# A generalization of primitive sets and a conjecture of Erdős 

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#### Abstract

A set of integers greater than 1 is primitive if no element divides another. Erdős proved in 1935 that the sum of $1 /(n \log n)$ for $n$ running over a primitive set $A$ is universally bounded over all choices for $A$. In 1988 he asked if this universal bound is attained by the set of prime numbers. We answer the Erdős question in the affirmative for 2-primitive sets. Here a set is 2-primitive if no element divides the product of 2 other elements.


Key words and phrases: primitive set, primitive sequence

## 1 Introduction and Statement of results

A set of integers greater than 1 is called primitive if no element divides any other. Erdős [4] showed that there is a constant $K$ such that for any primitive set $A$,

$$
f(A):=\sum_{n \in A} \frac{1}{n \log n} \leq K .
$$

Further, in 1988 he asked if $f(A)$ is maximized by the primes $A=\mathbb{P}$. This is now referred to as the Erdős conjecture for primitive sets:

$$
\text { For } A \text { primitive, we have } f(A) \leq f(\mathbb{P})=\sum_{p \in \mathbb{P}} \frac{1}{p \log p}=: C=1.636616 \cdots,
$$

where $\mathbb{P}$ is the set of prime numbers. By a simple argument, the Erdős conjecture is equivalent to the assertion that $f(A) \leq f(\mathcal{P}(A))$ for any primitive set $A$, where $\mathcal{P}(A)$ denotes the set of primes dividing some member of $A$.

Recently, the second and third authors [9] proved that

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Theorem 1. For any primitive set $A$,

$$
f(A)<e^{\gamma}=1.781072 \cdots
$$

where $\gamma=0.5772 \cdots$ is the Euler-Mascheroni constant. Further, if A does not contain a multiple of 8 , then

$$
f(A) \leq f(\mathcal{P}(A))+2.37 \times 10^{-7} .
$$

One can strengthen the notion of primitivity as follows. We say that a set $A$ of integers greater than 1 with $|A| \geq k+1$ is $k$-primitive if no element divides the product of $k$ distinct other elements. Note that $k$-primitive implies $j$-primitive for all $k \geq j \geq 1$.

In 1938, Erdős [6] first studied the maximal cardinality of 2-primitive sets in an interval. The first author together with Győri and Sárközy [3] extended it to all $k$ and it was subsequently improved in [2] and [10]. While the full conjecture remains out of reach, we prove the Erdős conjecture for 2-primitive sets (and hence $k$-primitive for all $k \geq 2$ ).

Theorem 2. For any 2-primitive set $A$,

$$
f(A) \leq f(\mathcal{P}(A)) .
$$

An immediate consequence is the following
Corollary 1. If $A$ is a primitive set with $f(A)>f(\mathcal{P}(A))$, then there exist three elements $a, b, c \in A$ with $a \mid b c$.

On the other hand, the primes are not optimal among primitive sets with respect to logarithmic density. Indeed, Erdős, Sárközy, and Szemerédi [8] obtained the best possible upper bound

$$
\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n} \leq\left(\frac{1}{\sqrt{2 \pi}}+o(1)\right) \frac{\log x}{\sqrt{\log \log x}}
$$

for any primitive set $A$, while Erdős [7] showed that

$$
\sum_{\substack{n \in A^{\prime} \\ n \leq x}} \frac{1}{n} \geq\left(\frac{1}{\sqrt{2 \pi}}+o(1)\right) \frac{\log x}{\sqrt{\log \log x}}
$$

where $A^{\prime}$ is the set of positive integers $a \leq x$ with $\Omega(a)=[\log \log x]$. (Here, $\Omega(a)$ is the number of prime factors of $a$, counted with multiplicity.) By contrast, the primes satisfy

$$
\sum_{p \leq x} \frac{1}{p}=\log \log x+O(1)
$$

Nevertheless, one may wonder if the primes still maximize the logarithmic density among 2-primitive sets. Indeed, we prove this to be the case.

## ERDŐs 2-PRIMITIVE SET CONJECTURE

Proposition 1. For all $x \geq 2$ and any 2-primitive set $A$,

$$
\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n} \leq \sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} \frac{1}{p}
$$

We use this to deduce Theorem 2.
Proof of Theorem 2 given Proposition 1. By Proposition 1, we have $F(x) \geq 0$ for all $x \geq 2$, where

$$
F(x):=\sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} \frac{1}{p}-\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n}
$$

Then by partial summation,

$$
\sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} \frac{1}{p \log p}-\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n \log n}=\frac{F(x)}{\log x}+\int_{2^{-}}^{x} \frac{F(u)}{u \log ^{2} u} d u \geq 0
$$

Hence taking $x \rightarrow \infty$ gives $f(\mathcal{P}(A)) \geq f(A)$ as desired.

In light of Proposition 1, it is natural to ask if there exists an exponent $\lambda<1$ for which

$$
\begin{equation*}
\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n^{\lambda}} \leq \sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} \frac{1}{p^{\lambda}} \tag{1.1}
\end{equation*}
$$

holds for all 2-primitive $A, x \geq 2$. Banks and Martin [1] settled the question in the setting of 1-primitive sets, proving (1.1) holds for all primitive $A$ if and only if

$$
\lambda \geq \tau_{1}:=1.1403659 \cdots
$$

where $t=\tau_{1}$ is the unique real solution to the equation

$$
\sum_{\mathbb{P}} p^{-t}=1+\left(1-\sum_{\mathbb{P}} p^{-2 t}\right)^{1 / 2}
$$

The fact that $\tau_{1}$ is markedly larger than 1 gives some indication as to why the full Erdős conjecture remains open.

In the setting of 2-primitive sets, we extend the range of valid exponents $\lambda$.
Theorem 3. For any $\lambda \geq 0.7983, x \geq 2$, and any 2 -primitive set $A$,

$$
\begin{equation*}
\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{n^{\lambda}} \leq \sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} \frac{1}{p^{\lambda}} \tag{1.2}
\end{equation*}
$$

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We remark it suffices to verify Theorem 3 with $\lambda=0.7983$. Indeed, suppose that $F_{\lambda}(x) \geq 0$ for all $x \geq 2$, where

$$
F_{t}(x)=\sum_{\substack{p \in \mathcal{P}(A) \\ p \leq x}} p^{-t}-\sum_{\substack{n \in A \\ n \leq x}} n^{-t}
$$

Then, by partial summation, for any $t>\lambda$,

$$
F_{t}(x)=x^{\lambda-t} F_{\lambda}(x)+(t-\lambda) \int_{2}^{x} u^{\lambda-t-1} F_{\lambda}(u) d u \geq 0
$$

Hence we may define the critical exponent $\tau_{2}$ for 2-primitive sets, as the infimum over all $\lambda$ for which (1.2) holds. Thus, Theorem 3 implies that $\tau_{2} \leq 0.7983$.

We also note that Theorem 3 with $\lambda=1$ gives us Proposition 1. However, Theorem 3 does not hold for every positive value of $\lambda$. Indeed, in [6], Erdős showed that there is a 2 -primitive set $A$ in $[1, x]$ of cardinality $\pi(x)-\pi\left(x^{1 / 3}\right)+c x^{2 / 3} /(\log x)^{2}$. It consists of primes in $\left(x^{1 / 3}, x\right]$ and a subset of $\left\{p_{1} p_{2} p_{3}: p_{i}\right.$ are primes $\left.\leq x^{1 / 3}\right\}$ where the triples $\left\{p_{1}, p_{2}, p_{3}\right\}$ form a Steiner triple system. Thus, by the prime number theorem,

$$
\sum_{n \in A} \frac{1}{n^{\lambda}} \geq \sum_{x^{1 / 3}<p \leq x} \frac{1}{p^{\lambda}}+\frac{c x^{2 / 3}}{(\log x)^{2}} \frac{1}{x^{\lambda}}>\sum_{p \leq x} \frac{1}{p^{\lambda}}
$$

when $\lambda<0.5$ and $x$ is sufficiently large. Hence the above argument and Theorem 3 together imply that the critical exponent lies in the interval

$$
\begin{equation*}
0.5 \leq \tau_{2} \leq 0.7983 \tag{1.3}
\end{equation*}
$$

In a sequel paper, we shall address the question of critical exponents for $k$-primitive sets, with $k \geq 3$.

## 2 Combinatorial Lemmas

Before proving Theorem 3, we need lemmas in counting the maximal number of elements in a $k$-primitive set.

We first recall the following famous result due to Erdős and Szekeres [5], whose proof we provide for completeness.

Lemma 1 (Erdős-Szekeres). A sequence of $(r-1)(s-1)+1$ real numbers has either a monotonic nondecreasing subsequence of length $r$ or a monotonic nonincreasing subsequence of length $s$.

Proof. Say the sequence is $a_{1}, a_{2}, \ldots, a_{n}$, where $n=(r-1)(s-1)+1$. For each $a_{i}$ consider the ordered pair $\left(b_{i}, c_{i}\right)$, where $b_{i}$ is the length of the longest nondecreasing subsequence ending at $a_{i}$ and $c_{i}$ is the length of the longest nonincreasing subsequence ending at $a_{i}$. Then no two pairs $\left(b_{i}, c_{i}\right)$ and $\left(b_{j}, c_{j}\right)$ can be equal, so for at least one choice of $i$ we have $b_{i} \geq r$ or $c_{i} \geq s$.

We next bound the size of a $k$-primitive set based on the number of prime factors used to generate its elements.

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Lemma 2. For $k \geq 2$, suppose $A$ is a $k$-primitive set and $T \subset A$ with $|\mathcal{P}(T)|=n$. If $n \leq k$, then $|T| \leq n$. If $n=k+1$, then $|T| \leq n+1$. Further, for $k=2, n=4$ we have $|T| \leq 19$.

Proof. We may assume that $|T| \geq n$. Let $\mathcal{P}(T)=\left\{q_{1}, \ldots, q_{n}\right\}$ and write each $t=\prod_{i} q_{i}^{e_{i}} \in T$ as an exponent vector $\vec{v}=\left(e_{1}, \ldots, e_{n}\right)$. Define the notation $\vec{v} \geq 0$ if $e_{i} \geq 0$ for all $i$, and define $\vec{v} \leq \vec{w}$ if $\vec{w}-\vec{v} \geq 0$. Take $\vec{v}_{1}$ with maximal entry $e_{1}$ among $T$. Then take $\vec{v}_{2}$ with maximal $e_{2}$ among the remaining vectors, and similarly define $\vec{v}_{3}, \ldots, \vec{v}_{n}$. Thus, the chosen vectors are distinct.

Case $n \leq k$ : If $|T| \geq n+1$ then $T$ has some vector $\vec{v} \neq \vec{v}_{i}$ for all $i$. But then $\vec{v} \leq \vec{v}_{1}+\cdots+\vec{v}_{n}$. This implies that $T$, and hence $A$, is not $n$-primitive, and since $n \leq k$, it implies that $A$ is not $k$-primitive, a contradiction. Hence we cannot have $|T| \geq n+1$ when $n \leq k$.

Case $n=k+1$ : If $|T| \geq n+2$ then $T$ has vectors $\vec{w}_{1} \neq \vec{w}_{2}$ with $\vec{w}_{j} \notin\left\{\vec{v}_{1}, \ldots \vec{v}_{n}\right\}$ for $j=1,2$. Write $\vec{w}_{j}=\left(f_{1}^{(j)}, \ldots, f_{n}^{(j)}\right)$. By the pigeonhole principle, we may assume

$$
f_{i}^{(1)} \leq f_{i}^{(2)}
$$

for at least $n / 2$ values of $i$, say $i=1, \ldots,\lceil n / 2\rceil$. Thus, we deduce

$$
\vec{w}_{1} \leq \vec{w}_{2}+\vec{v}_{[n / 2\rceil+1}+\cdots+\vec{v}_{n}
$$

contradicting $T$ as $k$-primitive, since $1+\lfloor n / 2\rfloor=1+\lfloor(k+1) / 2\rfloor \leq k$.
Now say $k=2, n=4$. Suppose there are 20 members in $T$ with corresponding vectors

$$
\vec{w}_{i}:=\left(e_{i, 1}, e_{i, 2}, e_{i, 3}, e_{i_{4}}\right) \text { for } 1 \leq i \leq 20
$$

Since $A$ is 2-primitive, so is $T$. Without loss of generality, say $\vec{w}_{18}$ has maximal first coordinate, $\vec{w}_{19} \neq \vec{w}_{18}$ has maximal second coordinate among the remaining 19 vectors, and $\vec{w}_{20}$ has maximal third coordinate among the remaining 18 vectors with $\vec{w}_{20} \neq \vec{w}_{18}, \vec{w}_{19}$. Arrange the remaining 17 vectors in ascending order of their first coordinate (i.e., $e_{1,1} \leq e_{2,1} \leq \ldots \leq e_{17,1}$ ). By Lemma 1, there is a monotonic sequence of length 5 among the $e_{i, 2}$ 's. Without loss of generality, say $e_{1,2}, e_{2,2}, e_{3,2}, e_{4,2}, e_{5,2}$ form such a sequence.

Case 1: $e_{1,2} \leq e_{2,2} \leq e_{3,2} \leq e_{4,2} \leq e_{5,2}$. Consider the numbers $e_{i, 3}$ for $i=1, \ldots, 5$. By Lemma 1, there is a monotonic sequence of length 3 among the $e_{i, 3}$ 's, without loss of generality, say it is $e_{1,3}, e_{2,3}, e_{3,3}$. If $e_{1,3} \leq e_{2,3} \leq e_{3,3}$, this forces $e_{2,4}>e_{1,4}+e_{3,4}$ for otherwise $\vec{w}_{2} \leq \vec{w}_{1}+\vec{w}_{3}$, contradicting $T$ being 2-primitive. But this implies that $\vec{w}_{1} \leq \vec{w}_{2}$ which contradicts $T$ being primitive. Hence, we must have $e_{1,3} \geq e_{2,3} \geq e_{3,3}$. Again, this forces $e_{2,4}>e_{1,4}+e_{3,4}$, which in turn implies that $\vec{w}_{1} \leq \vec{w}_{2}+\vec{w}_{20}$, again a contradiction.

Case 2: $e_{1,2} \geq e_{2,2} \geq e_{3,2} \geq e_{4,2} \geq e_{5,2}$. By Lemma 1, there is a monotonic sequence of length 3 among the $e_{i, 3}$ 's, without loss of generality, say it is $e_{1,3}, e_{2,3}, e_{3,3}$. If $e_{1,3} \leq e_{2,3} \leq e_{3,3}$, then again this forces $e_{2,4}>e_{1,4}+e_{3,4}$. But then $\vec{w}_{1} \leq \vec{w}_{2}+\vec{w}_{19}$. Hence, we must have $e_{1,3} \geq e_{2,3} \geq e_{3,3}$. This forces $e_{2,4}>e_{1,4}+e_{3,4}$. But then $\vec{w}_{3} \leq \vec{w}_{2}+\vec{w}_{18}$, again a contradiction.

Therefore, there can be at most 19 members in $T$.
Remark 2.1. It is not clear if the number " 19 " in Lemma 2 is optimal. We will not need it here, but by similar methods one can prove that if $T$ is a 2-primitive set of positive integers with $|\mathcal{P}(T)|=n \geq 3$, then $|T| \leq 9^{2^{n-3}}$.

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## 3 Proof of Theorem 3

Let $A \subset(1, x]$ be a 2-primitive set. Let $0.79 \leq \lambda<1$ be a parameter to be defined later. First, we partition $A$ into primes $S$ and composites $T$. Note $S$ and $\mathcal{P}(T)$ are disjoint since $A$ is primitive. For a prime $p$, define

$$
T_{p}:=\{t \in T: p \mid t\}
$$

If some prime $p \in \mathcal{P}(T)$ satisfies

$$
\begin{equation*}
\sum_{t \in T_{p}} \frac{1}{t^{\lambda}} \leq \frac{1}{p^{\lambda}} \tag{3.1}
\end{equation*}
$$

then we replace the members of $T_{p}$ with the prime $p$ (i.e., redefine $A=\left(T \backslash T_{p}\right) \cup\{p\}$ ). This would make $\Sigma_{T_{p}} t^{-\lambda}$ at least as big while keeping $A$ 2-primitive. Repeat the process with each prime $p \in \mathcal{P}(T)$ until no such prime satisfies (3.1). If $T=\emptyset$ after doing this, then $A=S$ consists of primes so Proposition 1 follows. Otherwise $T \neq \emptyset$, so we may assume

$$
\begin{equation*}
\sum_{t \in T_{p}} \frac{1}{t^{\lambda}}>\frac{1}{p^{\lambda}} \quad \text { for all } \quad p \in \mathcal{P}(T) \tag{3.2}
\end{equation*}
$$

Consider the set

$$
\begin{equation*}
D:=\{t / p: t \in T, p \mid t\} \tag{3.3}
\end{equation*}
$$

We record some useful properties of $T$ and $D$.
Lemma 3. Let $T$ be a 2-primitive set for which (3.2) holds and let $D$ be as in (3.3).
(i) For each $p \in \mathcal{P}(T), T_{p}$ has at least 3 elements.
(ii) The map sending ordered pairs $(t, p)$ with $t \in T$ and $p \mid t$ to $t / p \in D$ is injective.
(iii) Each $t \in T$ has at least 3 prime factors (counted with multiplicity).
(iv) $D$ is a primitive set of composite numbers.

Proof. (i) For $p \in \mathcal{P}(T)$, (3.2) implies that

$$
\sum_{t \in T_{p}} \frac{1}{(t / p)^{\lambda}}>1>2^{-0.79}+3^{-0.79}
$$

Thus (i) follows, since $t / p \in \mathbb{Z}_{>1}$ for all $t \in T_{p}$.
(ii) If not, then $t_{1} / p_{1}=t_{2} / p_{2}$ for some $t_{1}, t_{2}, p_{1} \mid t_{1}$, and $p_{2} \mid t_{2}$. If $t_{1} \neq t_{2}$, by (i) there exists some $p_{1} k \in T_{p_{1}}$ other than $t_{1}, t_{2}$. But then $t_{1}=\left(t_{1} / p_{1}\right) p_{1}=\left(t_{2} / p_{2}\right) p_{1} \mid t_{2}\left(p_{1} k\right)$, which contradicts $T$ as 2-primitive. Hence $t_{1}=t_{2}$, which forces $p_{1}=p_{2}$.
(iii) If not, say $t=p q$. Since $T_{p}, T_{q}$ each have at least 3 elements, there are some $p m$ and $q n$ other than $t \in T$. But then, $t=p q \mid(p m)(q n)$ which contradicts $T$ as 2-primitive. (This argument holds whether or not $p \neq q$.)

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(iv) If not, then $(t / p) \mid\left(t_{1} / p_{1}\right)$ for some $t, t_{1} \in T, p\left|t, p_{1}\right| t_{1}$, and $t / p \neq t_{1} / p_{1}$. If $p_{1}=p$, then $t \mid t_{1}$ which contradicts $T$ as primitive. And if $p_{1} \neq p$, then there is some $p l \in T_{p}$ other than $t$ and $t_{1}$. This implies $t \mid t_{1} \cdot p l$, and since $t \neq t_{1}$ (otherwise $p=p_{1}$ ), we have a contradiction to $T$ being 2-primitive. Thus $D$ is primitive, and also composite by (iii).

For Theorem 3, we must show

$$
\begin{equation*}
\sum_{t \in T} \frac{1}{t^{\lambda}}-\sum_{p \in \mathcal{P}(T)} \frac{1}{p^{\lambda}}<0 \tag{3.4}
\end{equation*}
$$

Suppose $\mathcal{P}(T)$ consists of primes $q_{1}<q_{2}<\cdots<q_{r}$. Let $2=p_{1}<p_{2}<\cdots<p_{r}$ be the first $r$ primes in $\mathbb{P}$. We are going to modify the set $T$ by the following process. First, if each $q_{i}=p_{i}$, we let $T$ stand as it is. Otherwise, let $i$ be the smallest index such that $q_{i}>p_{i}$. Then $q_{j}=p_{j}$ for all $j<i$ and we have $p_{i} \nmid t$ for all $t \in T$. Then replace each $t \in T_{q_{i}}$ with $p_{i} / q_{i} \cdot t$. This keeps $T$ as 2-primitive, and by (3.2),

$$
0<\sum_{t \in T_{q_{i}}} \frac{1}{t^{\lambda}}-\frac{1}{q_{i}^{\lambda}}<\frac{q_{i}^{\lambda}}{p_{i}^{\lambda}}\left(\sum_{t \in T_{q_{i}}} \frac{1}{t^{\lambda}}-\frac{1}{q_{i}^{\lambda}}\right)=\sum_{t \in T_{q_{i}}} \frac{1}{\left(p_{i} / q_{i} \cdot t\right)^{\lambda}}-\frac{1}{p_{i}^{\lambda}} .
$$

So replacing each $t \in T_{q_{i}}$ with $p_{i} / q_{i} \cdot t$ preserves (3.2). We repeat this process for each $i$ with $q_{i}>p_{i}$ and in the end we have $\mathcal{P}(T)=\left\{p_{1}, p_{2}, \ldots, p_{r}\right\}$. By showing (3.4) for this $T$ it would follow that (3.2) fails for some $p_{i}$, and this contradiction would prove the theorem.

We have reduced Theorem 3 to the following.
Theorem 3.1. Suppose $\lambda \geq 0.7983$ and $T$ is a 2 -primitive set of composite numbers satisfying (3.2) with $\mathcal{P}(T)=\mathbb{P} \cap(1, Y]$ for some $Y$. Then

$$
\begin{equation*}
\sum_{t \in T} \frac{1}{t^{\lambda}}-\sum_{p \leq Y} \frac{1}{p^{\lambda}}<0 . \tag{3.5}
\end{equation*}
$$

Our goal now is to prove Theorem 3.1. For a parameter $0<\theta<1$ to be chosen later, we define $\lambda$ as

$$
\begin{equation*}
\lambda=\tau(1-\theta), \text { where } \tau=1.140366 \tag{3.6}
\end{equation*}
$$

First consider those $t \in T$ with greatest prime factor $P(t) \geq t^{\theta}$. Then $t^{1-\theta} \geq t / P(t)$ and so $t^{-\lambda} \leq$ $(t / P(t))^{-\lambda /(1-\theta)}=(t / P(t))^{-\tau}$. Hence

$$
\begin{equation*}
\sum_{\substack{t \in T \\ P(t) \geq t^{\theta}}} t^{-\lambda} \leq \sum_{\substack{t \in T \\ P(t) \geq t^{\theta}}}\left(\frac{t}{P(t)}\right)^{-\tau} \leq \sum_{p \leq Y} p^{-\tau} \tag{3.7}
\end{equation*}
$$

by (1.1), since $\{t / P(t): t \in T\} \subset D$ is primitive by part (iii) of Lemma 3 .
For a positive integer $t$, we consider the following unique factorization

$$
t=m(t) M(t)
$$

into positive integers $m(t) \leq M(t)$ with ratio $M(t) / m(t)$ minimal. Let

$$
\mathcal{M}(T)=\{m(t): t \in T\} \cup\{M(t): t \in T\}
$$

We need two lemmas.

Lemma 4. For any 2-primitive set $T$, consider the graph on the integers with edges $\{t, m(t)\}$ and $\{t, M(t)\}$ for $t \in T$, where if $m(t)=M(t)$, there is just one edge containing $t$. This graph contains a matching from $T$ into $\mathcal{M}(T)$.

Proof. Let $t \in T$. If $m(t) \notin\left\{m\left(t^{\prime}\right), M\left(t^{\prime}\right)\right\}$ for all other $t^{\prime} \in T$, then we can match $t$ with $m(t)$. So assume $m(t) \in\left\{m\left(t^{\prime}\right), M\left(t^{\prime}\right)\right\}$ for some other $t^{\prime} \in T$. Then $M(t) \notin\left\{m\left(t^{\prime \prime}\right), M\left(t^{\prime \prime}\right)\right\}$ for all $t^{\prime \prime} \in T$ with $t^{\prime \prime} \neq t, t^{\prime}$, since otherwise $t \mid t^{\prime} t^{\prime \prime}$, contradicting $T$ being 2-primitive.

If $m(t)<M(t)$, then 2-primitive implies $M(t) \notin\left\{m\left(t^{\prime}\right), M\left(t^{\prime}\right)\right\}$ so we can match $t$ with $M(t)$.
Otherwise $m(t)=M(t)$, which means $t=m(t)^{2}$. Then $t^{\prime} \neq t$ forces $m\left(t^{\prime}\right)<M\left(t^{\prime}\right)$, so we make define $m^{\prime}=t^{\prime} / m(t)$ (that is $m^{\prime}$ is the singleton in $\left\{m\left(t^{\prime}\right), M\left(t^{\prime}\right)\right\} \backslash\{m(t)\}$ ). We would like to match $t^{\prime}$ with $m^{\prime}$ instead of $m(t)$, freeing up $m(t)$ to be matched with $t$. So suppose this is blocked by some $t^{\prime \prime}$ different from $t^{\prime}$ (and necessarily different from $t$ ) with $m^{\prime} \in\left\{m\left(t^{\prime \prime}\right), M\left(t^{\prime \prime}\right)\right\}$. But then $t^{\prime} \mid t t^{\prime \prime}$, a violation of 2-primitivity. Thus, the matching can be completed.
Lemma 5. Suppose $0<\theta<1 / 3$ and that $T$ is 2-primitive with $P(t)<t^{\theta}$ for each $t \in T$. Let $N(z)=$ $|T \cap[2, z]|$. Then, with $q$ running over primes in the interval $I:=\left[z^{(1+\theta) / 4}, z^{(1+\theta) / 2}\right)$, we have

$$
N(z)<z^{(1+\theta) / 2}-\sum_{q \in I}\left\lfloor\frac{z^{(1+\theta) / 2}}{q}\right\rfloor
$$

Proof. By Lemma 4, it suffices to bound $|\mathcal{M}(T \cap[2, z])|$. We first show that $\mathcal{M}(T \cap[2, z]) \subset\left[1, z^{(1+\theta) / 2}\right)$. Let $t \in T$ with $t \leq z$. Say $t=q_{1} q_{2} \ldots q_{r}$ where the primes $q_{i}$ are written in nondecreasing order. Let $d=q_{1} q_{2} \ldots q_{i}$ be maximal with $d \leq t^{(1-\theta) / 2}$. Then $d^{\prime}=d q_{i+1}$ satisfies $t^{(1-\theta) / 2}<d^{\prime}<t^{(1+\theta) / 2}$. Also, $d^{\prime \prime}=t / d^{\prime}$ satisfies the same double inequality. Thus,

$$
t^{(1-\theta) / 2}<m(t) \leq M(t)<t^{(1+\theta) / 2} \leq z^{(1+\theta) / 2}
$$

We further note that the members $m$ of $\mathcal{M}(T \cap[2, z])$ satisfy $P(m)<z^{\theta}$, since $m$ divides some member of $T \cap[2, z]$ and every $t$ in that set has $P(t)<z^{\theta}$. In particular, $m$ is not divisible by any prime $q \geq z^{\theta}$. Note that if $\theta<1 / 3$, then $\theta<(1+\theta) / 4$. So, $m$ is not divisible by any prime in the interval $I$. Since no integer below $z^{(1+\theta) / 2}$ is divisible by 2 primes from $I$, the lemma follows.

Set

$$
T^{p}=\{t \in T: P(t)=p\}
$$

so that $T^{p} \subset T_{p}$. We have the following variant of Lemma 5.
Lemma 6. For any 2-primitive set $T$ and prime $p$, let $N_{p}(z)$ denote the number of members $t$ of $T^{p}$ with $t \leq z$. With q running over the primes in $I_{p}:=\left(\max \left\{p, z^{1 / 4}\right\}, z^{1 / 2}\right)$, we have

$$
N_{p}(z) \leq z^{1 / 2}-\sum_{q \in I_{p}}\left\lfloor\frac{z^{1 / 2}}{q}\right\rfloor .
$$

Proof. Note that if $T$ is 2-primitive, so too is $T^{p} / p=\left\{t / p: t \in T^{p}\right\}$. Thus, we may apply Lemma 4 to obtain a matching from $T^{p} / p$ into $\mathcal{M}\left(T^{p} / p\right)$. The prime factors of each element $t / p \in T^{p} / p$ are at most $p$, so following the proof of Lemma 5, we have $m(t / p), M(t / p) \in\left[t^{1 / 2} / p, t^{1 / 2}\right)$. The lemma then follows in the same way as Lemma 5 .

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Lemma 7. For $x \geq 2$ we have

$$
\sum_{\substack{x^{1 / 2}<q<x \\ q \text { prime }}}\left\lfloor\frac{x}{q}\right\rfloor \geq\left(\log 2-\frac{1.25}{\log x}-\frac{2.5}{(\log x)^{2}}\right) x
$$

Proof. First suppose that $x \geq 286^{2}$. We have the sum is at least

$$
\sum_{x^{1 / 2}<q<x} \frac{x}{q}-\pi(x)
$$

From [11, (3.7)], we have that $\pi(x)<1.25 x / \log x$ and from [11, (3.17)] that

$$
\sum_{q<x} \frac{1}{q}>\log \log x+B-\frac{1}{2(\log x)^{2}}
$$

where $B$ is the Mertens constant. Further, from [11, (3.18)],

$$
\sum_{q \leq x^{1 / 2}} \frac{1}{q}<\log \log x^{1 / 2}+B+\frac{1}{2\left(\log x^{1 / 2}\right)^{2}}=\log \log x-\log 2+B+\frac{2}{(\log x)^{2}}
$$

This proves the lemma in the range $x \geq 286^{2}$ and direct calculation shows that it holds in the wider range $x \geq 2$.

We shall find it useful to use the following asymptotically weaker estimates in small cases. The proof follows by checking values of $x \leq 3213$ after which Lemma 7 is stronger.

Corollary 2. For $x \geq 185$, we have $\sum_{q \in\left(x^{1 / 2}, x\right]}\lfloor x / q\rfloor>0.5 x$. For $x \geq 67$, we have $\sum_{q \in\left(x^{1 / 2}, x\right\rfloor}\lfloor x / q\rfloor>0.45 x$.

Let

$$
\begin{equation*}
\theta=0.3, \quad \lambda=0.7982562, \quad v=1 / \theta=10 / 3 \tag{3.8}
\end{equation*}
$$

For each prime $p$, let

$$
S_{p}=\sum_{\substack{t \in T \\ P(t)=p<t^{\theta}}} \frac{1}{t^{\lambda}}
$$

With (3.7) it will suffice to prove Theorem 3.1 if we show under its hypotheses that for each $Y \geq 2$,

$$
\begin{equation*}
\sum_{p \leq Y} S_{p} \leq \sum_{p \leq Y}\left(\frac{1}{p^{\lambda}}-\frac{1}{p^{\tau}}\right) \tag{3.9}
\end{equation*}
$$

### 3.1 Small primes, $Y \leq 37$

We are going to estimate $S_{p}$ for various small primes $p$. Take $t \in T$ with $P(t)<t^{\theta}$. If $t \leq q^{v}$ for a prime $q$, then $P(t)<\left(q^{v}\right)^{\theta}=q$. If $q=3$, we see there can be at most one such $t$; that is, $T$ can contain at most one power of 2 . The values of $t \leq 5^{v}$ are supported on $\{2,3\}$, so by Lemma 2 with $k=n=2$ we see that there are at most 2 such members of $T$. Similarly, Lemma 2 with $k=2, n=3$ shows that $T$ has at most 4 members below $7^{v}$, and with $k=2, n=4, T$ has at most 19 members below $11^{v}$. Since members $t$ of $T$ with $P(t)<t^{\theta}$ have at least $\lceil v\rceil=4$ prime factors (counted with multiplicity), we have

$$
\begin{align*}
S_{2} & \leq \frac{1}{2^{4 \lambda}}<0.1093463 \\
S_{2}+S_{3} & <0.1093463+\frac{2-1}{3^{v \lambda}}<0.1631052 \\
S_{2}+S_{3}+S_{5} & <0.1631052+\frac{4-2}{5^{v \lambda}}<0.1907220 \\
S_{2}+S_{3}+S_{5}+S_{7} & <0.1907220+\frac{19-4}{7^{v \lambda}}<0.2753295 . \tag{3.10}
\end{align*}
$$

Computing $\sum_{p \leq Y}\left(1 / p^{\lambda}-1 / p^{\tau}\right)$ directly for $Y=2,3,5,7$ gives lower bounds

$$
0.121399,0.251741,0.368904,0.471733
$$

respectively. Thus we observe $\sum_{p \leq Y} S_{p}<\sum_{p \leq Y}\left(1 / p^{\lambda}-1 / p^{\tau}\right)$, so by (3.9), Theorem 3.1 holds when $Y=2,3,5,7$, respectively.

Now consider $11 \leq p \leq 37$. By partial summation, we have the equality

$$
\begin{equation*}
S_{p}=\int_{p^{v}}^{\infty} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z \tag{3.11}
\end{equation*}
$$

noting that the integral converges, since $N_{p}(z) \leq z^{(1+\theta) / 2}$ by Lemma 5 .
We use Lemmas 6 and 7 to get the upper estimates for $N_{p}(z)$ :

$$
\begin{align*}
& N_{p}(z) \leq\lfloor\sqrt{z}\rfloor-\sum_{\max \left(p, z^{1 / 4}\right)<q \leq \sqrt{z}}\left\lfloor\frac{\sqrt{z}}{q}\right\rfloor,  \tag{3.12}\\
& N_{p}(z) \leq \sqrt{z}\left(1-\log 2+\frac{2.5}{\log z}+\frac{10}{(\log z)^{2}}\right), \text { when } p \leq z^{1 / 4} \tag{3.13}
\end{align*}
$$

We split the integral in (3.11) at $p^{4}$. In the first range when $z<p^{4}$, we bound the contribution to (3.11) by splitting up into intervals $\left[m^{2},(m+1)^{2}\right]$ and using (3.12), which gives

$$
\begin{align*}
S_{p}^{\prime}:=\int_{p^{v}}^{p^{4}} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z \leq & \sum_{m_{0}<m<p^{2}} \int_{m^{2}}^{(m+1)^{2}} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z+\int_{p^{v}}^{\left(m_{0}+1\right)^{2}} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z \\
\leq & \sum_{m_{0}<m<p^{2}}\left(\frac{1}{m^{2 \lambda}}-\frac{1}{(m+1)^{2 \lambda}}\right)\left(m-\sum_{p<q \leq m}\left\lfloor\frac{m}{q}\right\rfloor\right) \\
& +\left(\frac{1}{p^{v \lambda}}-\frac{1}{\left(m_{0}+1\right)^{2 \lambda}}\right)\left(m_{0}-\sum_{p<q \leq p^{v / 2}}\left\lfloor\frac{m_{0}}{q}\right\rfloor\right) \tag{3.14}
\end{align*}
$$

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where $m_{0}=\left\lfloor p^{v / 2}\right\rfloor$.
For the second range when $z \geq p^{4}$, we use (3.13) when $z \geq 3213^{2}$ and for smaller values of $z$ we use Corollary 2. That is,

$$
\begin{aligned}
& S_{p}^{\prime \prime}:=\int_{p^{4}}^{\infty} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z \leq \int_{3213^{2}}^{\infty} \frac{\lambda}{z^{\lambda+1 / 2}}\left(1-\log 2+\frac{2.5}{\log z}+\frac{10}{(\log z)^{2}}\right) d z \\
&+0.5 \int_{\max \left(p^{4}, 185^{2}\right)}^{3213^{2}} \frac{\lambda}{z^{1 / 2+\lambda}} d z+0.55 \int_{p^{4}}^{\max \left(p^{4}, 185^{2}\right)} \frac{\lambda}{z^{1 / 2+\lambda}} d z .
\end{aligned}
$$

Denote the integrals

$$
\begin{aligned}
& f(y):=\int_{y}^{\infty} \frac{\lambda}{z^{\lambda+1 / 2}} d z \\
& g(y):=\int_{y}^{\infty} \frac{\lambda}{z^{\lambda+1 / 2}}\left(1-\log 2+\frac{2.5}{\log z}+\frac{10}{(\log z)^{2}}\right) d z
\end{aligned}
$$

So we obtain

$$
\begin{align*}
S_{p}^{\prime \prime} \leq & (1-\log 2) f\left(3213^{2}\right)+g\left(3213^{2}\right) \\
& +0.5\left[f\left(\max \left(p^{4}, 185^{2}\right)\right)-f\left(3213^{2}\right)\right]+0.55\left[f\left(p^{4}\right)-f\left(\max \left(p^{4}, 185^{2}\right)\right)\right] \\
& =(0.5-\log 2) f\left(3213^{2}\right)+g\left(3213^{2}\right)-0.05 f\left(\max \left(p^{4}, 185^{2}\right)\right)+0.55 f\left(p^{4}\right) . \tag{3.15}
\end{align*}
$$

Using the estimates in (3.14), (3.15), we bound $S_{p}=S_{p}^{\prime}+S_{p}^{\prime \prime}$ by the following.

| $p$ | $S_{p} \leq$ | $\sum_{q \leq p} S_{q} \leq$ | $\sum_{q \leq p}\left(q^{-\lambda}-q^{-\tau}\right) \geq$ |
| :---: | :---: | :---: | :---: |
| 11 | 0.13259 | 0.40792 | 0.55427 |
| 13 | 0.11241 | 0.52033 | 0.62966 |
| 17 | 0.08382 | 0.60415 | 0.69432 |
| 19 | 0.07601 | 0.68016 | 0.75484 |
| 23 | 0.06194 | 0.74210 | 0.80868 |
| 29 | 0.04757 | 0.78967 | 0.85521 |
| 31 | 0.04501 | 0.83468 | 0.89978 |
| 37 | 0.03680 | 0.87148 | 0.93950 |

Note that the first entry in the third column is found by adding $S_{11}$ to the estimate in (3.10). Since the entries in the fourth column exceed the entries in the third column, (3.9) implies Theorem 3.1 for $Y \leq 37$.
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### 3.2 Large primes, $Y \geq 41$

Now assume that $Y \geq 41$. We have via partial summation that

$$
\sum_{\substack{t \in T \\ P(t)<t^{\theta}}} \frac{1}{t^{\lambda}}=\sum_{p \leq 7} S_{p}+\sum_{11 \leq p \leq 23} \int_{p^{v}}^{29^{v}} \frac{\lambda}{z^{1+\lambda}} N_{p}(z) d z+\int_{22^{v}}^{\infty} \frac{\lambda}{z^{1+\lambda}} N(z) d z .
$$

(As before, the last integral converges.) From (3.10) the $S_{p}$ terms contribute at most 0.27533 . Using Lemmas 5, 6, and 7, and Corollary 2, we similarly obtain

$$
\begin{aligned}
& \sum_{\substack{t \in T \\
P(t)<t^{\theta}}} \frac{1}{t^{\lambda}} \\
&<0.27533+0.08455+0.06576+0.03756+0.02953+0.01487+0.45614=0.96374
\end{aligned}
$$

where the second to the sixth terms correspond to the five finite integrals, and the last term is our estimate for the tail integral. We also note that

$$
\sum_{p \leq Y}\left(\frac{1}{p^{\lambda}}-\frac{1}{p^{\tau}}\right) \geq \sum_{p \leq 41}\left(\frac{1}{p^{\lambda}}-\frac{1}{p^{\tau}}\right)>0.97661
$$

Since this estimate exceeds the prior one, this gives Theorem 3.1 with $\lambda=0.7982562$.

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