39. A Generalization of Local Class Field Theory by Using K-groups. I

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§0. **Introduction.** This note is a summary of our recent results on a generalization of local class field theory. Details will be published elsewhere.

Let F be a field which is complete with respect to a discrete valuation and with finite residue field. Let K be a field which is complete with respect to a discrete valuation and with residue field F. In this Part I, we shall study abelian extensions of K. The case in which F is a function field of one variable over a finite field and a generalization of our results will be studied in Part II ([1]).

§1. In Part I, let F denote a field which is complete with respect to a discrete valuation and with finite residue field, and let K denote a field which is complete with respect to a discrete valuation and with residue field F, and let K^{ab} denote the maximum abelian extension of K.

Theorem 1. (1) *There exists a canonical homomorphism*

$$\Phi : K_2(K) \longrightarrow \mathrm{Gal}\,(K^{ab}/K)$$

having the following property: For each finite abelian extension L of K, Φ *induces an isomorphism*

$$K_2(K)/N_{L/K}K_2(K) \cong \mathrm{Gal}\,(L/K),$$

where $N_{L/K}$ *denotes the norm map in* K_2*-theory.*

(2) $L \mapsto N_{L/K}K_2(L)$ *is a bijection from the set of all finite abelian extensions of K in a fixed algebraic closure of K to the set of all open subgroups of finite indices of* $K_2(K)$ *with respect to the topology defined later in* §4.

This is closely connected with the following result on the Brauer group of K.

Theorem 2. *There exists a canonical isomorphism*

$$\Psi : \mathrm{Br}\,(K) \xrightarrow{\sim} \mathrm{Hom}_c\,(K^*, \boldsymbol{Q}/\boldsymbol{Z})_{\mathrm{tor}}$$

having the following property, where K^* *denotes the multiplicative group of* K *and* $\mathrm{Hom}_c\,(K^*, \boldsymbol{Q}/\boldsymbol{Z})_{\mathrm{tor}}$ *denotes the torsion part of the group of all continuous homomorphism* $K^* \rightarrow \boldsymbol{Q}/\boldsymbol{Z}$ *with respect to the topology*

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defined later (§ 4): *For each central simple algebra A over K, the kernel of $\Psi(\{A\})$ is $Nrd_{A/K}A^*$ where Nrd denotes the reduced norm.*

§ 2. The definitions of the p-primary parts of Φ and Ψ in the mixed characteristic case. (Cf. Part II for the case $ch(K)=p$.)

Suppose that $ch(F)=p>0$ and $ch(K)=0$. (ch denotes the characteristic of a field.) Let K_{nr} be the maximum unramified extension of K and F_s be the residue field of K_{nr}, so that F_s is the separable closure of F. Put $G=\mathrm{Gal}\,(K_{nr}/K)\cong\mathrm{Gal}\,(F_s/F)$. Let r be any natural number. Consider the following diagram of G-modules:

$$(1)\qquad \begin{array}{ccc} K_2(K_{nr})/K_2(K_{nr})^{p^r} & \xrightarrow[g]{} & H^2(K_{nr},\mu_{p^r}\otimes\mu_{p^r}) \\ \downarrow t & & \\ F_s^*/F_s^{*p^r}. & & \end{array}$$

Here, we use the notation in [3] for the Galois cohomology group, t denotes the tame symbol, g denotes the Galois symbol (cf. [4]), and μ_{p^r} denotes the group of all p^r-th roots of 1. By Proposition 1 (1) below, g is an isomorphism. So, (1) induces a homomorphism

$$(2)\qquad H^1(G,H^2(K_{nr},\mu_{p^r}\otimes\mu_{p^r}))\longrightarrow H^1(G,F_s^*/F_s^{*p^r}).$$

On the other hand, $H^1(G,H^2(K_{nr},\mu_{p^r}\otimes\mu_{p^r}))\cong H^3(K,\mu_{p^r}\otimes\mu_{p^r})$ by Proposition 1 (2) below, and

$$H^1(G,F_s^*/F_s^{*p^r})\cong\frac{1}{p^r}Z/Z$$

by ordinary local class field theory. So, (2) induces a homomorphism

$$(3)\qquad H^3(K,\mu_{p^r}\otimes\mu_{p^r})\longrightarrow\frac{1}{p^r}Z/Z,$$

which is in fact an isomorphism.

Now, (3) induces two homomorphisms

$$(4)\qquad \begin{array}{l} K_2(K)/K_2(K)^{p^r}\otimes\mathrm{Hom}_c\,(\mathrm{Gal}\,K^{ab}/K),Z/p^r) \\ \xrightarrow{b} H^2(K,\mu_{p^r}\otimes\mu_{p^r})\otimes H^1(K,Z/p^r) \\ \xrightarrow{c} H^3(K,\mu_{p^r}\otimes\mu_{p^r})\longrightarrow\frac{1}{p^r}Z/Z, \end{array}$$

and

$$(5)\qquad \begin{array}{l} K^*/K^{*p^r}\otimes\mathrm{Br}(K)_{p^r} \\ \xrightarrow{b'} H^1(K,\mu_{p^r})\otimes H^2(K,\mu_{p^r}) \\ \xrightarrow{c'} H^3(K,\mu_{p^r}\otimes\mu_{p^r})\longrightarrow\frac{1}{p^r}Z/Z, \end{array}$$

where:

Hom_c is the group of continuous homomorphisms,

b is the tensor product of the Galois symbol and the canonical isomorphism $\mathrm{Hom}_c\,(\mathrm{Gal}(K^{ab}/K),Z/p^r)\cong H^1(K,Z/p^r)$,

b' is the tensor product of $K^*/K^{*p^r}\cong H^1(K,\mu_{p^r})$ and $\mathrm{Br}(K)_{p^r}\cong H^2(K,\mu_{p^r})$, where $\mathrm{Br}(K)_{p^r}$ denotes the group $\{w\in\mathrm{Br}(K)\mid p^r w=0\}$,

c and c' are the cup products.

Consequently, we have a homomorphism from $K_2(K)$ to the pro-p-part of $\mathrm{Gal}(K^{ab}/K)$ by (4) and a homomorphism from the p-primary part of $\mathrm{Br}(K)$ to $\mathrm{Hom}(K^*, \boldsymbol{Q}_p/\boldsymbol{Z}_p)$ by (5). These are the definitions of the p-primary parts of Φ and Ψ.

Proposition 1. *Let S be a field which is complete with respect to a discrete valuation and with residue field E. Suppose that $ch(E)=p>0$, $ch(S)=0$ and $[E:E^p]=p$. Then,*

(1) *the Galois symbol $K_2(S)/K_2(S)^{p^r} \to H^2(S, \mu_{p^r} \otimes \mu_{p^r})$ is an isomorphism for each $r \geqq 0$.*

(2) *Suppose further that E is separably closed. Then $cd_p(S)=2$. (Cf. [3] for the notation cd_p.)*

We need Proposition 2 (2) below to prove Proposition 1 (2).

Definition for Proposition 2. For each $i=0,1,2$, we call a field S a *B_i-field* if and only if for each finite extension T of S and for each finite extension T' of T, the norm map $N_{T'/T}: K_i(T') \to K_i(T)$ is surjective.

This is an analogy of the concept "C_i-field". We can prove that a C_i-field is a B_i-field for each $i=0,1,2$.

Proposition 2. *Let S be a field which is complete with respect to a discrete valuation and with residue field E. Suppose that E is a B_1-field. Then:*

(1) *For each central simple algebra A over S, $Nrd: A^* \to S^*$ is surjective.*

(2) *S is a B_2-field.*

Proposition 2 is an analogy of the following well known fact. "A field which is complete with respect to a discrete valuation is B_1 if its residue field is B_0 (i.e. algebraically closed)."

§3. The definitions of the "prime to p" parts of Φ and Ψ.

Let n be any natural number which is not divisible by $ch(F)$. Let G and K_{nr} be as in §2. Then we have

$$(6)\qquad \begin{aligned} H^3(K, \mu_n \otimes \mu_n) &\cong H^2(G, H^1(K_{nr}, \mu_n \otimes \mu_n)) \\ &\cong H^2(G, \mu_n) \cong \frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}, \end{aligned}$$

which can be easily deduced by the known facts in [3]. The composite of (6) induces a homomorphism from $K_2(K)$ to the "prime to p" part of $\mathrm{Gal}(K^{ab}/K)$ and a homomorphism from the "prime to p" part of $\mathrm{Br}(K)$ to $\mathrm{Hom}(K^*, \boldsymbol{Q}/\boldsymbol{Z})$ in the same way as in §2. These are the definitions of the "prime to p" parts of Φ and Ψ.

This simple argument cannot be adopted in case of §2. The main difficulty in our theory lies in the p-primary part in the mixed characteristic case.

§4. The topologies of K^* and $K_2(K)$. In case $ch(F)=0$, we take the discrete topologies of K^* and $K_2(K)$. In what follows, suppose that $ch(F)=p>0$.

Let R be the ring of integers of K, and m be the maximal ideal of R. First, we define the canonical topology of R/m^n for each n. Let $W(F)$ be the Witt ring of F (cf. [2]). Choose r such that $r \geqq n-1$. Then there exists a unique ring-homomorphism $w_r: W(F) \rightarrow R/m^n$ such that

$$w_r(\bar{a}_0, \bar{a}_1, \bar{a}_2, \cdots) \equiv \sum_{i=0}^{r} p^i a_i^{p^{r-i}} \bmod m^n$$

for all $a_i \in R$, where $\bar{a}_i$ denotes the residue class of a_i. By w_r, R/m^n becomes a finitely generated $W(F)$-module. We define the topology of R/m^n by regarding R/m^n as a quotient $W(F)$-module of a finite product of $W(F)$. (Here the topology of $W(F)$ is the product topology of the valuation topology of F.) This topology of R/m^n is independent of the choice of r. In this way, R/m^n becomes a topological ring and $(R/m^n)^*$ becomes a topological group for the induced topology.

We define the topology of R^* by regarding R^* as the inverse limit of $(R/m^n)^*$ as $n \rightarrow \infty$. We define the topology of K^* in such a way that R^* becomes as open subgroup of K^*.

Finally, we define the topology of $K_2(K)$ by the following characterization. For each commutative topological group H and for each group-homomorphism $h: K_2(K) \rightarrow H$, h is continuous if and only if the composite map

$$K^* \times K^* \longrightarrow H : (x, y) \longmapsto h(\{x, y\})$$

is continuous.

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