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Section A :
Physique théorique.

A generalization of metric tensor and its application

by

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ABSTRACT. — The definition of metric tensor and the well known Ricci theorem are modified. The Einstein tensor and Maxwell equations are generalized. The field equations of gravitation and electricity are unified. The introduction of a metric in the space with symmetric connection is defined.

1. INTRODUCTION

The unified field theories as introduced by Einstein and others [*1*]-[*8*] are still unsolved today. These theories, in general, are different modifications of Einstein's gravitation theory and Riemann geometry. Therefore, the author believe that the modifications of these theories only may be the source of difficulties. Hence the unified field equation derived in this paper is based also on the generalization of Maxwell equations, the modifications of Ricci theorem and the definition of metric tensor. It will be shown below that the simplest coefficient of symmetric connection under which the unified field equation can be derived is

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} + 2æ\varepsilon_{\mu\nu}\varepsilon^{\sigma s}\varphi_s$$

where $\left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\}$ is the Christoffel symbol formed with respect to the tensor $\varepsilon_{\rho\sigma}$,

$æ$ is an important universal constant and φ_s is the electromagnetic vector potential. Consequently there are four postulates:

i) In the fields of gravitation and electricity, the coefficient of connection of the space is symmetric and is defined by

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} + 2æ\varepsilon_{\mu\nu}\varepsilon^{\sigma s}\varphi_s \tag{1}$$

ii) The line element in this space is given by

$$ds^2 = \varepsilon_{mk}dx^m dx^k, \qquad \varepsilon_{mk} = \frac{1}{2}(A_{mk} + A_{km}) \tag{2}$$

$$m, k = 1, 2, 3, 4 \qquad x^4 = ict$$

where A_{ik} is given by the solution of the equation:

$$A_{ik;m} = \frac{\partial A_{ik}}{\partial x^m} - A_{hk}\Gamma^h_{im} - A_{ih}\Gamma^h_{km} = 0 \tag{3}$$

The condition of integrability is

$$A_{hk}B^h_{iml} + A_{ih}B^h_{kml} = 0 \tag{4}$$

(2) and (3) are the evident modifications of the well known Ricci theorem and the definition of metric tensor.

The semi-colon always indicates covariant differentiation with respect to $\Gamma^{\alpha}_{\beta\gamma}$. The determinant of A_{mk} does not vanish so that

$$A^{ik}A_{mk} = A^{ki}A_{km} = \delta^i_m.$$

In the following sections the tensors A_{mk} and A^{mk} always play the role of lowering and raising the indices respectively.

iii) Since the energy and mass are equivalent, the electromagnetic field is therefore equivalent to a gravitational field. Hence in the absence of gravitation, the space-time of an electromagnetic field is not « flat » *i. e.* the line element $ds^2 \neq \delta_{ik}dx^i dx^k$, it is valid only in empty space. The author therefore suppose that the line element is $ds^2 = X_{ik}dx^i dx^k$.

Hence the order of magnitude of the tensor X_{ik} is $X_{ik} \approx \delta_{ik}$. Consequently the space-time of electromagnetic field is « quasi-flat ».

iv) In the presence of gravitation, Maxwell equations are

$$F_{mk} = \varphi_{m;k} - \varphi_{k;m} = \frac{\partial \varphi_m}{\partial x^k} - \frac{\partial \varphi_k}{\partial x^m} \qquad F^{mk}_{;k} = 0 \tag{5}$$

To remove the arbitrariness of φ_μ, it is natural to impose the condition

$$\varphi^\sigma_{;\sigma} = 0 \tag{6}$$

It is evidently a modification of Lorenz subsidiary condition of electromagnetic potential. According to *iii)* the Maxwell equations for electromagnetic field alone muste be

$$\overline{F}_{mk} = \overline{\varphi}_{m|k} - \overline{\varphi}_{k|m} = \frac{\partial \overline{\varphi}_m}{\partial x^k} - \frac{\partial \overline{\varphi}_k}{\partial x^m}, \qquad \overline{F}^{mk}_{|k} = 0 \tag{7}$$

where a solidus « | » indicate covariant differenciation with respect to $\overline{\Gamma^\alpha_{\beta\gamma}}$

$$\overline{\Gamma}^\alpha_{\beta\gamma} = \overline{\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}} + 2X æ_{\beta\gamma} X^{\alpha s} \overline{\varphi}_s, \qquad X_{ik} = \frac{1}{2}(\overline{A}_{ik} + \overline{A}_{ki}), \qquad \overline{A}_{ik|m} = 0, \qquad \overline{\varphi}^\sigma_{|\sigma} = 0$$

and $\overline{\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}}$ is the Christoffel symbol with respect to the tensor X_{ik}

$$\because \overline{F}^{\mu\sigma}_{|\sigma} = \frac{\partial \overline{F}^{\mu\sigma}}{\partial x^\sigma} + \overline{\Gamma}^\sigma_{\alpha\sigma} \overline{F}^{\mu\alpha} = 0 \qquad \because X_{ik} \approx \delta_{ik}, \qquad \frac{\partial X_{ik}}{\partial x^m} \approx 0 \qquad \therefore \overline{\left\{ \begin{matrix} \sigma \\ \alpha\sigma \end{matrix} \right\}} \approx 0$$

$$\because æ \ll 1 \qquad \therefore 2æ\varphi_\alpha \ll 1 \qquad \therefore \overline{\Gamma}^\sigma_{\alpha\sigma} \ll 1$$

$$\therefore \overline{F}^{\mu\sigma}_{|\sigma} \approx \frac{\partial \overline{F}^{\mu\sigma}}{\partial x^\sigma} \approx 0$$

Since φ_μ, $\overline{\varphi}_\mu$ must be determined from (5) (7) therefore (5) is multiplied by $A_{m\mu}$

$$\begin{aligned} \therefore A_{m\mu} F^{\mu\sigma}_{;\sigma} &= F^\sigma_{m;\sigma} = (A^{s\sigma} F_{ms})_{;\sigma} = A^{s\sigma} F_{ms;\sigma} = A^{s\sigma}(\varphi_{m;s} - \varphi_{s;m})_{;\sigma} \\ &= A^{s\sigma}(\varphi_m)_{;s\sigma} - (A^{s\sigma}\varphi_s)_{;m\sigma} = A^{s\sigma}(\varphi_m)_{;s\sigma} - \varphi^\sigma{}_{;m\sigma} \\ &= A^{s\sigma}(\varphi_m)_{;s\sigma} - \varphi^\sigma{}_{;\sigma m} + \varphi^\sigma B_{\sigma m} = A^{s\sigma}(\varphi_m)_{;s\sigma} + \varphi^\sigma B_{\sigma m} = 0, \text{ by (6).} \end{aligned}$$

$$\therefore \boxed{\square}\,\varphi_m = -\varphi^\sigma B_{\sigma m}, \qquad \boxed{\square} \equiv A^{s\sigma}(\ldots)_{;s\sigma} \tag{8}$$

The tensors $A_{\mu\sigma}$, $B_{\mu\sigma}$ and $X_{\mu\sigma}$ will be determined below.

Similarly, if there is no gravitation

$$\overline{\boxed{\square}}\,\overline{\varphi}_m = -\overline{\varphi}^\sigma \overline{B}_{\sigma m} \tag{9}$$

where

$$\overline{\boxed{\square}} \equiv \overline{A}^{s\sigma}(\ldots)_{|s\sigma}$$

It should be noted that the first equation in (5) are the same as the ordinary Maxwell's equation, this means that the electromagnetic field strength is the same whether the space if flat or « quasi-flat ».

2. A GENERALIZATION OF EINSTEIN TENSOR

By the generalized Bianchi's identity

$$B^i_{jkl;m} + B^i_{jlm;k} + B^i_{jmk;l} = 0 \tag{10}$$

we get from (3) (10)

$$B_{ijkl;m} + B_{ijlm;k} + B_{ijmk;l} = 0, \qquad B_{ijkl} \equiv A_{is}B^s_{jkl} \tag{11}$$

(11) is multiplied by $A^{il}A^{jk}$, we get

$$A^{jk}B_{jk;m} - A^{jk}B_{jm;k} + A^{jk}B^l_{jmk;l} = 0$$

$$\delta^k_m B_{;k} - B^k_{m;k} + A^{jl}B^k_{jml;k} = 0, \qquad B^k_m \equiv A^{jk}B_{jm}$$

$$[\delta^k_m B - B^k_m + A^{jl}B^k_{jml}]_{;k} = 0 \tag{12}$$

From (4) we get

$$A^{sh}A^{jl}(A_{kh}B^k_{jml} + A_{jk}B^k_{hml}) = 0$$

$$A^{jl}B^k_{jlm} = A^{kh}B_{hm} \tag{13}$$

(12) becomes

$$[\delta^k_m B - B^k_m - A^{hk}B_{hm}]_{;k} = 0 \tag{14}$$

Let

$$P^k_m \equiv B^k_m + A^{kh}B_{hm} - \delta^k_m B$$

then

$$P^k_{m;k} = 0 \tag{15}$$

If

$$\varphi_\mu = 0 \qquad \Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \qquad B_{hm} = R_{hm}$$

$$\therefore P^k_m = 2\left(R^k_m - \frac{1}{2}\delta^k_m R\right) \equiv 2G^k_m$$

$$\therefore G^k_m \equiv R^k_m - \frac{1}{2}\delta^k_m R$$

hence the solution of the equation $P^k_{m;k} = 0$ is $G^k_m = \lambda\delta^k_m$.

The tensor G^k_m is the well known Einstein tensor. Hence $\varepsilon_{\mu\nu}$ represents the gravitational potential. Consequently the tensor P^k_m is evidently a generalization of Einstein tensor. The equation $G^k_m = \lambda\delta^k_m$ has been taken by Einstein as the field equation of gravitation or

$$R^k_m - \frac{1}{2}\delta^k_m R = \lambda\delta^k_m \tag{16}$$

This fact suggest that the solution of (15) is the possible field equation of gravitation and electricity, that is,

$$P^k_m = C\delta^k_m$$

or:

$$B^k_m + A^{kh}B_{hm} - \delta^k_m B = C\delta^k_m \tag{17}$$

3. THE UNIFIED FIELD EQUATION

From $(17)_1$ put $k = m$ we get $B = -2C. \therefore B^k_m + A^{kh}B_{hm} = -C\delta^k_m$. To determine C, let $\varphi_\mu = 0$ and compare with (16) we get $C = 2\lambda$

$$\therefore B^k_m + A^{kh}B_{hm} = -2\lambda\delta^k_m \tag{18}$$

(18) can be written

$$B_{km} + A_{ks}A^{sh}B_{hm} = -2\lambda A_{km} \tag{19}$$

or

$$(\delta^h_k + A_{ks}A^{sh})B_{hm} = -2\lambda A_{km} \tag{20}$$

Let

$$\alpha^h_k \equiv \delta^h_k + A_{ks}A^{sh} \qquad \overline{\alpha}^h_k \equiv \delta^h_k + \overline{A}_{ks}\overline{A}^{sh}$$

$$\therefore \alpha^h_k B_{hm} = -2\lambda A_{km} \tag{21}$$

Suppose the determinant of α^h_k, $|\alpha^h_k| \neq 0$ so that $\alpha^m_r a^i_m = \alpha^i_m a^m_r = \delta^i_r$, (21) is multiplied by a^k_s

$$\therefore B_{sm} = -2\lambda a^k_s a_{km}$$

Let

$$\mathfrak{A}_{sm} \equiv a^k_s A_{km} \qquad \overline{\mathfrak{A}}_{sm} \equiv \overline{a}^k_s \overline{A}_{km}$$

$$\therefore B_{sm} = -2\lambda\mathfrak{A}_{sm} \tag{22}$$

This equation is taken by the author as the unified field equation of gravitation and electricity.

(22) can be written in the form

$$F_{km} + \frac{1}{æ} S_{km} = -\frac{2\lambda}{æ}\mathfrak{A}_{km} \tag{23}$$

where

$$S_{km} = \frac{1}{2}\left(\frac{\partial\Gamma^h_{hk}}{\partial x^m} + \frac{\partial\Gamma^h_{hm}}{\partial x^k}\right) - \frac{\partial\Gamma^h_{km}}{\partial x^h} + \Gamma^h_{ki}\Gamma^i_{hm} - \Gamma^h_{km}\Gamma^i_{hi}$$

Let

$$Q_{km} \equiv \frac{1}{æ} S_{km} + \frac{2\lambda}{æ} \mathfrak{A}_{km}$$

than

$$F_{km} + Q_{km} = 0, \qquad F^{km} + Q^{km} = 0 \tag{24}$$

$$\therefore\ F^{km}_{;m} + Q^{km}_{;m} = 0 \tag{25}$$

This equation is satisfied by

$$F^{km}_{;m} = 0 \quad (26) \qquad\qquad Q^{km}_{;m} = 0 \quad (27)$$

(27) is just a restriction of the choice of the coordinates, such a restriction is evidently an analogue of the choice of coordinates given by Einstein in his general approximative solution of gravitational field equation [8]. Therefore, Maxwell equation is a consequence of the unified field equation. From (8) (22) we get

$$\boxed{\Box}\, \varphi_m = 2\lambda \mathfrak{A}_{\sigma m} \varphi^\sigma \tag{28}$$

Since (22) (26) (27) must valid when there is no gravitation

$$\overline{B}_{\sigma m} = -\,2\lambda \overline{\mathfrak{A}}_{\sigma m} \tag{29}$$

Hence

$$\boxed{\overline{\Box}}\, \overline{\varphi}_m = 2\lambda \overline{\mathfrak{A}}_{\sigma m} \overline{\varphi}^\sigma \tag{30}$$

$$\overline{F}^{km}_{|m} = 0 \qquad \overline{Q}^{km}_{|m} = 0$$

The tensors φ_m, $A_{\mu\nu}$ are completely determined by

$$B_{\sigma m} = -\,2\lambda \mathfrak{A}_{\sigma m}, \qquad \boxed{\Box}\, \varphi_m = -\,\lambda \mathfrak{A}_{\sigma m} \varphi^\sigma \tag{31}$$

for there are 20 partial differential equations (31) satisfied by 20 unknowns φ_m, $A_{\mu\nu}$. Similarly, the equations (29) (30) with the boundary conditions

$$\varphi_m = 0 \qquad X_{\mu\sigma} = \delta_{\mu\sigma}$$

completely determine the tensors $\overline{\varphi}_m$, $\overline{A}_{\sigma\mu}$.

4. DISCUSSION

The theory developed in this paper can be extended evidently to N-dimensional symmetric connected space and is equivalent to an introduction of a metric in this space. Consequently, in a N-dimensional symmetric connected space with

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} + C^{\sigma}_{\mu\nu}$$

where the tensor $C^{\sigma}_{\mu\nu}$ is known, we can define the metric tensors of this space from the solutions of the partial differential equations

$$B_{\rho\sigma} = - 2\lambda\mathfrak{A}_{\rho\sigma}.$$

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