potential $U(x)$, since

$$\left.\begin{aligned}\varphi(x,k)&\equiv F_2(x,k)C_{12}^{-1}(k)\\&\to e^{-ikx}I+e^{ikx}R(k),\quad x\to\infty,\\&\to e^{-ikx}C_{12}^{-1}(k),\quad x\to-\infty,\end{aligned}\right\}\tag{2·63}$$

$$\left.\begin{aligned}\varphi_j(x,k_j)&\equiv F_2(x,k_j)N_j\\&=e^{ik_jx}R_j,\quad x\to\infty,\\&=e^{-ik_jx}N_j,\quad x\to-\infty.\end{aligned}\right\}\tag{2·64}$$

We call the aggregate of quantities

$$R(k);\ k_j^2, R_j \qquad (j=1,2,\cdots,N)$$

from which the kernel (i.e., the matrix $F(x)$) of Gelfand-Levitan-Marchenko equation (2·61) is constructed, the scattering data of the problem (2·1).

§ 3. Applications

The system of the differential equations (2·1) is rewritten in the form

$$L\psi=\lambda\psi,\quad \lambda=k^2,\tag{3·1}$$

where

$$L=-\partial^2 I+U(x),\tag{3·2}$$

$$\partial=\frac{\partial}{\partial x}\tag{3·3}$$

and

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\\\vdots\\\psi_n\end{pmatrix}.\tag{3·4}$$

In the preceding section, we have derived Gelfand-Levitan-Marchenko equation on the assumption of the Hermitian $U(x)$. However, if we assume that the pole of the matrix $C_{12}(k)$ is simple, we expect Gelfand-Levitan-Marchenko equation in a similar form also holds for the non-Hermitian potential $U(x)$.

We extend the inverse scattering method for solving nonlinear differential equations into the matrix form. Given a nonlinear evolution of the form

$$U_t=S[U],\tag{3·5}$$

where S is in general a nonlinear operator and the matrix function $U(x,t)$ is a square matrix of n-th order. We shall study a C^∞ solution defined for all x in $(-\infty,\infty)$ which tends to zero as $x\to\pm\infty$. Assuming the operator L is linear with respect to U(for example, Eq. (3·2)), we rewrite Eq. (3·5) in the operator form

$$L_t=[B,L]=BL-LB,\tag{3·6}$$

by choosing an appropriate operator B. Here the operator B is a square matrix of n-th order.

Instead of solving Eq. (3·6), we introduce eigenvalue problem

$$L\psi = \lambda\psi\,, \tag{3·7}$$

where the time evolution of the eigenfunction is given by

$$\psi_t = B\psi\,. \tag{3·8}$$

It is readily seen from the time evolution properties of L and ψ that the eigenvalue λ of Eq. (3·7) does not depend on time:

$$\lambda_t = 0\,. \tag{3·9}$$

Conversely, we can derive Eq. (3·6) from Eq. (3·7) with Eqs. (3·8) and (3·9). Then, a non-linear equation (3·5), which is equivalent to Eq. (3·6), is also equivalent to a set of Eqs. (3·7), (3·8) and (3·9). The study on the inverse scattering problem indicates that the sought matrix $U(x, t)$ can be reconstructed only from the knowledge of the scattering data. The time dependence of the scattering data is determined by Eq. (3·8) at the limit $x \to \pm\infty$. This procedure for finding a solution of the system (3·5) is the extension of the so-called inverse scattering method. It is remarked that, if the operator L is Hermitian, the operator B is antisymmetric, i.e., $B^* = -B$.[14]

By choosing the operator L in the form of Eq. (3·2), we shall demonstrate some of the examples which may be solved by our extension of the inverse scattering method.

(3-1) *Generalized Korteweg-de Vries equation in the matrix form*

The operator L is a $n\times n$ matrix in this case.

$$L = -\partial^2 I + U(x), \tag{3·10}$$

where

$$\partial \equiv \frac{\partial}{\partial x}\,.$$

The simplest choice may be

$$B = B_0\partial, \quad B_0\text{: constant matrix.} \tag{3·11}$$

Since

$$BL - LB = (B_0U - UB_0)\partial + B_0U_x\,, \tag{3·12}$$

Eq. (3·6) yields

$$U_t = B_0U_x \tag{3·13}$$

with the condition

$$B_0U = UB_0\,. \tag{3·14}$$

Equation (3·13) is a linear differential equation and this is a trivial case.

Next we try the operator B such that

$$B=B_2\partial^3+B_1\partial+\partial B_1,\quad B_2\text{: constant matrix.} \tag{3·15}$$

Since

$$\begin{aligned}BL-LB=&(B_2U-UB_2)\partial^3+(3B_2U_x+4B_{1x})\partial^2\\&+(3B_2U_{xx}+4B_{1xx}+2B_1U-2UB_1)\partial\\&+(B_2U_{xxx}+2B_1U_x+B_{1x}U-UB_{1x}+B_{1xxx}),\end{aligned} \tag{3·16}$$

Eq. (3·6) yields

$$U_t=\tfrac{1}{4}B_2U_{xxx}-\tfrac{3}{4}B_2(U^2)_x \tag{3·17}$$

with the conditions:

$$\left.\begin{aligned}&B_iU=UB_i,\quad (i=1,2)\\&3B_2U=-4B_1.\end{aligned}\right\} \tag{3·18}$$

Equation (3·17) is the Korteweg-de Vries equation in the matrix form.

Further, if we choose the operator B in the following form:

$$B=B_3\partial^5+B_2\partial^3+\partial^3B_2+B_1\partial+\partial B_1,\quad B_3\text{: constant matrix,} \tag{3·19}$$

Eq. (3·6) yields

$$U_t=\frac{1}{16}B_3(U_{xxxxx}-10UU_{xxx}-15U_xU_{xx}-5U_{xx}U_x+15U^2U_x+15U_xU^2) \tag{3·20}$$

with the conditions:

$$\left.\begin{aligned}&B_iU=UB_i,\quad (i=1,2,3)\\&B_2=-\frac{5}{4}B_3U,\\&B_1=\frac{5}{16}B_3(U_{xx}+3U^2).\end{aligned}\right\} \tag{3·21}$$

(3-2) 2×2 *matrix formalism*

Here we show that 2×2 matrix formalism contains the Modified Korteweg-de Vries equation, Nonlinear Schrödinger equation, Sine-Gordon equation and Reduced Maxwell Bloch equation for the system of two-level atoms.

(a) The Modified Korteweg-de Vries equation:

$$u_t+6u^2u_x+u_{xxx}=0. \tag{3·22}$$

If we choose

$$L=-\begin{pmatrix}1&0\\0&1\end{pmatrix}\partial^2+U, \tag{3·23}$$

$$B=-4I\partial^3+3U\partial+3\partial U, \tag{3·24}$$

where

$$U=\begin{pmatrix}-u^2 & iu_x \\ iu_x & -u^2\end{pmatrix}, \tag{3·25}$$

substitution of Eqs. (3·23), (3·24) and Eq. (3·25) into Eq. (3·6) gives Eq. (3·22). This case turns out to be a special case of Eq. (3·17).
(b) Nonlinear Schrödinger equation:

$$iu_t+u_{xx}+2\bar{u}u^2=0. \tag{3·26}$$

If we choose

$$L=-\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\partial^2+\begin{pmatrix}-u\bar{u} & iu_x \\ i\bar{u}_x & -u\bar{u}\end{pmatrix}, \tag{3·27}$$

$$B=2i\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\partial^2+2\begin{pmatrix}0 & u \\ -\bar{u} & 0\end{pmatrix}\partial+\begin{pmatrix}iu\bar{u} & u_x \\ -\bar{u}_x & -iu\bar{u}\end{pmatrix}, \tag{3·28}$$

substitution of Eq. (3·27) and Eq. (3·28) into Eq. (3·6) gives Eq. (3·26).
(c) Sine-Gordon equation:

$$u_{XX}-u_{TT}=\sin u, \tag{3·29}$$

or

$$u_{xt}=\sin u. \tag{3·30}$$

If we choose

$$L=-\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\partial^2+\begin{pmatrix}-\frac{1}{4}u_x^2 & -\frac{1}{2}u_{xx} \\ \frac{1}{2}u_{xx} & -\frac{1}{4}u_x^2\end{pmatrix}, \tag{3·31}$$

$$B=\frac{i}{4}\begin{pmatrix}\cos u & \sin u \\ \sin u & -\cos u\end{pmatrix}L^{-1/2}, \tag{3·32}$$

where

$$L^{1/2}=i\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\partial+\begin{pmatrix}0 & \frac{1}{2}iu_x \\ \frac{1}{2}iu_x & 0\end{pmatrix}, \tag{3·33}$$

substitution of Eqs. (3·31), (3·32) and (3·33) into Eq. (3·6) gives Eq. (3·30).
(d) Reduced Maxwell Bloch equation for the system of two-level atoms:

$$\left.\begin{aligned} &E_t(x,t)=s(x,t), \\ &s_x(x,t)=E(x,t)u(x,t)+\mu r(x,t), \\ &r_x(x,t)=-\mu s(x,t), \\ &u_x(x,t)=-E(x,t)s(x,t), \end{aligned}\right\} \tag{3·34}$$

or

$$\left.\begin{aligned} E_{tx} &= Eu + \mu r\,, \\ r_x &= -\mu E_t\,, \\ u_x &= -EE_t\,. \end{aligned}\right\} \tag{3·35}$$

If we choose

$$L = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial^2 + \begin{pmatrix} -\frac{1}{4}E^2 & -\frac{1}{2}E_x \\ \frac{1}{2}E_x & -\frac{1}{4}E^2 \end{pmatrix}, \tag{3·36}$$

$$B = \begin{pmatrix} u & s \\ s & -u \end{pmatrix}\frac{iL^{1/2}}{4L-\mu^2} + \begin{pmatrix} 0 & -\frac{1}{2}r \\ \frac{1}{2}r & 0 \end{pmatrix}\frac{\mu}{4L-\mu^2}, \tag{3·37}$$

where

$$L^{1/2} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial + \begin{pmatrix} 0 & \frac{1}{2}iE \\ \frac{1}{2}iE & 0 \end{pmatrix}, \tag{3·38}$$

substitution of Eqs. (3·36), (3·37) and (3·38) into Eq. (3·6) gives Eq. (3·35).

Recently Ablowitz et al.[9] and Gibbon et al.[8] have shown that the four nonlinear equations mentioned above and the Korteweg-de Vries equation can be solved by the inverse scattering method for the system

$$\left.\begin{aligned} \frac{\partial\psi_1}{\partial x} + i\zeta\psi_1 &= q(x,t)\psi_2\,, \\ \frac{\partial\psi_2}{\partial x} - i\zeta\psi_2 &= r(x,t)\psi_1\,. \end{aligned}\right\} \tag{3·39}$$

Since the system (3·39) can be rewritten in the form

$$\left[-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial^2 + \begin{pmatrix} qr & q_x \\ r_x & rq \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \zeta^2\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3·40}$$

what we have shown here are rather natural results.

§ 4. Concluding remarks

In this paper we have extended the idea of the inverse scattering method into the matrix form so as to include wider classes of nonlinear differential equations. For the purpose we have derived Gelfand-Levitan-Marchenko equation, Eq. (2·61) with Eq. (2·62), for the system (2·1) and explicitly shown several nonlinear differential equations which may be solvable by our extension of the theory. In some of the examples, however, the potential $U(x, t)$ is not Hermitian. Since we have assumed the Hermitian potential in the discussions of § 2, we might have to re-examine the inverse scattering problem depending on the properties of the potentials.

One of our next task will be to investigate what kinds of nonlinear differential equations are included in our formalism. Here we only point out the connection between our approach and the extension of Ablowitz et al.'s approach. In the case of 2×2 matrix, the connection between Eq. (3·39) and Eq. (3·40) is straightforward. In the case of 3×3 matrix, the system

$$\left.\begin{aligned}&\frac{\partial\psi_1}{\partial x}+i\zeta\psi_1=q\psi_3\,,\\&\frac{\partial\psi_2}{\partial x}+i\zeta\psi_2=r\psi_3\,,\\&\frac{\partial\psi_3}{\partial x}-i\zeta\psi_3=s\psi_1+u\psi_2\end{aligned}\right\}\tag{4·1}$$

reduces to

$$\left[-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\partial^2+\begin{pmatrix}qs&qu&q_x\\rs&ru&r_x\\s_x&u_x&sq+ur\end{pmatrix}\right]\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}=\zeta^2\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}.\tag{4·2}$$

Also, in the case of 4×4 matrix, the system

$$\left.\begin{aligned}&\frac{\partial\psi_1}{\partial x}+i\zeta\psi_1=q\psi_3+r\psi_4\,,\\&\frac{\partial\psi_2}{\partial x}+i\zeta\psi_2=s\psi_3+u\psi_4\,,\\&\frac{\partial\psi_3}{\partial x}-i\zeta\psi_3=v\psi_1+w\psi_2\,,\\&\frac{\partial\psi_4}{\partial x}-i\zeta\psi_4=y\psi_1+z\psi_2\,,\end{aligned}\right\}\tag{4·3}$$

reduces to

$$\left[-\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\partial^2+U(x)\right]\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}=\zeta^2\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix},\tag{4·4}$$

where

$$U(x)=\begin{pmatrix}qv+ry&qw+rz&q_x&r_x\\sv+uy&sw+uz&s_x&u_x\\v_x&w_x&vq+ws&vr+wu\\y_x&z_x&yq+zs&yr+zu\end{pmatrix}.\tag{4·5}$$

Therefore, our formalism seems to include the extension of Ablowitz et al.'s approach.

Acknowledgements

The authors would like to thank Professor T. Kotera for valuable comments on this work and critical reading of the manuscript. They also would like to thank Dr. Y. Tagami and Dr. H. Sanuki for helpful discussions. One of the authors (M.W.) is partially supported by the Sakkokai Foundation.

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