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A GENERALIZATION OF MAGNUS' THEOREM

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Let $f(x, y)$ and $g(x, y)$ be polynomials in two variables with integral coefficients. O.H. Keller raised the problem in [1]: If the functional determinant $\partial(f, g)/\partial(x, y)$ is equal to 1, then is it possible to represent x and y as polynomials of f and g with integral coefficients? This problem drew many mathematicians' attension and several attempts have been made by enlarging the coefficient domain to the complex number field $\boldsymbol{C}$. But no success has been reported yet. On the other hand A. Magnus studied the volume preserving transformation of complex planes and obtained a result which is relevant to Keller's problem ([2]). From his results it is immediately deduced that Keller's problem is answered affirmativiely provided one of $f(x, y)$ and $g(x, y)$ has prime degree. For the proof Maguns used recursive formulas. But these formulas are complicated and not easy to handle. In this paper we shall give a simple proof of his theorem based on the notion of quasi-homogeneity for generalized polynomials. Moreover we shall go one step further than he did. Our results ensure that Keller's problem is valid provided one of $f(x, y)$ and $g(x, y)$ has degree 4 or larger degree is of the form $2p$ with an odd prime p. Since a complete solution of Keller's problem is not found yet our paper will be of some interest and worth-while publication.

1. Quasi-homogeneous generalized polynomials

Let x and y be two indeterminates. We shall set $\tilde{A}=\sum_{i,j\in \boldsymbol{Z}} \boldsymbol{C}\, x^i y^j$ where $\boldsymbol{C}$ is the complex number field and $\boldsymbol{Z}$ is the ring of rational integers. $\tilde{A}$ is a graded ring and the polynomial ring $\boldsymbol{C}[x, y]$ is a graded subring. Hereafter we shall call an element $f(x, y)$ of $\tilde{A}$ a generalized polynomial or simply a g-polynomial. We shall denote by $S(f)$ the set of lattice points (i, j) in the real two space $\boldsymbol{R}^2$ such that the monomial $x^i y^j$ appears in $f(x, y)$ with a non-zero coefficient. $S(f)$ will be called the *supoprt* of $f(x, y)$. A g-polynomial $f(x, y)$ is called a homogeneous g-polynomial or a g-form if $S(f)$ lies in the straight line of the form $X+Y=m$ where $m\in \boldsymbol{Z}$ and is called the degree of the g-form $f(x, y)$.

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We shall use the symbol $S[f]$ to denote the set of monomials $x^i y^j$ such that the lattice point (i, j) is in $S(f)$.

Proposition 1. *Let $f(x, y)$ and $g(x, y)$ be non-constant g-forms of degrees m and n respectively such that the functional deteriminant $\partial(f, g)/\partial(x, y)$ is equal to zero. We shall define an integer d by the rule:* (a) *d is equal to the* GCD of *$|m|$ and $|n|$ if one of m and n is postive,* (*b*) *d is equal to the negative of* GCD $(|m|, |n|)$ *if both of m aud n are negative. We shall set $m/d=m'$ and $n/d=n'$. Then we have the following:*

(i) *If one of m and n is zero, so is the other and $f(x, y)$ and $g(x, y)$ are g-polynomials in one variable (y/x).*

(ii) *If $mn<0$, then both of $f(x, y)$ and $g(x, y)$ are monomials and there exist a monomial $h(x, y)$ of degree d such that $f=c_1h^{m'}$, and $g=c_2h^{n'}$ where $c_i(i=1, 2)$ are constants.*

(iii) *If $mn>0$, there exists a g-form $h(x, y)$ of degree d such that $f=c_1h^{m'}$ and $g=c_2h^{n'}$.*

Proof. Assume first $m=0$ and $n\neq 0$. It follows from $\partial(f, g)/\partial(x, y)=0$ that we have $\partial f/\partial x=\partial f/\partial y=0$. This is against the assumption. Since a g-form of degree zero is necessarily of the form $\sum_{i\in Z} a_i(y/x)^i$ we get the assertion (i). To prove (ii) we assume $m>0$ and $n<0$ and let $f_1=f^{-n}$ and $g_1=g^m$. Then $\partial(f_1, g_1)/\partial(x_1, y_1)=0$. Since the degrees of f_1 and g_1 differ only in sign we see immediately that $f_1\frac{\partial g_1}{\partial x}+g_1\frac{\partial f_1}{\partial x}=0$, or equivalently, $\partial(f_1g_1)/\partial x=0$. Similarly we have $\partial(f_1g_1)/\partial y=0$. Hence f_1g_1 must be a constant. But such a case can occur only when f_1, hence f, is a monomial because g_1 is a g-polynomial. The rest follows easily from this. The proof of (iii) will be carried out by a similar device and the detailed proof will be omitted.

DEFINITION. A g-polynomial $f(x, y)$ is called a quasi homogeneous g-polynomial (or simply a quasi g-form) if the support $S(f)$ of $f(x, y)$ is contained in the straight line. When the equation of that straight line has the form $Y+\alpha X=\lambda$. We shall say that the quasi g-form $f(x, y)$ is (α)-homogeneous of degree λ.

It should be noticed that if α is an irrational number, monomials only can be (α)-homogeneous g-forms.

Proposition 2. *Let $f(x, y)$ and $g(x, y)$ be (α)-homogeneous g-forms of positive degrees λ and μ respectively such that $\partial(f, g)/\partial(x, y)=0$. Assume that α is a rational nnmber q/p with comprime integers $p\,(>0)$ and q.* Let $d=$ GCD$(p\lambda, p\mu)$. *Then there exists an (α)-homogeneous g-form $h(x, y)$ of degree d/p such that $f=c_1h^{m'}$ and $g=c_2h^{n'}$ where $m'=p\lambda/d$, $n'=p\mu/d$ and $c_i(i=1, 2)$ are constants.*

Proof. Let u, v be new indeterminates and let $x=u^p$ and $y=v^q$. Then $F(u, v)=f(u^p, v^q)$ and $G(u, v)=g(u^p, v^q)$ are g-forms of degrees $p\lambda$ and $p\mu$ respectively. The rest follows easily from Proposition 1.

Let γ be an arbitrary real number. Then we can define a grading on $\tilde{A}$ in the following way. Let λ be a real number and let $\tilde{A}_\lambda$ be the vector space over $\boldsymbol{C}$ generated by the set of g-monomials $x^i y^j$ such that $j+\gamma i=\lambda$. Then we have $\tilde{A}=\bigoplus_\lambda \tilde{A}_\lambda$ where the sum is extended over all real numbers contained in the additive subgroup of $\boldsymbol{R}$ generated by 1 and γ. In case $\gamma=1$ we have the standard grading and its degree function is the ordinary function. The term "homogeneous" is reserved for this standard grading.

Proposition 3. *Let $f(x, y)$ and $g(x, y)$ be g-polynomials in x and y such that $\partial(f, g)/\partial(x, y)\in\boldsymbol{C}$. Let α be any real number and fet $f=\bigoplus f_\lambda$ and $g=\bigoplus g_\mu$ be the direct sum decomposition by the (α)-grading. Then we have*

$$\sum_{\substack{\lambda+\mu=s\\ 1+\alpha\neq s}} \frac{\partial(f_\lambda, g_\mu)}{\partial(x, y)}=0 .$$

The proof is immediate and will be mitted.

2. Magnus' Theorem

For future reference we shall give Magnus' Theorem in a slightly different formulation from Magnus' original one.

Theorem 1. *Let $f(x, y)$ and $g(x, y)$ be polynomials iu two ariables x and y with complex coefficients and let m and n be the degrees of $f(x, y)$ and $g(x, y)$ respectivzely. Assume that the functional determinant $\partial(f, g)/\partial(x, y)$ is a nonzero constant. If Min $(m, n)>1$, then we have* GCD$(m, n)>1$.

Proof. Assume that GCD$(m, n)=1$. Let f_m and g_n be the degree forms of f and g respectively. From proposition 1, there is a linear form, say h, such that $f_m=\varepsilon_1 h^m$ and $g=\varepsilon_2 h^n$. Without loss of generalities we can assume that $h=x$ and $\varepsilon_i=1$. We shall pick up a point $P=(p_1, p_2)$ in $S(f)$ in the following way. Let L be the line defined by the equation $X=m$ and let L rotate around the point $M=(m, 0)$ counterclockwise until L meets a point in $S(f)$ other than M. Let l be the line thus obtained. The point in $S(f)\cap l$ with the smallest X-coordinate is the desired point P. Pick up a point $Q=(q_1, q_2)$ in $S(g)$ in a similar way.

Now ssume we have either $(m>)\ p_2>0$ or $(n>)\ q_2>0$. Then we easily verify that one of the following situation takes place.

(1) The lines MP and NQ are not parallel where $N=(n, 0)$.

(2) The three points P, Q and the origin are not collinear.

If the case (a) occurs let

$$Y+aX = am,\ Y+bx = bn$$

be the equations of the lines MP and NQ respectively. Then we have $a \neq b$. If $a>b$ let γ be a real number such that $a>\gamma>b$. If we choose γ near enough to a, then $x^{p_1}y^{p_2}$ will have the highest (γ)-degree in $S[f]$ and x^n will have the highest (γ)-degree in $S[g]$. Hence by Proposition 3, $\partial(x^{p_1}y^{p_2}, x^n)/\partial(x, y)= np_2x^{n+p_1-1}\,y^{p_2-1}=0$. But this is impossible. Similarly we have a ontradicition if $a<b$.

Now assume the lines MP and NQ are parallel, *i.e.*, $a=b$ then we have the case (2), *i.e.*, $p_2q_1 \neq q_1p_2$. Let $\gamma=a-\varepsilon$ with $\varepsilon<0$. If we choose ε small enough, then $x^{p_1}y^{p_2}$ will have the highest (γ)-degree in $S[f]$ and $x^{q_1}y^{q_2}$ will have the highest (γ)-degree in $S[g]$. But this contradicts Proposition 3 because we have $q_1p_2 \neq q_2p_1$.

Thus we have seen that $p_2=q_2=0$, *i.e.*, $f(x, y)$ and $g(x, y)$ are polynomials in x alone. But this is impossible because $\partial(f,g)/\partial(x,y)$ is a non-zero constant, and the proof of Theorem 1 is complete.

For the sake of reference we shall call the method adopted in this proof "*the method of rotation of lines around the points M and N*".

3. A generalization of Magnus' Theorem

Theorem 2. *Under the same notations and assumptions as Theorem* 1, *we hove the following*: *If* $\mathrm{Min}(m, n)>2$, *then we have* $\mathrm{GCD}(m, n)>2$.

Proof. Assume that $\mathrm{Min}(m, n)>2$ and $\mathrm{GCD}(m, n)=2$ and we shall draw a contradiction. Let f_m and g_n be degree forms of f and g respectively. From Proposition 2 it follows that there exists a quadratic form $h(x, y)$ such that $f_m= ah^{m'}$ and $g_n=bh^{n'}$, where $m=2m'$ and $n=2n'$. There are two possibilites.

(I) h is a product of two independent linear forms. In this case we can assume without loss of generalities that $f_m=(xy)^{m'}$ and $g_n=(xy)^{n'}$. Apply the method of rotation of lines around the points $M_1=(m', m')$ and $N_1=(n', n')$. Then we can easily see that any point (i, j) in $S(f)$ satisfies the condition $j\leq m'$, and any point (s, t) in $S(g)$ satisfies the condition $t\leq n'$.

Now consider the (0)-grading in A. The degree forms of f and g are respectively of the forms

$$f^{(0)}_{m'} = y^{m'}(a_0+a_1x+\cdots+a_{m'-1}x^{m'-1}+x^{m'})$$
$$g^{(0)}_{n'} = y^{n'}(b_0+b_1x+\cdots+b_{n'-1}x^{n'-1}+x^{n'})$$

From Propositions 2 and 3 there is a linear form $c+x$ such that

$$f^{(0)}_{m'} = y^{m'}(c+x)^{m'} \text{ and } g^{(0)}_{n'} = y^{n'}(c+x)^{n'}$$

If we set $x_1=c+x$ and consider f and g as polynomials in new variables x_1 and

$y_1=y$,then the support $S_1(f)$ have no point (i, j) with $j\geq m'$ except the point (m', m'). Similarly $S_1(g)$ have no point (s, t) with $t\geq n'$. Apply again the method of rotation of lines around the points M_1 and N_1. Then we can see finally that no point (i, j) with $i<j$ is in $S(f)$ and no point (s, t) with $s<t$ is in $S(g)$. This means that $f(x, y)$ and $g(x, y)$ lack the terms $y^s (s\geq 1)$. This is impossible because of the assumption $\partial(f, g)/\partial(x, y)$ is an element of $\boldsymbol{C}^*$.

(II) h is a power of a linear form: In this case we can assume as before that the degree forms are of the forms $f_m=x^m$ and $g_n=x^n$ respectively. Then we can see, following the method of rotations of lines around the point $M=(m, 0)$ and $N(n, 0)$, that $S(f)$ is contained in the region defined by the inequality $Y+\frac{1}{2}X\leq\frac{m}{2}$ and (g) is in the region $Y+\frac{1}{2}X\leqq\frac{n}{2}$. Consider (1/2)-grading and apply Propositions 2 and 3. Then we see that degree forms of f and g by this grading are

$$(ay+x^2)^{m'} \text{ and } (ay+x^2)^{m'}$$

respectively. If $a=0$ we can proceed further and we see that no point (i, j) with $j>0$ is in $S(f)$ and no point (s, t) with $t>0$ is in $S(g)$. This is a contradiction. Hence we must have $a\neq 0$. Then apply de Jonquiere transformation

$$Y_1=ay+x^2,\ x_1=x\,.$$

Since we have

$$f(x, y)=(ay+x^2)^{m'}+\sum_{j+i/2<m'} a_{ij}x^i y^j$$

and

$$g(x, y)=(ay+x^2)^{n'}+\sum_{j+i/2<n'} b_{ij}x^i y^j$$

We easily see that

$$f_1(x_1, y_1)=y_1^{m'}+\sum_{j+i/2<m'} a'_{ij}x_1^i y_1^j$$

and

$$g_1(x_1, y_1)=y_1^{n'}+\sum_{j+i/2<n'} b'_{ij}x_1^i y_1^j$$

where

$$f_1(x_1, y_1)=f(x_1, a^{-1}(y_1-x_1^2)) \text{ and } g_1(x_1, y_1)=g_1(x, a^{-1}(y_1-x_1^2))\,.$$

By the method of (clockwise) rotation of lines around the points $(0, m')$ and $(0, n')$ applied to the pair of polynomials $f_1(x_1, y_1)$ and $g_1(x_1, y_1)$, we see that

$S(f_1)$ is in the half plane $X+Y\leq m'$ and $S(g_1)$ is in the half plane $X+Y\leq n'$. This means that $f_1(x_1, y_1)$ is of degree m' and $g_1(x_1, y_1)$ is of degree n'. Moreover $\frac{\partial(f_1, g_1)}{\partial(x_1, y_1)}=a^{-1}\frac{\partial(f, g)}{\partial(x, y)}$ is in $\boldsymbol{C}^*$. Since $\mathrm{Min}(m, n)>2$, we have $\mathrm{Min}(m', n')>1$. Moreover $\mathrm{GCD}(m', n')=1$. This is the situation negated in Theorem 1.

4. Application to Keller's problem

Theorem 3. *Let $f(x, y)$ and $g(x, y)$ be polynomials of degrees m and n respectively with complex coefficients and assume that the functional determinant $\partial(f, g)/\partial(x, y)$ is a non-zero constant. Then we have $\boldsymbol{C}[x, y]=\boldsymbol{C}[f, y), g(x, y)]$ in the following three cases:*

(1) *m or n is a prime number;*

(2) *m or n is 4;*

(3) *$m=2p\geqq n$ where p is an odd prime.*

Proof. In any case it follows from Theorems 1 and 2 that smaller degree, say n, divides larger degree m. Then from Proposition 2 and 3 the degree forms f_m and g_n are related like this, $f_m=\varepsilon g_n^{m/n}$. Then

$$f_1=f-(\varepsilon^{n/m}g)^{m/n}$$

has lower degree than f and $\partial(f_1, g)/\partial(x, y)=\partial(f, g)/\partial(x, y)$ is a non-zero constant. Thus we can use induction on the sum $m+n$ of degrees to get the conclusion. q.e.d.

Keller's Original problem is also settled in these three cases cited in Theorem 3 because of the following

Proposition 4. *Let $f(x, y)$ and $g(x, y)$ be the polynomials in x and y with integer coefficients such that the functional determinant is equal to 1 and $\boldsymbol{C}[f, g]=\boldsymbol{C}[x, y]$. Then we hove necessarily $\boldsymbol{Z}[f, g]=\boldsymbol{Z}[x, y]$.*

Proof. It suffices to prove that x and y are in $\boldsymbol{Z}[f, g]$. By assumption we have

$$x=\sum c_{ij}f^i g^j,\ c_{ij}\in\boldsymbol{C}.$$

If we set

$$f(x, y)=f_{10}x+f_{01}y+\cdots$$

$$g(x, y)=g_{10}x+g_{01}y+\cdots$$

then the assumption implies that $f_{10}g_{01}-f_{01}g_{10}=1$. Apply the unimodular transformation of variables

$$x' = f_{10}x + f_{01}y$$

$$y' = g_{10}x + g_{01}y\,.$$

Then $\boldsymbol{Z}[x, y] = \boldsymbol{Z}[x', y']$ and $f = x' +$(higher degree terms) and $g = y' +$ (higher degree terms). Hence to prove the assertion we can assume without loss of generalities that linear parts of f and g are x and y respectively. We shall define a linear order in the set (i, j) of lattice points in $\boldsymbol{R}^2$ by the way: $(i, j) > (i', j')$ if (i) $i+j > i'+j'$ or (ii) $i+j = i'+j'$ and $i > i'$. We shall show that every c_{ij} is in Z by induction on the linear order just defined. Assume every $c_{i'j'}$ with $(i', j') < (i, j)$ is integer. Then the coefficients of the polynomial

$$c_{ij} f^i g^j + c_{i+1\,j-1} f^{i+1} g^{j-1} + \cdots + c_{0\,i+j+1} g^{i+j+1} + \cdots$$

are integers. In this polynomial $x^i y^j$ appears once with the coefficient c_{ij}. Hence c_{ij} must be an integer. Similarly y is in $\boldsymbol{Z}[f, g]$ and the assertion in proved completely.

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