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A GENERALIZATION OF LIOUVILLE'S THEOREM ON INTEGRATION IN FINITE TERMS

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Abstract. A generalization of Liouville's theorem on integration in finite terms, by enlarging the class of fields to an extension called Ei-Gamma extension is established. This extension includes the $\mathcal{EL}$-elementary extension of Singer, Saunders and Caviness and contains the Gamma function.

1. Introduction

The problem of expressing certain entities "in finite terms", such as the computation of roots of polynomials in terms of radicals, the solving of differential equations in terms of elementary functions, arises frequently in Mathematics. One such problem known as "integration in finite terms" is dealt with in this paper. Roughly speaking, the problem of integration in finite terms is that given a γ in a differential field F with derivation D, we ask when a solution of $D(y) = \gamma$ can be expressed in certain special forms. Historically, Joseph Liouville (see e.g. [4]) first systematically worked on the question of when an algebraic function has an algebraic integral and he later gave conditions relating to when an algebraic function has an integral of a special form called "elementary" ; this particular result is generally known as Liouville's theorem on integration in finite terms. In its simplified form, it reads : if $\gamma(x)$ is an algebraic function whose integral is elementary, then

$$\int \gamma(x)\,dx = \nu_0(x) + c_1 \log \nu_1(x) + \cdots + c_n \log \nu_n(x),$$

where n is a positive integer, each $\nu_i(x)$ an algebraic function, and each c_i a constant. The works of Liouville were subsequently extended by

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a number of other people such as D. D. Mordukhai-Boltovskoi [2], A. Ostrowski [3], J. F. Ritt [4], and M. Rosenlicht [5], [6], [7]. A proof of Liouville's theorem can be found in Ritt's classic exposition [4]; the proof is a combination of clever observations and is analytic in nature. In 1946, Ostrowski gave for the first time in [3] a proof of Liouville's theorem in the context of differential fields of complex meromorphic functions. In 1968, M. Rosenlicht found a completely algebraic proof of Liouville's theorem as described in his series of papers [5], [6], and [7].

To date, one of the most generalized Liouville's theorem is due to M. F. Singer, B. D. Saunders and B. F. Caviness [8]. Singer, Saunders and Caviness generalized Liouville's theorem by enlarging the class of fields from elementary to a special class of fields, call $\mathcal{EL}$ -elementary, which includes elementary functions as well as special functions such as error function and logarithmic integral.

A natural question arises whether extensions to other classes of fields containing functions not previously covered are possible.

The objective of this paper is to affirmatively answer this question by establishing a special class of fields, called Ei - Gamma extensions, which contains the $\mathcal{EL}$-elementary and two more functions such as the exponential integral and the Gamma function.

In Section 2, we state some results of Lang [1], Rosenlicht [7] and Singer, Saunders and Caviness [8] that are used in the proof of the main result and define the extended class of functions.

In Section 3, we give the main result of this paper.

All fields are assumed to be of characteristic zero. $\mathbb{Q}, \mathbb{Z}$ and $\mathbb{Z}^+$ stand for the set of rational numbers, the set of integers and the set of positive integers, respectively.

2. Some preliminaries

In this section we state some results from Lang [1], Rosenlicht [7] and Singer, Saunders and Caviness [8] that will be used later.

LEMMA 2.1 ([1, Theorem 16]). *Let F be a field and n an integer ≥ 2. Let $a \in F$, $a \neq 0$. Assume that for all prime numbers p such that $p|n$ we have $a \notin F^p$, and if $4|n$ then $a \notin -4F^4$. Then $X^n - a$ is irreducible in $F[X]$.*

LEMMA 2.2 ([2, Theorem 2]). *Let F be a differential field, K a differential extension field of F with the same subfield of constants, with K algebraic over $F(t)$ for some given $t \in K$. Suppose that $c_1, ..., c_n$ are* *constants of F that are linearly independent over $\mathbb{Q}$, that $u_1, \ldots, u_n$, v are elements of K, with $u_1, \ldots, u_n$ nonzero, and that for each given derivation D of K we have*

$$\sum_{i=1}^{n} c_i D(u_i)/u_i + D(v) \in F.$$

If for each given derivation D of K we have $D(t) \in F$, then $u_1, ..., u_n$ are algebraic over F and there exists a constant c of F such that $v - ct$ is algebraic over F. If for each given derivation D of K we have $D(t)/t \in F$, then v is algebraic over F and there are integers $m_0, m_1, ..., m_n$ with $m_0 \neq 0$, such that each $u_i^{m_0} t^{m_i}$ $(i = 1, ..., n)$ is algebraic over F.

LEMMA 2.3 ([8, Lemma 3.1]). *Let F be a field containing the algebraic closure of the rationals and let X and Y be indeterminates. Let $A(Y)$ and $B(Y)$ $(\neq 0)$ be relatively prime elements of $F[Y]$. Furthermore, assume A/B is not an n-th power in $F(Y)$ for any positive integer $n \geq 2$. Then the polynomial $B(Y)X^m - A(Y)$ is irreducible in $F(X)[Y]$ for any positive integer m.*

LEMMA 2.4 ([3, Lemma 3.2]). *Let F be a field, X and Y indeterminates, and $A(Y)$ and $B(Y)$ relatively prime elements of $F[Y]$. If a and b are elements of F with $a \neq 0$, then $A(Y) - (aX + b)B(Y)$ is irreducible in $F(X)[Y]$.*

Next we define our generalized class of functions.

Let F be a differential field with derivation D and the subfield of constants C. We say that a differential extension E of F is an Ei- Gamma extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ such that for each $i = 1, ..., n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) t_i is an exponential over F_{i-1} (i.e., $D(t_i) = D(u)t_i$ for some u in F_{i-1}, and we write $t_i = \exp(u)$),

(iii) t_i is a logarithmic over F_{i-1} (i.e., $D(t_i) = D(u)/u$ for some nonzero u in F_{i-1}, and we write $t_i = \log(u)$),

(iv) there are $G \in C(Y)$, u and nonzero v in F_{i-1}, $r \in \mathbb{Q}$ with $-1 \leq r \leq 1$ such that $D(t_i) = D(u^r)G(v)$, where $v = \exp(u)$ (i.e., $t_i = \int G(\exp(u)) r u^{r-1} D(u)$),

(v) there are $G \in C(Y)$, nonzero u and v in F_{i-1} such that $D(t_i) = D(u)G(v)$ or $D(t_i) = (D(u)/u)\ G(v)$ where $v = \exp R(u)$ for some $R \in C(Y)$ (i.e., $t_i = \int G\ (\exp R(u))\ D(u)$, or $t_i = \int G\ (\exp R(u))\ \frac{D(u)}{u}$),

(vi) there are $H \in C(Y)$, with deg(numerator of H) $\leq$ deg(denominator of H), nonzero u, v in F_{i-1} such that $D(t_i) = D(u)H(v)$ or $D(t_i) = (D(u)/u)H(v)$ where $v = \log S(u)$ for some nonzero $S \in C(Y)$ (i.e., $t_i = \int H(\log S(u))D(u)$, or $t_i = \int H\big(\log S(u)\big)\frac{D(u)}{u}\big)$.

We say that a $\gamma \in F$ is an Ei-Gamma element in F if there exist

(1) $a_i \in C, v_o$ algebraic over F and nonzero elements v_i algebraic over F for all $i \in I$,
(2) $b_i \in C$, $r_i \in \mathbb{Q}$ with $-1 \leq r_i \leq 1$, nonzero elements w_i, x_i algebraic over F and $P_i \in C(Y)$ for all $i \in J$,
(3) $c_{i\alpha}$, $d_{i\alpha} \in C$, nonzero elements $w_{i\alpha}$, $x_{i\alpha}$ algebraic over F and $G_\alpha \in C(Y)$ for all $i \in I_\alpha$, $\alpha \in A$,
(4) $e_{i\beta}, f_{i\beta} \in C$, nonzero elements $y_{i\beta}, z_{i\beta}$ algebraic over F and $H_\beta \in C(Y)$ with deg(numerator of H_β) $\leq$ deg(denominator of H_β) for all $i \in J_\beta$, $\beta \in B$, such that

$$\begin{aligned}\gamma =& D(v_o) + \sum_{i\in I} a_i D(v_i)/v_i + \sum_{i\in J} b_i D(w_i^{r_i}) P_i(x_i)\\ &+ \sum_{\alpha\in A}\sum_{i\in I_\alpha} \big(c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}\big)G_\alpha(x_{i\alpha})\\ &+ \sum_{\beta\in B}\sum_{i\in J_\beta} \big(e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta}\big)H_\beta(z_{i\beta}),\end{aligned}$$

where A, B, I, J, I_α and J_β are all finite indexing sets,

$x_i = \exp(w_i)$ for all $i \in I$,
$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ with $R_\alpha \in C(Y)$ for all $i \in I_\alpha$, $\alpha \in A$, and
$z_{i\beta} = \log S_\beta(y_{i\beta})$ with $S_\beta \in C(Y) \setminus \{0\}$ for all $i \in J_\beta$, $\beta \in B$.

REMARKS.

1. The differential extension E of F equipped with cases (i)-(iii) is an elementary extension of F ([5], [6], [7]).
2. The Ei-Gamma extension embraces the $\mathcal{EL}$-elementary extension of F of Singer, Saunder and Caviness [8].

3. The Ei-Gamma extension contains the exponential integral and the Gamma function which are defined, respectively, by

$$Ei(u) = \int (D(u)/u)\exp(u),$$
$$\Gamma(u) = \int D(u^r)\exp(-u),$$

where $r \in \mathbb{Q}$ and $0 < r \leq 1$.

EXAMPLE. Let C be the field of complex numbers and let $F = C(x)$ be the set of rational functions with coefficients in C. Then F is a differential field under the usual derivation $D = d/dx$. Hence

$$F\Big(\exp(x),\ \log(x+1),\ \int \frac{\exp(x)}{x} D(x),$$
$$\int \frac{1}{x(\log(x+1)+2)} D(x),\ \int \exp(x) D(\sqrt{x})\Big)$$

is an Ei-Gamma extension of F.

3. The main theorem

THEOREM 3.1. *Let F be a differential field with derivation D and an algebraically closed subfield of constants C. Let $\gamma \in F$. Assume that there exist an Ei-Gamma extension E of F whose subfield of constants is C and $y \in E$ such that $D(y) = \gamma$. Then γ is an Ei-Gamma element in F.*

The proof of Theorem is by induction on the transcendence degree of E over F and it suffices to prove the following lemmas. Before proving the lemma, it will be convenient to define the following term:

If f and g are polynomials over a field F, and $g \neq 0$, then there exist unique polynomials $q(X) = a_o + a_1 X + \cdots + a_n X^n$ and $r(X)$ over F such that $f(X)/g(X) = q(X) + r(X)/g(X)$, where $r(X) = 0$ or deg $r(X) <$ deg $g(X)$. Call the unique element a_o the head of f/g.

LEMMA 3.2. *Let F be a differential field with derivation D and C its algebraically closed subfield of constants. Let t be transcendental over F such that $D(t) = D(u)t$ for some u in F. Assume that the subfield of constants of $F(t)$ is C. Let $\gamma \in F$. If γ is an Ei-Gamma element in $F(t)$ then γ is also an Ei-Gamma element in F.*

Proof. Since γ is an Ei-Gamma element in $F(t)$, then there exist

(1) $a_i \in C$, v_o algebraic over $F(t)$ and nonzero elements v_i algebraic over $F(t)$ for all $i \in I$,
(2) $b_i \in C$, $r_i \in \mathbb{Q}$ with $-1 \leq r_i \leq 1$, nonzero elements w_i, x_i algebraic over $F(t)$ and $P_i \in C(Y)$ for all $i \in J$,
(3) $c_{i\alpha}$, $d_{i\alpha} \in C$, nonzero elements $w_{i\alpha}$, $x_{i\alpha}$ algebraic over $F(t)$ and $G_\alpha \in C(Y)$ for all $i \in I_\alpha$, $\alpha \in A$,
(4) $e_{i\beta}$, $f_{i\beta} \in C$, nonzero elements $y_{i\beta}$, $z_{i\beta}$ algebraic over $F(t)$ and $H_\beta \in C(Y)$ with deg(numerator of H_β) $\leq$ deg(denominator of H_β) for all $i \in J_\beta$, $\beta \in B$, such that

$$\begin{aligned}\gamma =& D(v_o) + \sum_{i\in I} a_i D(v_i)/v_i + \sum_{i\in J} b_i D(w_i^{r_i}) P_i(x_i) \\ &+ \sum_{\alpha\in A}\sum_{i\in I_\alpha} \big(c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}\big) G_\alpha(x_{i\alpha}) \\ &+ \sum_{\beta\in B}\sum_{i\in J_\beta} \big(e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}\big) H_\beta(z_{i\beta}),\end{aligned}$$

where A, B, I, J, I_α and J_β are all finite indexing sets,

$x_i = \exp(w_i)$ for all $i \in J$,
$x_{i\alpha} = \exp R_\alpha(W_{i\alpha})$ with $R_\alpha \in C(Y)$ for all $i \in I_\alpha$, $\alpha \in A$, and
$z_{i\beta} = \log S_\beta(y_{i\beta})$ with $S_\beta \in C(Y) \setminus \{0\}$ for all $i \in J_\beta$, $\beta \in B$.

We may assume that F is algebraically closed, for if F is not algebraically closed, then we work in $\bar{F}$, an algebraic closure of F. Note that $\bar{F}(t)$, $\bar{F}$, F have the same subfield of constants C. Since γ is an Ei-Gamma element in $F(t)$ and $F(t) \subset \bar{F}(t)$, γ is also an Ei-Gamma element in $\bar{F}(t)$. In this case, we could replace F by $\bar{F}$. It is easy to see that if γ is an Ei-Gamma element in $\bar{F}$, then γ is also an Ei-Gamma element in F.

Step 1. We may assume that for all α in A, $R_\alpha \notin C$; for if $R_{\alpha_0} \in C$ for some $\alpha_0 \in A$, then for each $i \in I_{\alpha_0}$, $G_{\alpha_0}(x_{i\alpha_0}) \in C$. Thus

$$\sum_{i\in I_{\alpha_0}} \big(c_{i\alpha_0} D(w_{i\alpha_0}) + d_{i\alpha_0} D(w_{i\alpha_0})/w_{i\alpha_0}\big) G_{\alpha_0}(x_{i\alpha_0})$$

is of the form $D(v_o) + \sum b_i D(v_i)/v_i$ which can be included into the first two terms of γ.

Step 2. For each $i \in J$, we have $D(x_i) = D(w_i)x_i$, then by Lemma 2.2, we have that $w_i \in F$ and there exist rational numbers ν_i and p_i in F such that $x_i = p_i t^{\nu_i}$.

For each $\alpha \in A,\ \ i \in I_\alpha$ we have $D(x_{i\alpha}) = D(R_\alpha(w_{i\alpha}))x_{i\alpha}$, then by Lemma 2.2 we have that $R_\alpha(w_{i\alpha}) \in F$ and there exist rational integers $r_{i\alpha}$ and $p_{i\alpha}$ in F such that $x_{i\alpha} = p_{i\alpha}t^{r_{i\alpha}}$. Since $R_\alpha(w_{i\alpha}) \in F$ and F is algebraically closed, $w_{i\alpha} \in F$.

Step 3. For each $\beta \in B,\ \ i \in J_\beta$, we have

$$D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta}).$$

We may assume that for all β in B, $S_\beta(Y)$ is not an mth power in $C(Y)$ for any positive integer m. For if $S_\beta(Y) = (\bar{S}_\beta(Y))^m$ then

$$D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta}) = mD(\bar{S}_\beta(y_{i\beta}))/\bar{S}_\beta(y_{i\beta}).$$

So we could replace $S_\beta(Y)$ by $\bar{S}_\beta(Y)$ and $z_{i\beta}$ by $z_{i\beta}/m$. By Lemma 2.2, we have that $z_{i\beta} \in F$ and there exist rational numbers $s_{i\beta}$ and $q_{i\beta}$ in F such that $S_\beta(y_{i\beta}) = q_{i\beta}t^{s_{i\beta}}$.

Note that we can arrange so that ν_i, $r_{i\alpha}$, and $s_{i\beta}$ are actually integers. To see this, let $\nu_i = g_i/m$, $r_{i\alpha} = g_{i\alpha}/m$ and $s_{i\beta} = k_{i\beta}/m$, where $g_i, g_{i\alpha}$, $k_{i\beta}$ and m are integers.

Let $\bar{t} = t^{\frac{1}{m}}$. Hence $D(\bar{t}) = D(u/m)\bar{t}$ and $F \subset F(\bar{t})$. If we replace t by $\bar{t}$, we still have fields of the appropriate form and furthermore, $x_i = p_i(\bar{t})^{g_i}$, $x_{i\alpha} = p_{i\alpha}\bar{t}^{g_{i\alpha}}$ and $S_\beta(Y) = q_{i\beta}(\bar{t})^{k_{i\beta}}$, where $g_i, g_{i\alpha}$ and $k_{i\beta}$ are integers.
We shall use the old notation but from now on assume that $\nu_i, r_{i\alpha}$, and $s_{i\beta}$ are integers.

Step 4. Let K be a finite Galois extension over $F(t)$ and let σ be an element of the Galois group of K over $F(t)$. Then

$$\begin{aligned}\gamma = \sigma(\gamma) =& D(\sigma v_0) + \sum_{i\in I} a_i D(\sigma v_i)/(\sigma v_i)\\ &+ \sum_{i\in J} b_i D(w_i^{r_i})P_i(x_i)\\ &+ \sum_{\alpha\in A}\sum_{i\in I_\alpha}\big(c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}\big)G(x_{i\alpha})\\ &+ \sum_{\beta\in B}\sum_{i\in J_\beta}\big(e_{i\beta}D(\sigma y_{i\beta}) + f_{i\beta}D(\sigma y_{i\beta})/(\sigma y_{i\beta})\big)H_\beta(z_{i\beta}).\end{aligned}$$

Summing over all σ yields, for some M in $\mathbb{Z}$,

$$
\begin{aligned}
M\gamma = D(Tv_0) &+ \sum_{i\in I} a_i D(Nv_i)/(Nv_i) \\
&+ M\sum_{i\in J} b_i D(w_i^{r_i})P_i(x_i) \qquad (3.1)\\
&+ M\mathcal{E}_1 + \mathcal{E}_2,
\end{aligned}
$$

where $\mathcal{E}_1 = \sum_{\alpha\in A}\sum_{i\in I_\alpha}\big(c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}\big)G(x_{i\alpha})$,

$\mathcal{E}_2 = \sum_{\beta\in B}\sum_{i\in J_\beta}\big(e_{i\beta}D(Ty_{i\beta}) + f_{i\beta}D(Ny_{i\beta})/(Ny_{i\beta})\big)H_\beta(z_{i\beta})$,

and T and N denote the trace and norm of K into $F(t)$, respectively.

Step 5. Write $Tv_0 = \sum_{i=0}^{n} h_i t^i + \sum\sum(a_{ij}/(t-t_i)^j)$, where h_i, a_{ij} and t_i are in F. Hence the head of $D(Tv_0)$ is $D(h_0)$.

Step 6. For each $i\in I$ write $Nv_i = k_i\prod_{j=1}^{\alpha_i}(t-\mu_j)^{n_{ij}}$ where the $\alpha_i\in\mathbb{Z}^+$, the $k_i\in F\backslash\{0\}$, the $\mu_j\in F$ and the $n_{ij}\in\mathbb{Z}$.
Therefore the head of $\sum_{i\in I} a_i D(Nv_i)/(Nv_i)$ is $\sum_{i\in I} a_i D(k_i)/(k_i) + \sum_{i\in I}\sum_{j=1}^{\alpha_i} a_i n_{ij} D(u)$.

Step 7. For each $i\in J$, recall $x_i = p_i t^{\nu_i}$. Write

$$\sum_{i\in J} b_i D(w_i^{r_i})P_i(x_i) = \sum_{\substack{i\in J\\ \nu_i=0}} b_i D(w_i^{r_i})P_i(x_i) + \sum_{\substack{i\in J\\ \nu_i\neq 0}} b_i D(w_i^{r_i})P_i(x_i).$$

Clearly, $\sum_{\substack{i\in J\\ \nu_i=0}} b_i D(w_i^{r_i})P_i(x_i)\in F$. It is easy to see that

$$\sum_{\substack{i\in J\\ \nu_i\neq 0}} b_i D(w_i^{r_i})P_i(x_i) = \sum_{\substack{i\in J\\ \nu_i\neq 0}} \bar{b}_i D(w_i^{r_i}) + \text{ elements in } F(t)\backslash F,$$

where the $\bar{b}_i\in C$.

Step 8. We find the head of $\mathcal{E}_1$. For each $i \in I_\alpha, \alpha \in A$, recall $x_{i\alpha} = p_{i\alpha} t^{r_{i\alpha}}$. If $r_{i\alpha} = 0$, then $x_{i\alpha} \in F$ and hence $G_\alpha(x_{i\alpha}) \in F$. Assume that $r_{i\alpha} \neq 0$. Let d_{α_0} be the head of $G_\alpha(Y)$. Hence d_{α_0} is in C. So the head of $G_\alpha(x_{i\alpha})$ is d_{α_0}. Therefore the head of $\mathcal{E}_1$ is

$$\sum \sum_{r_{i\alpha}=0} \big(c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}\big) G_\alpha(x_{i\alpha})$$
$$+ \sum \sum_{r_{i\alpha}\neq 0} d_{\alpha_0} \big(c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}\big).$$

Step 9. We find the head of $\mathcal{E}_2$. For each $i \in J_\beta, \beta \in B$, recall $S_\beta(y_{i\beta}) = q_{i\beta} t^{s_{i\beta}}$.
Case 9.1. If $s_{i\beta} = 0$, then $S_\beta(y_{i\beta}) \in F$. Since F is algebraically closed and $y_{i\beta}$ is algebraic over F, $y_{i\beta} \in F$. Thus $Ty_{i\beta} = My_{i\beta}$ and $Ny_{i\beta} = y_{i\beta}^M$. So $D(Ty_{i\beta}) = MD(y_{i\beta})$ and $D(Ny_{i\beta})/Ny_{i\beta} = MD(y_{i\beta})/y_{i\beta}$.

Case 9.2. Assume that $s_{i\beta} \neq 0$. Calculate the trace and norm of the $y_{i\beta}$. Write $S_\beta(Y) = A_\beta(Y)/B_\beta(Y)$ where $A_\beta, B_\beta \in C[Y]$, $B_\beta \neq 0$ and A_β and B_β are relatively prime in $C[Y]$. Each $y_{i\beta}$ satisfies $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y) = 0$. By Lemma 2.3, $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y)$ is irreducible in $F(t)[Y]$. So the trace and norm can be read of from its coefficients. The coefficients of powers of Y in $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y)$ are all of the form $\delta_{i\beta} q_{i\beta} t^{s_{i\beta}} + \varepsilon_{i\beta}$ where $\delta_{i\beta}, \varepsilon_{i\beta} \in C$.

Dividing by the leading coefficient, we get

$$Ty_{i\beta} = m_{i\beta} \left(\frac{\delta_{i\beta} q_{i\beta} t^{s_{i\beta}} + \varepsilon_{i\beta}}{\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \nu_{i\beta}} \right), \quad \text{and}$$
$$Ny_{i\beta} = \left(\frac{\omega_{i\beta} q_{i\beta} t^{s_{i\beta}} + \zeta_{i\beta}}{\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \nu_{i\beta}} \right)^{m_{i\beta}},$$

where $m_{i\beta} \in \mathbb{Z}^+, \delta_{i\beta}, \mu_{i\beta}, \omega_{i\beta}, \varepsilon_{i\beta}, \nu_{i\beta}, \zeta_{i\beta} \in C$. Hence

$$D(Ty_{i\beta}) = \frac{m_{i\beta}\big(\delta_{i\beta}\nu_{i\beta} - \varepsilon_{i\beta}\mu_{i\beta}\big)\big(D(q_{i\beta}) + q_{i\beta} s_{i\beta} D(u)\big) t^{s_{i\beta}}}{(\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \nu_{i\beta})^2}$$

and

$$\frac{D(Ny_{i\beta})}{Ny_{i\beta}} = \frac{m_{i\beta}\big(\omega_{i\beta}\nu_{i\beta} - \zeta_{i\beta}\mu_{i\beta}\big)\big(D(q_{i\beta}) + q_{i\beta} s_{i\beta} D(u)\big) t^{s_{i\beta}}}{(\omega_{i\beta} q_{i\beta} t^{s_{i\beta}} + \zeta_{i\beta})(\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \nu_{i\beta})}.$$

Thus the head of $D(Ty_{i\beta})$ is 0 and the head of $\frac{D(Ny_{i\beta})}{Ny_{i\beta}}$ can be put in the form $\bar{m}_{i\beta}D(z_{i\beta})$ where $\bar{m}_{i\beta} \in \mathbb{Z}$. Hence $\sum \sum_{s_{i\beta}\neq 0} f_{i\beta}\bar{m}_{i\beta}D(z_{i\beta})H_\beta(z_{i\beta})$ can be put in the form $D(\hat{v}_0) + \sum \hat{a}_i D(\hat{v}_i)/\hat{v}_i$ where $\hat{v}_0 \in F$ and $\hat{v}_i \in F\backslash\{0\}, \hat{a}_i \in C$.

Therefore the head of $\mathcal{E}_2$ is

$$M\sum \sum_{s_{i\beta}=0} (e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta})+D(\hat{v}_0)+\sum \hat{a}_i D(\hat{v}_i)/\hat{v}_i.$$

Step 10. We conclude that the head of the right hand side of (3.1) is

$$\begin{aligned}
&D(\bar{v}_0) + \sum \bar{a}_i D(\bar{v}_i)/\bar{v}_i \\
&\quad + M\sum_{\substack{i\in J\\ \nu_i=0}} b_i D(w_i^{r_i})P_i(x_i) + M\sum_{\substack{i\in J\\ \nu_i\neq 0}} \bar{b}_i D(w_i^{r_i}) \\
&\quad + M\sum\sum (c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha})G_\alpha(x_{i\alpha}) \\
&\quad + M\sum\sum (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta}),
\end{aligned}$$

where $v_0 \in F$, $v_i \in F\ \{0\}, \bar{a}_i \in C$. Then comparing the head of (3.1) and dividing by M, we get the correct sum of γ.

LEMMA 3.3. *Let F be a differential field with derivation D and C its algebraically closed subfield of constants. Let t be transcendental over F satisfying one of the following conditions:*

(i) *t is a logarithmic over F*

(ii) *there are $G \in C(Y)$, u and nonzero v in F, $r \in \mathbb{Q}$ with $-1 \leq r \leq 1$ such that $D(t) = D(u^r)G(v)$, where $v = \exp(u)$*

(iii) *there are $G \in C(Y)$, nonzero u and v in F such that*

$$D(t) = D(u)G(v) \text{ or } D(t) = (D(u)/u)G(v)$$

where $v = \exp R(u)$ for some $R \in C(Y)$

(iv) *there are $H \in C(Y)$, with deg(numerator of H) $\leq$ deg(denominator of H), nonzero u, v in F such that*

$$D(t) = D(u)H(v) \text{ or } D(t) = (D(u)/u)H(v)$$

where $v = \log S(u)$ for some nonzero $S \in C(Y)$.

Assume that the subfield of constants of $F(t)$ is C. Let $\gamma \in F$. If γ is an Ei-Gamma element in $F(t)$ then γ is also an Ei-Gamma element in F.

Proof. Since γ is an Ei- Gamma form over $F(t)$, there exist

(1) $a_i \in C$, v_0 algebraic over $F(t)$ and nonzero elements v_i algebraic over $F(t)$ for all $i \in I$,

(2) $b_i \in C$, $r_i \in \mathbb{Q}$ with $-1 \leq r_i \leq 1$, nonzero elements w_i, x_i algebraic over $F(t)$ and $P_i \in C(Y)$ for all $i \in J$,

(3) $c_{i\alpha}, d_{i\alpha} \in C$, nonzero elements $w_{i\alpha}, x_{i\alpha}$ algebraic over $F(t)$ and $G_\alpha \in C(Y)$ for all $i \in I_\alpha, \alpha \in A$,

(4) $e_{i\beta}, f_{i\beta} \in C$, nonzero elements $y_{i\beta}, z_{i\beta}$ algebraic over $F(t)$ and $H_\beta \in C(Y)$ with deg(numerator of H_β) $\leq$ deg(denominator of H_β) for all $i \in J_\beta$, $\beta \in B$, such that

$$\begin{aligned}\gamma = D(v_0) &+ \sum_{i\in I} a_i D(v_i)/v_i + \sum_{i\in J} c_i D(w_i^{r_i})P_i(x_i)\\ &+ \sum_{\alpha\in A}\sum_{i\in I_\alpha}(c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha})G_\alpha(x_{i\alpha})\\ &+ \sum_{\beta\in B}\sum_{i\in J_\beta}(e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta}),\end{aligned}$$

where A, B, I, J, I_α and J_β are all finite indexing sets,

$x_i = \exp(w_i)$ for all $i \in J$,

$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ with $R_\alpha \in C(Y)$ for all $i \in I_\alpha, \alpha \in A$, and

$z_{i\beta} = \log S_\beta(y_{i\beta})$ with $S_\beta \in C(Y)\backslash\{0\}$ for all $i \in J_\beta, \beta \in B$.

Similar to Lemma 3.2, we may assume that F is algebraically closed.

Step 1. We may assume that $R_\alpha \neq C$ for all $\alpha \notin A$, by the same reasoning as in Lemma 3.2.

Step 2. For each $i \in J$, we have that $D(x_i) = D(w_i)x_i$, then by Lemma 2.2, we get $x_i \in F$ and there exist $\lambda_i \in C$, $p_i \in F$ such that $w_i = \lambda_i t + p_i$. For each $\alpha \in A$, $i \in I_\alpha$, we have that $D(x_{i\alpha}) = D(R_\alpha(w_{i\alpha}))x_{i\alpha}$, then by Lemma 2.2, we get $x_{i\alpha} \in F$ and there exist $\lambda_{i\alpha} \in C$, $p_{i\alpha} \in F$ such that $R_\alpha(w_{i\alpha}) = \lambda_{i\alpha}t + p_{i\alpha}$.

Step 3. For each $\beta \in B$, $i \in J_\beta$, we have that $D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta})$, then by Lemma 2.2, we get $S_\beta(y_{i\beta}) \in F$ and there exist $\bar{\lambda}_{i\beta} \in$

C, $q_{i\beta} \in F$ such that $z_{i\beta} = \bar{\lambda}_{i\beta} t + q_{i\beta}$. Since $S_\beta(y_{i\beta}) \in F$ and F is algebraically closed, $y_{i\beta} \in F$.

Step 4. Let K be a finite Galois extension over $F(t)$ containing $\{w_i^{r_i}; i \in J\}$ and let σ be an element of the Galois group of K over $F(t)$. Then

$$\begin{aligned}\gamma = \sigma(\gamma) =& D(\sigma v_0) + \sum_{i\in I} a_i D(\sigma v_i)/(\sigma v_i) + \sum_{i\in J} b_i D(\sigma w_{i\alpha}) P_i(x_i) \\ &+ \sum_{\alpha\in A}\sum_{i\in I_\alpha} (c_{i\alpha} D(\sigma w_{i\alpha}) + d_{i\alpha} D(\sigma w_{i\alpha})/(\sigma w_{i\alpha})) G(x_{i\alpha}) \\ &+ \sum_{\beta\in B}\sum_{i\in J_\beta} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) H_\beta(z_{i\beta}).\end{aligned}$$

Summing over all σ yields, for some M in $\mathbb{Z}$,

$$\begin{aligned}M\gamma = & D(Tv_0) + \sum_{i\in I} a_i D(Nv_i)/(Nv_i) + \sum_{i\in J} b_i D(Tw_i^{r_i}) P_i(x_i) \\ &+ \mathcal{E}_1 + M\mathcal{E}_2,\end{aligned} \tag{3.2}$$

where

$$\mathcal{E}_1 = \sum_{\alpha\in A}\sum_{i\in I_\alpha} (c_{i\alpha} D(Tw_{i\alpha}) + d_{i\alpha} D(Nw_{i\alpha})/(Nw_{i\alpha})) G(x_{i\alpha}),$$

$$\mathcal{E}_2 = \sum_{\beta\in B}\sum_{i\in J_\beta} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) H_\beta(z_{i\beta}),$$

and T and N denote the trace and norm of K into $F(t)$, respectively.

Step 5. Consider $\sum_{i\in I} a_i D(Nv_i)/(Nv_i)$. Write $Nv_i = k_i \prod_{j=1}^{\alpha_i} (t-\mu_j)^{n_{ij}}$ where the $\alpha_i \in \mathbb{Z}^+$, the $k_i \in F\backslash\{0\}$, the $\mu_j \in F$ and the $n_{ij} \in \mathbb{Z}$. So $\sum_{i\in I} a_i D(Nv_i)/(Nv_i) = \sum_{i\in I} a_i D(k_i)/k_i +$ an element in $F(t)\backslash F[t]$.

Step 6. Now consider $\sum_{i\in J} b_i D(Tw_i^{r_i}) P_i(x_i)$. For each $i \in J$, recall $w_i = \lambda_i t + p_i$. Write

$$\sum_{i\in J} biD(Tw_i^{r_i}) P_i(x_i) = \sum_{\substack{i\in J\\ \lambda_i = 0}} b_i D(Tw_i^{r_i}) P_i(x_i) + \sum_{\substack{i\in J\\ \lambda_i \neq 0}} b_i D(Tw_i^{r_i}) P_i(x_i).$$

Consider $i \in J$ for which $\lambda_i = 0$. Hence $w_i \in F$ and also $w_i^{r_i} \in F$. So $Tw_i^{r_i} = Mw_i^{r_i}$. Thus

$$\sum_{\substack{i \in J \\ \lambda_i = 0}} b_i D(Tw_i^{r_i}) P_i(x_i) = \sum_{\substack{i \in J \\ \lambda_i = 0}} b_i M D(w_i^{r_i}) P_i(x_i).$$

Now consider $i \in J$ for which $\lambda_i \neq 0$. For these i, $-1 \leq r_i \leq 1$. If $r_i = 1$, then $Tw_i = Mw_i$ and $D(Tw_i) = MD(x_i)/x_i$. If $r_i = -1$, then $Tw_i^{-1} = Mw_i^{-1}$ and $D(Tw_i^{-1}) = -MD(w_i)/(\lambda_i t + p_i)^2$. If $-1 < r_i < 1$, then write $r_i = s_i/h_i$ where s_i and $h_i (> 1)$ are relatively prime in $\mathbb{Z}$. Here $w_i^{r_i}$ satisfy $Y^{h_i} - (\lambda_i t + p_i)^{s_i} = 0$. By Lemma 2.1, $Y^{h_i} - (\lambda_i t + p_i)^{s_i}$ is irreducible in $F(t)[Y]$. Hence $Tw_i^{r_i} = 0$. Thus $D(Tw_i^{r_i}) = 0$. Therefore

$$\sum_{\substack{i \in J \\ \lambda_i \neq 0}} b_i D(Tw_i^{r_i}) P_i(x_i) = \sum_{\substack{i \in J \\ \lambda_i \neq 0, r_i = 1}} b_i M \frac{D(x_i)}{x_i} P_i(x_i) + \text{ elements in } F(t) \backslash F[t].$$

Note that $P_i(x_i)/x_i$ is a rational function of x_i with constant coefficients. It follows that

$$\sum_{\substack{i \in J \\ \lambda_i \neq 0, r_i = 1}} b_i M \frac{D(x_i)}{x_i} P_i(x_i) = D(u_o) + \sum_{i \in \bar{J}} d_i D(u_i)/u_i + \text{ elements in } F(t) \backslash F[t],$$

where $u_o \in F$, $d_i \in C$, $u_i \in F \backslash \{0\}$ for all $i \in \bar{J}$ and $\bar{J}$ is a finite indexing set. So

$$\begin{aligned} &\sum_{i \in J} b_i D(Tw_i^{r_i}) P_i(x_i) \\ &= \sum_{\substack{i \in J \\ \lambda_i = 0}} b_i M D(w_i^{r_i}) P_i(x_i) + D(u_0) + \sum_{i \in \bar{J}} d_i D(u_i)/u_i \\ &\quad + \text{ elements in } F(t) \backslash F[t]. \end{aligned}$$

Step 7. Next, consider $\mathcal{E}_1$. Recall $R_\alpha(w_{i\alpha}) = \lambda_{i\alpha} t + p_{i\alpha}$ for all $i \in I_\alpha, \alpha \in A$.

Case 7.1. Assume that $\lambda_{i\alpha} = 0$. For these $\alpha, i, R_\alpha(w_{i\alpha}) \in F$ and thus $w_{i\alpha} \in F$. So $Tw_{i\alpha} = Mw_{i\alpha}$ and $Nw_{i\alpha} = w_{i\alpha}^M$. Hence $D(Tw_{i\alpha}) = MD(w_{i\alpha})$ and $D(Nw_{i\alpha})/(Nw_{i\alpha}) = MD(w_{i\alpha})/w_{i\alpha}$.

Case 7.2. Assume that $\lambda_{i\alpha} \neq 0$. Write $R_\alpha(Y) = A_\alpha(Y)/B_\alpha(Y)$ where A_α and B_α are relatively prime in $C[Y]$ and $B_\alpha \neq 0$. Each $w_{i\alpha}$ satisfies $A_\alpha(Y) - (\lambda_{i\alpha}t + p_{i\alpha})B_\alpha(Y) = 0$. By Lemma 2.4, $A_\alpha(Y) - (\lambda_{i\alpha}t + p_{i\alpha})B_\alpha(Y)$ is irreducible in $F(t)[Y]$. So the trace and norm can be read of its coefficients. Therefore

$$Tw_{i\alpha} = m_{i\alpha}\Big(\frac{\delta_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \varepsilon_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \nu_{i\alpha}}\Big)$$

and

$$Nw_{i\alpha} = \Big(\frac{\zeta_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \eta_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \nu_{i\alpha}}\Big)^{m_{i\alpha}},$$

where $\delta_{i\alpha}$, $\varepsilon_{i\alpha}$, $\zeta_{i\alpha}$, $\eta_{i\alpha}$, $\mu_{i\alpha}$, $\nu_{i\alpha} \in C$ and $m_{i\alpha} \in \mathbb{Z}^+$.

Therefore

$$D(Tw_{i\alpha}) = m_{i\alpha}\frac{(\nu_{i\alpha}\delta_{i\alpha} - \varepsilon_{i\alpha}\mu_{i\alpha})(\lambda_{i\alpha}D(t) + D(p_{i\alpha}))}{(\mu_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \nu_{i\alpha})^2}$$

and

$$\frac{D(Nw_{i\alpha})}{Nw_{i\alpha}} = m_{i\alpha}\left(\frac{\zeta_{i\alpha}(\lambda_{i\alpha}D(t) + D(p_{i\alpha}))}{\zeta_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \eta_{i\alpha}} - \frac{\mu_{i\alpha}(\lambda_{i\alpha}D(t) + D(p_{i\alpha}))}{\mu_{i\alpha}(\lambda_{i\alpha}t + p_{i\alpha}) + \nu_{i\alpha}}\right).$$

The head of $D(Nw_{i\alpha})/(Nw_{i\alpha})$ is 0. Now, consider $D(Tw_{i\alpha})$. If $\mu_{i\alpha} \neq 0$, then the head of $D(Tw_{i\alpha})$ is 0. Hence

$$D(Tw_{i\alpha}) = \Big(\frac{m_{i\alpha}\delta_{i\alpha}}{\nu_{i\alpha}}\Big)\Big(\frac{D(x_{i\alpha})}{x_{i\alpha}}\Big),$$

and so

$$\begin{aligned}&\sum_{\substack{\alpha\in A\\ \lambda_{i\alpha}\neq 0}}\ \sum_{\substack{i\in I_\alpha\\ \mu_{i\alpha}=0}} c_{i\alpha}D(Tw_{i\alpha})G_\alpha(x_{i\alpha})\\ =&\sum_{\substack{\alpha\in A\\ \lambda_{i\alpha}\neq 0}}\ \sum_{\substack{i\in I_\alpha\\ \mu_{i\alpha}=0}} \Big(\frac{c_{i\alpha}m_{i\alpha}\delta_{i\alpha}}{\nu_{i\alpha}}\Big)\Big(\frac{D(x_{i\alpha})}{x_{i\alpha}}\Big)G_\alpha(x_{i\alpha}).\end{aligned}$$

Note that $\frac{G_\alpha(x_{i\alpha})}{(x_{i\alpha})}$ is a rational function $x_{i\alpha}$ with constant coefficients. It follows that

$$\begin{aligned}
&\sum_{\substack{\alpha\in A\\ \lambda_{i\alpha}\neq 0}} \sum_{\substack{i\in I_\alpha\\ \mu_{i\alpha}=0}} \Big(\frac{c_{i\alpha}m_{i\alpha}\delta_{i\alpha}}{\nu_{i\alpha}}\Big)\Big(\frac{D(x_{i\alpha})}{x_{i\alpha}}\Big) G_\alpha(x_{i\alpha}) \\
&= D(\bar{w}_0) + \sum_{i\in\bar{I}} \bar{c}_i D(\bar{w}_i)/(\bar{w}_i),
\end{aligned}$$

where $\bar{c}_i \in C$, the $\bar{w}_i$ are in F and $\bar{I}$ is the finite indexing set. Therefore

$$\begin{aligned}
\mathcal{E}_1 &= \sum_{\alpha\in A}\sum_{i\in I_\alpha} (c_{i\alpha}D(Tw_{i\alpha}) + d_{i\alpha}D(Nw_{i\alpha})/(Nw_{i\alpha}))G_\alpha(x_{i\alpha}) \\
&= M\sum_{\alpha\in A} \sum_{\substack{i\in I_\alpha\\ \lambda_{i\alpha}=0}} (c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/(w_{i\alpha}))G_\alpha(x_{i\alpha}) \\
&\quad + D(\bar{w}_0) + \sum_{i\in\bar{I}} \bar{c}_i D(\bar{w}_i)/(\bar{w}_i) \\
&\quad + \text{ an element in } F(t)\backslash F[t].
\end{aligned}$$

Step 8. Finally, consider $\mathcal{E}_2$. For each $i \in J_\beta$, $\beta \in B$, recall $z_{i\beta} = \bar{\lambda}_{i\beta} + q_{i\beta}$.

Step 8.1. Assume $\bar{\lambda}_{i\beta} = 0$. Hence $z_{i\beta} \in F$ and so $H_\beta(z_{i\beta}) \in F$. Clearly,

$$\sum_{\beta\in B} \sum_{\substack{i\in J_\beta\\ \bar{\lambda}_{i\beta}=0}} (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})H_{i\beta}(z_{i\beta}) \in F.$$

Step 8.2. Assume $\bar{\lambda}_{i\beta} \neq 0$. Since deg(numerator of H_β) $\leq$ deg(denominator of H_β),

$$H_\beta(Y) = \sum_{i=1}^{n_\beta}\sum_{j=1}^{r_i} \Big(\frac{a_{ij}}{(Y-\alpha_i)^j}\Big) + q_\beta,$$

where $n_\beta, r_i \in \mathbb{Z}^+$, and $a_{ij}, \alpha_i, q_\beta \in C$, $a_{ij} \neq 0$. Hence

$$\begin{aligned}&\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar\lambda_{i\beta}\neq 0}}(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})H_{i\beta}(z_{i\beta})\\ =&\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar\lambda_{i\beta}\neq 0}}\big(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta}\big)q_\beta\\ &+\text{ an element in } F(t)\backslash F[t].\end{aligned}$$

Step 9. From (3.2), we conclude that

$$\begin{aligned}M\gamma = {}& D(T\nu_0)+\sum_{i\in I}a_iD(k_i)/k_i + M\sum_{\substack{i\in J\\ \lambda_i=0}}b_iD(w_i^{r_i})P_i(x_i)\\ &+D(u_0)+\sum_{i\in\bar J}d_iD(u_i)/u_i\\ &+M\sum_{\alpha\in A}\sum_{\substack{i\in I_\alpha\\ \lambda_{i\alpha}=0}}\big(c_{i\alpha}D(w_{i\alpha})+d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}\big)G_\alpha(x_{i\alpha})\\ &+D(\bar w_0)+\sum_{i\in\bar I}\bar c_iD(\bar w_i)/\bar w_i \qquad (3.3)\\ &+M\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar\lambda_{i\beta}=0}}(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})H_{i\beta}(z_{i\beta})\\ &+M\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar\lambda_{i\beta}\neq 0}}(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})q_\beta\\ &+\text{ an element in } F(t)\backslash F[t].\end{aligned}$$

Step 10. Now consider $D(Tv_0)$. Write $Tv_0 = \sum_{j=0}^{n}\bar v_j t^j +$ an element in $F(t)\backslash F[t]$, where n is nonnegative integer and the $\bar v_j \in F$ and $\bar v_n \neq 0$. So

$$\begin{aligned}D(Tv_0) = {}& D(T\bar v_n)t^n + \sum_{j=1}^{n}(j\bar v_jD(t)+D(\bar v_{j-1}))t^{j-1} \qquad (3.4)\\ &+\text{ an element in } F(t)\backslash F[t].\end{aligned}$$

We now prove that $n \leq 1$, suppose that $n > 1$. Replacing (3.4) in (3.3), we have that the right hand side of (3.3) would contain an expression of the form t^i with $i \geq 2$. Comparing terms of degree n and $n-1$ in (3.3), $D(\bar{v}_n) = 0$ and $(n\bar{v}_n D(t) + D(\bar{v}_{n-1})) = 0$. Since $D(\bar{v}_n) = 0$, $\bar{v}_n \in C$. Thus $D(n\bar{v}_n t + \bar{v}_{n-1}) = n\bar{v}_n D(t) + D(\bar{v}_{n-1}) = 0$. So $\bar{v}_n t + \bar{v}_{n-1} \in C$. Thus t is algebraic over F, a contradiction. So we have the claim. From (3.4), we get $D(Tv_0) = D(\bar{v}_1)t + (\bar{v}_1 D(t) + D(\bar{v}_0))+$ an element in $F(t)\backslash F[t]$. Clearly, $\bar{v}_1 \in C$. Hence $D(Tv_0) = \bar{v}_1 D(t) + D(\bar{v}_0)+$ an element in $F(t)\backslash F[t]$.

Step 11. Replacing $D(Tv_0)$ in (3.3) and comparing the head, we get

$$
\begin{aligned}
M\gamma = {} & \bar{v}_1 D(t) + D(\bar{v}_0 + u_0 + \bar{w}_0) \\
& + \sum_{i \in I} a_i D(k_i)/k_i + \sum_{i \in \bar{J}} d_i D(u_i)/u_i \\
& + \sum_{i \in \bar{I}} \bar{c}_i D(\bar{w}_i)/\bar{w}_i \\
& + M \sum_{\substack{i \in J \\ \lambda_i = 0}} b_i D(w_i^{r_i}) P_i(x_i) \\
& + M \sum_{\alpha \in A} \sum_{\substack{i \in I_\alpha \\ \lambda_{i\alpha} = 0}} \big(c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}\big) G_\alpha(x_{i\alpha}) \\
& + M \sum_{\beta \in B} \sum_{\substack{i \in J_\beta \\ \bar{\lambda}_{i\beta} = 0}} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) H_{i\beta}(z_{i\beta}) \\
& + M \sum_{\beta \in B} \sum_{\substack{i \in J_\beta \\ \bar{\lambda}_{i\beta} \neq 0}} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) q_\beta.
\end{aligned}
$$

Dividing by M, we obtain the correct sum γ. □

Proof of Theorem 3.1. Let $m =$ tr. deg. E/F. The proof is by induction on m. If $m = 0$, then E is algebraic over F, and the theorem is trivially true. Assume that $m > 0$. Suppose that the theorem is true for any Ei-Gamma extension L of a field F' such that tr. deg. $L/F' < m$. Since tr. deg. $E/F = m$, we can choose a transcendence basis $t_1, \ldots, t_m$ of E over F such that $F = F_0 \subset F(t_1) = F_1 \subset \cdots \subset F(t_1, \ldots, t_m) = F_m \subset E$ and each t_i satisfies one of the following conditions :

(1) $t_i = \exp(u)$ for some nonzero u in $\bar{F}_{i-1} \cap E$,

(2) $t_i = \log(u)$ for some nonzero u in $\bar{F}_{i-1} \cap E$,

(3) there are $G \in C(Y)$, u and nonzero v in $\bar{F}_{i-1} \cap E$, $r \in \mathbb{Q}$ with $-1 \leq r \leq 1$ such that $D(t_i) = D(u^r)G(v)$, where $v = \exp(u)$

(4) there are $G \in C(Y)$, nonzero u and v in $\bar{F}_{i-1} \cap E$ such that $D(t_i) = D(u)G(v)$ or $D(t_i) = (D(u)/u)G(v)$ where $v = \exp(u)$ for some $R \in C(Y)$

(5) there are $H \in C(Y)$, with deg(numerator of H) $\leq$ deg(denominator of H), nonzero u and v in $\bar{F}_{i-1} \cap E$ such that $D(t_i) = D(u)H(v)$ or $D(t_i) = (D(u)/u)H(v)$ where $v = \log S(u)$ for some $S \in C(Y)$ ($\bar{F}_{i-1}$ denote the algebraic closure of F_{i-1}).

Note that E is also a Ei-Gamma extension of F_1 and tr. deg. $E/F_1 = m - 1 < m$. So by induction hypothesis, we set that γ is an Ei-Gamma element in F_1. By Lemma 3.2 and Lemma 3.3, we get the result of the theorem. □

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