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A generalization of k-Cohen–Macaulay simplicial complexes

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Abstract. For a positive integer k and a non-negative integer t, a class of simplicial complexes, to be denoted by k-CM_t, is introduced. This class generalizes two notions for simplicial complexes: being k-Cohen–Macaulay and k-Buchsbaum. In analogy with the Cohen–Macaulay and Buchsbaum complexes, we give some characterizations of CM_t $(=1-\mathrm{CM}_t)$ complexes, in terms of vanishing of some homologies of its links, and in terms of vanishing of some relative singular homologies of the geometric realization of the complex and its punctured space. We give a result on the behavior of the CM_t property under the operation of join of two simplicial complexes. We show that a complex is k-CM_t if and only if the links of its non-empty faces are k-CM_{t-1}. We prove that for an integer $s \leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM_t complex is $(k+s)$-CM_t. This result generalizes Hibi's result for Cohen–Macaulay complexes and Miyazaki's result for Buchsbaum complexes.

1. Introduction

Let K be a fixed field. The Stanley–Reisner ring of a simplicial complex over K provides a "bridge" to transfer properties in commutative algebra such as being Cohen–Macaulay or Buchsbaum into simplicial complexes. The main advantage in the study of simplicial complexes is the interplay between their algebraic, combinatorial, homological and topological properties. Stanley's book [17] is a suitable reference for a comprehensive introduction to the subject. The aim of this paper is to introduce and develop basic properties of a new class of simplicial complexes, called k-CM_t complexes, which generalizes two notions for simplicial complexes: being k-Cohen–Macaulay, and being k-Buchsbaum. Recall that a Cohen–Macaulay (resp. Buchsbaum) complex is k-Cohen–Macaulay (resp. k-Buchsbaum) if it retains

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its dimension and is still Cohen–Macaulay (resp. Buchsbaum) whenever $k-1$ or fewer vertices are removed.

In Section 2, we introduce CM_t complexes and discuss their basic properties. We show that for a pure simplicial complex Δ of dimension $d-1$ the following are equivalent (see Theorems 2.6 and 2.8):

(i) Δ is CM_t;

(ii) $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ for all $\sigma\in\Delta$ with $\#\sigma\geq t$ and $i<d-\#\sigma-1$;

(iii) $H_i(|\Delta|,|\Delta|\setminus p;K)=0$ for all $p\in|\Delta|\setminus|\Delta_{t-2}|$ and all $i<d-1$, where $|\Delta|$ is the geometric realization of Δ and Δ_{t-2} is the $(t-2)$-skeleton of Δ.

We also study the behavior of the CM_t property under the operation of join of two simplicial complexes. We prove that if Δ and Δ' are simplicial complexes of dimensions $d-1$ and $d'-1$, respectively, then $\Delta*\Delta'$ is CM_t if and only if Δ is $\mathrm{CM}_{t-d'}$ and Δ' is CM_{t-d}.

In Section 3, k-CM_t complexes are introduced and some of their basic properties are studied. We show that a complex is k-CM_t if and only if the links of its non-empty faces are k-CM_{t-1} (see Proposition 3.6). We consider a simplicial complex Δ and certain faces $\sigma_1,\ldots,\sigma_\ell$ of Δ such that

(i) $\sigma_i\cup\sigma_j\notin\Delta$ if $i\neq j$; and

(ii) if $\Delta_1=\{\tau\in\Delta\,|\,\tau\not\supseteq\sigma_i$ for all $i\}$ then $\dim\Delta_1<\dim\Delta$.

In [7] Hibi showed that Δ_1 is 2-Cohen–Macaulay of dimension $\dim\Delta-1$ provided that Δ is Cohen–Macaulay and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen–Macaulay for all i. In [11] Miyazaki extended this result for Buchsbaumness by showing that if Δ is a Buchsbaum complex of dimension $d-1$, and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen–Macaulay for all i, then Δ_1 is 2-Buchsbaum. We prove that a similar result is valid for CM_t complexes (see Theorem 3.8). This leads to a proof of the fact that for an integer $s\leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM_t complex is $(k+s)$-CM_t (see Corollary 3.10). This generalizes a result of Terai and Hibi [19] (also see [2]) which asserts that the 1-skeleton of a simplicial $(d-1)$-sphere with $d\geq2$ is d-connected. It also generalizes a result of Hibi [7] (see the introduction in [11]) which says that if Δ is a Cohen–Macaulay complex of dimension $d-1$, then the $(d-2)$-skeleton of Δ is 2-Cohen–Macaulay.

2. The CM_t simplicial complexes

In this section we introduce CM_t complexes and discuss their basic properties. We give some characterizations of CM_t complexes, in terms of vanishing of some homologies of its links (see Theorem 2.6), and, in terms of vanishing of some relative singular homologies of the geometric realization of the complex and its punctured space (see Theorem 2.8). We then study the behavior of the CM_t property under the operation of the join of two simplicial complexes (see Proposition 2.10).

First recall that for any face σ of the simplicial complex Δ, the link of σ is defined as follows:

$$\mathrm{lk}_\Delta(\sigma)=\{\tau\in\Delta \mid \tau\cup\sigma\in\Delta \text{ and } \tau\cap\sigma=\varnothing\}.$$

Definition 2.1. Let K be a field and let Δ be a simplicial complex of dimension $d-1$ over K. Let t be an integer, $0\le t\le d-1$. Then Δ is called CM_t *over* K if Δ is pure and $\mathrm{lk}_\Delta(\sigma)$ is Cohen–Macaulay over K for any $\sigma\in\Delta$ with $\#\sigma\ge t$.

We will adopt the convention that for $t\le 0$, CM_t means CM_0. Note that from the results by Reisner [14] and Schenzel [16] it follows that CM_0 is the same as Cohen–Macaulayness and CM_1 is identical with the Buchsbaum property. It is also clear that for any $j\ge i$, CM_i implies CM_j.

Example 2.2. Let Δ be the union of two $(d-1)$-simplices that intersect in a $(t-2)$-dimensional face, where $1\le t\le d-1$. Then Δ is a CM_t complex which is not a CM_{t-1} complex. In fact, if Γ is a finite union of $(d-1)$-simplices where any two of them intersect in a face of dimension at most $t-2$, then Γ is a CM_t complex, and if at least two of the simplices have a $(t-2)$-dimensional face in common, then Γ is not CM_{t-1}. These include simplicial complexes corresponding to the transversal monomial ideals [20].

Note that the condition $t<d-1$ is necessary because the union of two $(d-1)$-simplices which intersect in a $(d-2)$-dimensional face, is Cohen–Macaulay.

It is known that the links of a Cohen–Macaulay simplicial complex are also Cohen–Macaulay, see [9]. As the first result of this section we show that a similar property holds for CM_t complexes. In the rest of this paper we freely use the following fact:

$$\text{for all } \sigma\in\Delta \text{ and all } \tau\in\mathrm{lk}_\Delta(\sigma),\quad \mathrm{lk}_{\mathrm{lk}_\Delta(\sigma)}(\tau)=\mathrm{lk}_\Delta(\sigma\cup\tau).$$

Lemma 2.3. *Let Δ be a simplicial complex. Then the following are equivalent*:

(i) *Δ is a CM_t complex*;
(ii) *Δ is pure and $\mathrm{lk}_\Delta(\{x\})$ is CM_{t-1} for all $\{x\}\in\Delta$.*

Proof. (i)⇒(ii) Let $\{x\}\in\Delta$ and $\tau\in\mathrm{lk}_\Delta(\{x\})$ with $\#\tau\ge t-1$. Since Δ is CM_t and $\#(\{x\}\cup\tau)\ge t$ we see that $\mathrm{lk}_{\mathrm{lk}_\Delta(\{x\})}(\tau)=\mathrm{lk}_\Delta(\{x\}\cup\tau)$ is Cohen–Macaulay. In addition, since Δ is pure it follows that $\mathrm{lk}_\Delta(\{x\})$ is pure for all $x\in\Delta$.

(ii)⇒(i) Let $\sigma\in\Delta$ with $\#\sigma\geq t$. Let $x\in\sigma$ and $\tau=\sigma\setminus\{x\}$. Then $\#\tau\geq t-1$ and $\mathrm{lk}_\Delta\,\sigma=\mathrm{lk}_\Delta(\{x\}\cup\tau)=\mathrm{lk}_{\mathrm{lk}_\Delta(\{x\})}(\tau)$ is Cohen–Macaulay. □

Remark 2.4. In Lemma 2.3(ii), purity is not necessary. In fact it is sufficient to assume that all connected components of Δ have the same dimension (see [3, Lemma 3]). It may also be useful to mention that the condition (ii) for $t=1$ is usually called *locally Cohen–Macaulay* [8].

We recall Reisner's characterization of Cohen–Macaulay simplicial complexes [14, Theorem 1].

Theorem 2.5. *Let Δ be a simplicial complex of dimension $d-1$. Then the following are equivalent*:

(i) *Δ is Cohen–Macaulay over K*;

(ii) $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ *for any* $\sigma\in\Delta$ *and all* $i<\dim(\mathrm{lk}_\Delta(\sigma))$.

In analogy with the above result, the following theorem provides equivalent conditions for CM_t complexes.

Theorem 2.6. *Let Δ be a simplicial complex of dimension $d-1$. Then the following are equivalent*:

(i) *Δ is* CM_t *over K*;

(ii) *Δ is pure and* $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ *for all* $\sigma\in\Delta$ *with* $\#\sigma\geq t$ *and* $i<d-\#\sigma-1$.

Proof. (i)⇒(ii) Suppose that Δ is CM_t over K. Then Δ is pure and $\mathrm{lk}_\Delta(\sigma)$ is Cohen–Macaulay for all $\sigma\in\Delta$ with $\#\sigma\geq t$. Therefore, $\widetilde{H}_i(\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma)}(\tau);K)=0$ for all $\tau\in\mathrm{lk}_\Delta(\sigma)$ and all $i<\dim(\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma)}(\tau))$. In particular, for $\tau=\varnothing$, $\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma)}(\varnothing)=\mathrm{lk}_\Delta(\sigma)$ and we have $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ for all $i<\dim(\mathrm{lk}_\Delta(\sigma))\leq d-\#\sigma-1$.

(ii)⇒(i) We use induction on t. Use [16, Theorem 3.2] for the case $t=1$. Assume that the assertion holds for $t-1$. Let $\{x\}\in\Delta$ and $\tau\in\mathrm{lk}_\Delta\{x\}$ with $\#\tau\geq t-1$. Then by the purity of Δ, $\dim\mathrm{lk}_\Delta\{x\}=d-2$. But $\tau\cup\{x\}\in\Delta$ and hence by (ii), $\widetilde{H}_i(\mathrm{lk}_\Delta(\tau\cup\{x\});K)=0$ for all $i<d-\#\tau-2$. This yields that $\widetilde{H}_i(\mathrm{lk}_{\mathrm{lk}_\Delta(\{x\})}(\tau);K)=0$ for all $\tau\in\mathrm{lk}_\Delta\{x\}$ with $\#\tau\geq t-1$, and all $i<d-1-\#\tau-1$. By the induction hypothesis $\mathrm{lk}_\Delta\{x\}$ is CM_{t-1} for all $\{x\}\in\Delta$. Now by Lemma 2.3 we are done. □

We state a result due to Munkres [12, Corollary 3.4] which shows that Cohen–Macaulayness is a topological property.

Theorem 2.7. *Let Δ be a pure simplicial complex of dimension $d-1$. Then the following are equivalent*:

(i) Δ *is Cohen–Macaulay over* K;

(ii) $\widetilde{H}_i(|\Delta|;K)=0=H_i(|\Delta|,|\Delta|\backslash p;K)$ *for all* $p\in|\Delta|$ *and all* $i<d-1$, *where* $|\Delta|$ *is the geometric realization of* Δ.

The following theorem may lead one to believe that the property CM_t is also a topological invariant.

Theorem 2.8. *Let* Δ *be a pure simplicial complex of dimension* $d-1$. *Then the following are equivalent*:

(i) Δ *is* CM_t *over* K;

(ii) $H_i(|\Delta|,|\Delta|\backslash p;K)=0$ *for all* $p\in|\Delta|\backslash|\Delta_{t-2}|$ *and all* $i<d-1$, *where* Δ_{t-2} *is the* $(t-2)$*-skeleton of* Δ *and* $|\Delta_{t-2}|$ *is induced from a fixed geometric realization of* Δ.

Proof. First note that by Theorem 2.6, Δ is CM_t if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ for all $\sigma\in\Delta$ with $\#\sigma\geq t$ and all $i<d-\#\sigma-1$. Now by [12, Lemma 3.3], for any interior point p of σ we have

$$H_i(|\Delta|,|\Delta|\backslash p;K)\cong\widetilde{H}_{i-\#\sigma}(\mathrm{lk}_\Delta(\sigma);K).$$

Therefore, $H_i(|\Delta|,|\Delta|\backslash p;K)=0$ for any $\sigma\in\Delta$ with $\#\sigma\geq t$, any interior point p of σ and any $i<d-1$ if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);K)=0$ for all $\sigma\in\Delta$ with $\#\sigma\geq t$ and $i<d-\#\sigma-1$. But the set of such points is precisely $|\Delta|\backslash|\Delta_{t-2}|$ when some geometric realization is fixed. □

Let Δ and Δ' be two simplicial complexes whose vertex sets are disjoint. The simplicial join $\Delta*\Delta'$ is defined to be the simplicial complex whose faces are of the form $\sigma\cup\sigma'$, where $\sigma\in\Delta$ and $\sigma'\in\Delta'$.

The algebraic and combinatorial properties of the simplicial join $\Delta*\Delta'$ through the properties of Δ and Δ' have been studied by a number of authors (see [1], [4], [6] and [13]). For instance, in [6], Fröberg showed that the simplicial join $\Delta*\Delta'$ is Cohen–Macaulay (resp. Gorenstein) if and only if both of them are Cohen–Macaulay (resp. Gorenstein). One can see that the simplicial join of the triangulation of a cylinder (which is Buchsbaum [18, Example II.2.13(i)]) with a simplicial complex with only one vertex (which is Cohen–Macaulay [18, Example II.2.14(ii)] and so Buchsbaum) is not Buchsbaum. In [15, Corollary 2.9] it is shown that $\Delta*\Delta'$ is Buchsbaum (over K) if and only if Δ and Δ' are Cohen–Macaulay (over K). Therefore, it is natural to ask about Δ and Δ' when $\Delta*\Delta'$ is CM_t.

After preparing this paper, the authors of [10] brought our attention to their recent work on simplicial complexes with singularities. These authors study Cohen–Macaulay complexes in a fixed codimension (see [10, Definition 6.3]). For a simplicial complex of dimension $d-1$, CM_t implies Cohen–Macaulayness in codimension $d-t$ in their sense and the two concepts coincide if Δ is pure. These authors also provided an answer to our question (with two proofs) on the behavior of the join of two simplicial complexes with respect to CM_t which will be given here by their permission. One of the proofs is based on their characterization of CM_t in terms of Ext modules which is interesting in its own.

Proposition 2.9. [10, Corollary 7.4] *Let Δ be a simplicial complex of dimension $d-1$ on n vertices and let $R=k[x_1, ..., x_n]$. Then Δ is CM_t if and only if Δ is pure and*

$$\dim \operatorname{Ext}^i_R(k[\Delta], R) \leq t$$

for all $i>n-d$, where dim *refers to the Krull dimension.*

Proposition 2.10. *Let Δ and Δ' be two simplicial complexes of dimensions $d-1$ and $d'-1$, respectively. Then $\Delta * \Delta'$ is CM_t if and only if Δ is $\mathrm{CM}_{t-d'}$ and Δ' is CM_{t-d}. Here we use the convention that for $s<0$, CM_s is just CM_0.*

Proof. First note that $\Delta * \Delta'$ is pure if and only if both Δ and Δ' are pure. Assume that $\Delta * \Delta'$ is CM_t and let F be a face of Δ such that $\#F \geq t-d'$. Let G be a facet of Δ'. Then $F*G$ is a face of $\Delta * \Delta'$ with at least t elements. Hence $\mathrm{lk}(F*G)= \mathrm{lk}(F)*\mathrm{lk}(G)$ is Cohen–Macaulay. Therefore $\mathrm{lk}\, F$ is Cohen–Macaulay. Hence Δ is $\mathrm{CM}_{t-d'}$. Similarly Δ' is CM_{t-d}. Conversely, assume that Δ and Δ' are $\mathrm{CM}_{t-d'}$ and CM_{t-d}, respectively. Let H be a face of $\Delta * \Delta'$ with $\#H \geq t$. Then there exist faces $F \in \Delta$ and $G \in \Delta'$ such that $H=F*G$. Therefore $\#F = \#H - \#G \geq \#H - d' \geq t-d'$. Hence $\mathrm{lk}\, F$ is Cohen–Macaulay. Similarly $\mathrm{lk}\, G$ is Cohen–Macaulay. Thus $\mathrm{lk}\, H = \mathrm{lk}\, F * \mathrm{lk}\, G$ is Cohen–Macaulay.

The second part of the proof is based on the Künneth tensor formula and the above characterization of CM_t in terms of Ext modules. Indeed, by the Künneth formula (see, e.g., [15, Lemma 2.1]), for all j,

$$\operatorname{Ext}^j_{R''}(k[\Delta * \Delta'], R'') \cong \bigoplus_{p+q=j} \operatorname{Ext}^p_R(k[\Delta], R) \otimes_k \operatorname{Ext}^q_{R'}(k[\Delta'], R'),$$

where R and R' are polynomial rings corresponding to the vertex sets of Δ and Δ', respectively, and $R''=R \otimes_k R'$. Assume that Δ has n vertices and Δ' has m vertices. It follows that $\dim \operatorname{Ext}^j_{R''}(k[\Delta''], R'') \leq t$ for $j>n-d+m-d'$ if and only if $\dim \operatorname{Ext}^p_R(k[\Delta], R) \leq t-d'$ for $p>n-d$ and $\dim \operatorname{Ext}^q_{R'}(k[\Delta'], R') \leq t-d$ for $q>m-d'$. We skip the details. $\square$

3. The k-CM$_t$ simplicial complexes

In this section k-CM$_t$ complexes are introduced and some of their basic properties are given. We show that a complex is k-CM$_t$ if and only if the links of its non-empty faces are k-CM$_{t-1}$ (see Proposition 3.6). The main result of this section is Theorem 3.8 which states that a certain subcomplex of a CM$_t$ complex is 2-CM$_t$. This leads to a proof of the fact that for an integer $s \leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM$_t$ complex is $(k+s)$-CM$_t$ (see Corollary 3.10).

Definition 3.1. Let K be a field. For a positive integer k and a non-negative integer t, a simplicial complex Δ with vertex set V is called k-CM$_t$ *of dimension* r *over* K if for any subset W of V (including $\varnothing$) with $\#W<k$, $\Delta_{V\setminus W}$ is CM$_t$ of dimension r over K. The complex Δ is k-CM$_t$ over K if Δ is k-CM$_t$ of some dimension r over K.

Note that for any $\ell \leq k$, k-CM$_t$ implies ℓ-CM$_t$. In particular, any k-CM$_t$ is CM$_t$.

In the rest of this paper we will often need the following lemma [11, Lemma 2.3].

Lemma 3.2. *Let Δ be a simplicial complex with vertex set V. Let $W \subseteq V$ and let σ be a face in Δ. If $W \cap \sigma = \varnothing$, then* $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma) = \mathrm{lk}_\Delta(\sigma)_{V\setminus W}$.

Lemma 3.3. *Let Δ be a simplicial complex. Then the following are equivalent*:

(a) *Δ is k-CM$_t$;*

(b) *for all $\sigma \in \Delta$ with $\#\sigma \geq t$, $\mathrm{lk}_\Delta(\sigma)$ is k-Cohen–Macaulay.*

Proof. Indeed both properties require that for all $W \subset V$ such that $\#W<k$, $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma) = \mathrm{lk}_\Delta(\sigma)_{V\setminus W}$ is Cohen–Macaulay. □

Lemma 3.4. *Let Δ be a k-CM$_t$ complex and let $\sigma \in \Delta$ be an arbitrary face with $\#\sigma = s$. Then $\mathrm{lk}_\Delta(\sigma)$ is k-CM$_{t-s}$.*

Proof. Let V_1 be the vertex set of $\mathrm{lk}_\Delta(\sigma)$ and consider $W \subset V_1$ with $\#W<k$. We need to show that, $(\mathrm{lk}_\Delta(\sigma))_{V_1\setminus W}$ is CM$_{t-s}$. Observe that since $\sigma \cap W = \varnothing$, $\mathrm{lk}_\Delta(\sigma)_{V_1\setminus W} = \mathrm{lk}_\Delta(\sigma)_{V\setminus W} = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$. Put $\Gamma = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$ and let $\tau \in \Gamma$ with $\#\tau \geq t-s$. Then $\#(\sigma \cup \tau) \geq t$ and $\mathrm{lk}_\Gamma(\tau) = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma \cup \tau)$, which is Cohen–Macaulay by assumption. □

Corollary 3.5. *Let Δ be a k-Buchsbaum (k-CM$_2$) complex and let $\sigma \in \Delta$ be a non-empty face. Then $\mathrm{lk}_\Delta(\sigma)$ is k-Cohen–Macaulay (resp. k-Buchsbaum).*

Proposition 3.6. *Let Δ be a pure complex of dimension $d-1$ with vertex set V. Then for all positive integers k and t the following are equivalent*:

(i) Δ *is* k-CM_t;

(ii) *for any non-empty face σ in Δ*, $\mathrm{lk}_\Delta(\sigma)$ *is* k-CM_{t-1}.

Proof. (i)$\Rightarrow$ (ii) Use Lemma 3.4.

(ii)$\Rightarrow$ (i) For any subset W of V with $\#W<k$, we need to show that $\Delta_{V\setminus W}$ is CM_t of dimension $d-1$. Let $\sigma\in\Delta_{V\setminus W}$ with $\#\sigma\geq t$. Then $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)=(\mathrm{lk}_\Delta(\sigma))_{V\setminus W}$. Since $\mathrm{lk}_\Delta(\sigma)$ is k-CM_{t-1} we have that $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$ is Cohen–Macaulay.

Now we show that $\Delta_{V\setminus W}$ is pure of dimension $d-1$. Let τ be an arbitrary facet in $\Delta_{V\setminus W}$. Since $\mathrm{lk}_\Delta(\tau)$ is a k-CM_{t-1} complex, we have

$$\dim(\mathrm{lk}_\Delta(\tau)_{V\setminus W})=\dim(\mathrm{lk}_\Delta(\tau)).$$

On the other hand, since Δ is pure, we have $\dim(\mathrm{lk}_\Delta(\tau))=d-\#\tau-1$. In addition,

$$\dim(\mathrm{lk}_\Delta(\tau)_{V\setminus W})=\dim(\mathrm{lk}_{\Delta_{V\setminus W}}(\tau))=\dim(\{\varnothing\})=-1.$$

Therefore, we have $\dim(\tau)=d-1$. $\square$

Corollary 3.7. (See [11, Lemma 4.2]) *Let Δ be a pure complex of dimension $d-1$ with vertex set V. Then for all positive integers k the following are equivalent*:

(i) Δ *is k-Buchsbaum*;

(ii) *for any non-empty face σ in Δ*, $\mathrm{lk}_\Delta(\sigma)$ *is a k-Cohen–Macaulay complex.*

Now we are ready to give one of the main results of this paper which generalizes results due to Hibi [7] and Miyazaki [11].

Let Δ a simplicial complex and let $\sigma_1,\dots,\sigma_\ell$ be faces of Δ such that

(i) $\sigma_i\cup\sigma_j\notin\Delta$ if $i\neq j$; and

(ii) if $\Delta_1=\{\tau\in\Delta|\tau\not\supseteq\sigma_i$ for all $i\}$ then $\dim\Delta_1<\dim\Delta$.

In [7] Hibi showed that Δ_1 is 2-Cohen–Macaulay of dimension $\dim\Delta-1$ provided that $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen–Macaulay for all i. In [11] Miyazaki extended this result for Buchsbaumness by showing that if Δ is a Buchsbaum complex of dimension $d-1$, and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen–Macaulay for all i, then Δ_1 is 2-Buchsbaum. Therefore, it is natural to ask whether a similar result is valid for CM_t complexes. In the following result we give an affirmative answer to this question.

Theorem 3.8. *Let Δ be a* CM_t *complex and let $\sigma_1,\dots,\sigma_\ell$ be faces of Δ satisfying the above conditions* (i) *and* (ii). *If* $\mathrm{lk}_\Delta(\sigma_i)$ *is* 2-CM_{t-1} *for all i, then Δ_1 is a* 2-CM_t *complex of dimension* $\dim\Delta-1$.

Proof. We use induction on t. If $t=0,1$ the assertion hold by [7] and [11, Theorem 7.4]. Assume that the assertion holds for $t-1$. By Lemma 3.6 we need to show that Δ_1 is pure and for any non-empty face τ in Δ_1, $\mathrm{lk}_{\Delta_1}(\tau)$ is 2-CM_{t-1}. By [11, Lemma 7.2], Δ_1 is pure. Let τ be a non-empty face in Δ_1. We may reorder the σ_i's so that $\sigma_i\cup\tau\in\Delta$ if and only if $i\leq s$. Then

$$\begin{aligned}\mathrm{lk}_{\Delta_1}(\tau) &= \{\sigma\in\Delta \mid \sigma\cup\tau\in\Delta_1 \text{ and } \sigma\cap\tau=\varnothing\}\\ &= \{\sigma\in\Delta \mid \sigma\cup\tau\in\Delta,\ \sigma\cap\tau=\varnothing \text{ and } \sigma\cup\tau\not\supseteq\sigma_i,\ 1\leq i\leq\ell\}\\ &= \{\sigma\in\Delta \mid \sigma\cup\tau\in\Delta,\ \sigma\cap\tau=\varnothing \text{ and } \sigma\not\supseteq\tau_i,\ 1\leq i\leq s\},\end{aligned}$$

where $\tau_i=\sigma_i-\tau$ for $1\leq i\leq s$. Thus if we put $\Gamma=\mathrm{lk}_\Delta(\tau)$ then

$$\mathrm{lk}_{\Delta_1}(\tau)=\{\sigma\in\Gamma \mid \sigma\not\supseteq\tau_i,\ 1\leq i\leq s\}.$$

On the other hand,

$$\mathrm{lk}_\Gamma(\tau_i)=\mathrm{lk}_\Delta(\tau\cup\tau_i)=\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma_i)}(\tau-\sigma_i).$$

By assumption $\mathrm{lk}_\Delta(\sigma_i)$ is 2-CM_{t-1}. Then Lemma 3.4 shows that $\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma_i)}(\tau-\sigma_i)$ is 2-CM_{t-2} and hence $\mathrm{lk}_\Gamma(\tau_i)$ is 2-CM_{t-2}. Applying the induction hypothesis for Γ and $\tau_1,\dots,\tau_s$ it follows that $\mathrm{lk}_{\Delta_1}(\tau)$ is 2-CM_{t-1}. Since τ is an arbitrary non-empty face of Δ_1, by Lemma 3.6 it follows that Δ_1 is a 2-CM_t complex of dimension $\dim\Delta-1$. □

The condition on $\mathrm{lk}_\Delta(\sigma_i)$ in the above theorem cannot be weakened in the sense that one cannot replace CM_{t-1} by CM_t for these links. This can be seen in the following example.

Example 3.9. (See [11, Example 7.5]) If

$$\Delta_1=\langle\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{1,5\},\{4,5\}\rangle,$$

which has dimension 1, and $\Delta_2=\langle\{x,y\}\rangle$, then $\Delta=\Delta_1*\Delta_2$ is Cohen–Macaulay. If we put $\sigma_1=\{x,y\}$ and $t=1$, then $\mathrm{lk}_\Delta(\sigma_1)=\Delta_1$ is a 2-Buchsbaum complex and $\Delta\backslash\sigma_1=\Delta_1*\langle\{x\},\{y\}\rangle$. So $\mathrm{lk}_{\Delta\backslash\sigma_1}(\{x\})=\Delta_1$ is not 2-Cohen–Macaulay and we see that $\Delta\backslash\sigma_1$ is not 2-Buchsbaum.

Corollary 3.10. *Let Δ be a k-CM_t complex of dimension $d-1$. If $s\leq d$ and Δ' is the $(d-s-1)$-skeleton of Δ, then Δ' is $(k+s)$-CM_t.*

Proof. We may assume that $s{=}1$. Let V be the vertex set of Δ and W be a subset of V such that $0<\#W<k+1$. If we take $x\in W$ and put $W'=W\setminus\{x\}$, then $\Delta_{V\setminus W'}$ is CM_t of dimension $d-1$ by assumption. On the other hand, since

$$\Delta'_{V\setminus W'}=\{\sigma\in\Delta\,|\,\dim(\sigma)<d-1 \text{ and } \sigma\cap W'=\varnothing\}$$

and this is equal to the $(d-2)$-skeleton of $\Delta_{V\setminus W'}$, by Theorem 3.8, $(\Delta')_{V\setminus W'}$ is 2-CM_t of dimension $d-2$. So $(\Delta')_{(V\setminus W')\setminus\{x\}}=(\Delta')_{V\setminus W}$ is a CM_t complex of dimension $d-2$. □

Remark 3.11. The above corollary generalizes a result of Terai and Hibi [19] (see also [2]) which states that the 1-skeleton of a simplicial $(d-1)$-sphere with $d\geq 2$ is d-connected (topologically). This is just due to the fact that a simplicial $(d-1)$-sphere is 2-Cohen–Macaulay and $(d-1)$-Cohen–Macaulayness implies $(d-1)$-connectedness. This corollary also generalizes a result of Hibi [7] (see the introduction in [11]) which says that if Δ is a Cohen–Macaulay complex of dimension $d-1$, then the $(d-2)$-skeleton of Δ is 2-Cohen–Macaulay. On the other hand, Fløystad has proved the above corollary for the case $t{=}0$ in the more general setting of cell complexes [5, Theorem 2.1 and Remark 2.2].

Example 3.12. If Γ is a finite union of $(d-1)$-simplices where any two of them intersect in a face of dimension at most $t-2$ and Λ is the $(d-2)$-skeleton of Γ, then Λ is 2-CM_t. If at least two facets in Γ intersect in a $(t-2)$-dimensional face, then Λ is not 2-CM_{t-1}.

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