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A GENERALIZATION OF ROBERTS' COUNTEREXAMPLE TO THE FOURTEENTH PROBLEM OF HILBERT

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Abstract. We generalize Roberts' counterexample to the fourteenth problem of Hilbert, and give a sufficient condition for certain invariant rings not to be finitely generated. It shows that there exist a lot of counterexamples of this type. We also determine the initial algebra of Roberts' counterexample for some monomial order.

1. Introduction. The fourteenth problem of Hilbert asks whether the K-algebra $L\cap A$ is finitely generated. Here, K is a field, A is a polynomial ring over K, and L is a subfield of the quotient field of A containing K. The first counterexample to this problem was found by Nagata in 1958. It was given as the invariant subring of a polynomial ring in 32 variables for a linear action of the 13-dimensional additive group (cf. [12]). Recently, Mukai [11] showed that there exists a similar counterexample which is the invariant subring of a polynomial ring in 18 variables for a linear action of the three-dimensional additive group.

In 1990, Roberts gave a simple new counterexample of different type as follows.

THEOREM 1.1 (Roberts [14, Theorem 1]). *Let $A = K[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$ be a polynomial ring in seven variables over a field K of characteristic zero. For each nonnegative integer t, let L_t be the subfield of the quotient field of A generated by*

$$x_1\,,\quad x_2\,,\quad x_3\,,\quad x_1y_4 - x_2^tx_3^ty_1\,,\quad x_2y_4 - x_1^tx_3^ty_2\,,\quad x_3y_4 - x_1^tx_2^ty_3 \tag{1.1}$$

over K. If $t \geq 2$, then the K-algebra $L_t \cap A$ is not finitely generated.

Following this result, Deveney and Finston [2] showed that this counterexample can be obtained as the invariant subring of A for a nonlinear action of the one-dimensional additive group G_a. Kojima and Miyanishi [6] generalized Roberts' counterexample. They constructed a G_a-invariant subring of the polynomial ring of each dimension greater than or equal to seven which is not finitely generated. Furthermore, Freudenburg [4] gave a counterexample in dimension six, while Daigle and Freudenburg [1] gave one in dimension five.

In the present paper, we will generalize Roberts' counterexample further, and show that there exist a lot of counterexamples of this type. We give in Theorems 1.3 and 1.4 sufficient conditions for a certain kind of G_a-invariant subring of a polynomial ring not to be finitely generated. In Section 3, we will discuss Roberts' counterexample $L_t\cap A$ in terms of the theory

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of SAGBI (Subalgebra Analogue to Gröbner Bases for Ideals) bases. As a consequence, we determine a generating set of it in Theorem 3.3. We also remark on a sufficient condition for finite generation in Section 4.

Throughout this paper, let K denote a field of characteristic zero. Assume that R is a commutative K-algebra, and A is a commutative R-algebra. An R-homomorphism $D : A \to A$ is called an *R-derivation* on A if $D(ab) = D(a)b + aD(b)$ holds for any $a, b \in A$. Then, its kernel

$$A^D = \{a \in A \mid D(a) = 0\}$$

is an R-subalgebra of A. An R-derivation D on A is said to be *locally nilpotent* if, for each $a \in A$, there exists $r \in \mathbf{Z}_{\geq 0}$ such that $D^r(a) = 0$. Here, we denote by $\mathbf{Z}_{\geq 0}$ the set of nonnegative integers. We remark that a locally nilpotent R-derivation D on A defines an action $A \to A \otimes_R R[t]$ of the one-dimensional additive group scheme $G_a = \operatorname{Spec} R[t]$ over R on A by $a \mapsto \sum_{k \geq 0} D^k(a) \otimes (t^k/k!)$. The invariant subring A^{G_a} of A for this action of G_a is equal to A^D (cf. [10]).

Let $R = K[\boldsymbol{x}] = K[x_1, \ldots, x_m]$ be the polynomial ring in m variables over K, and $A = K[\boldsymbol{x}][\boldsymbol{y}] = K[\boldsymbol{x}][y_1, \ldots, y_n]$ that in n variables over $K[\boldsymbol{x}]$. A $K[\boldsymbol{x}]$-derivation D on $K[\boldsymbol{x}][\boldsymbol{y}]$ is said to be *elementary* if $D(y_j)$ is in $K[\boldsymbol{x}]$ for each j. Note that an elementary $K[\boldsymbol{x}]$-derivation is locally nilpotent. An elementary $K[\boldsymbol{x}]$-derivation D on $K[\boldsymbol{x}][\boldsymbol{y}]$ is said to be *monomial* if each $D(y_i)$ is a monomial, i.e., $x_1^{a_1} \cdots x_m^{a_n}$ for some $(a_1, \ldots, a_m) \in (\mathbf{Z}_{\geq 0})^m$. In this paper, we discuss the problem of finite generation of the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^D$ of an elementary monomial $K[\boldsymbol{x}]$-derivation D. As we remarked above, it is equal to the invariant subring of $K[\boldsymbol{x}][\boldsymbol{y}]$ for an action of G_a, since D is locally nilpotent. Note that $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is finitely generated over K if and only if it is so over $K[\boldsymbol{x}]$.

In the case of $n = m + 1$, the $K[\boldsymbol{x}]$-derivation

$$D_{t,m} = x_1^{t+1} \frac{\partial}{\partial y_1} + \cdots + x_m^{t+1} \frac{\partial}{\partial y_m} + (x_1 \cdots x_m)^t \frac{\partial}{\partial y_{m+1}} \tag{1.2}$$

on $K[\boldsymbol{x}][\boldsymbol{y}]$ is elementary and monomial. The kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of this $K[\boldsymbol{x}]$-derivation has been studied well. Deveney and Finston [2] showed that Roberts' K-algebra $L_t \cap A$ in Theorem 1.1 is equal to the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ for $m = 3$ (see also Maubach's result found in [3, Section 9.6]). Furthermore, Kojima and Miyanishi showed the following.

THEOREM 1.2 (Kojima-Miyanishi [6]). *Assume that $n = m + 1$. If $t \geq 2$ and $m \geq 3$, then the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of the $K[\boldsymbol{x}]$-derivation $D_{t,m}$ is not finitely generated over K.*

We will study the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^D$ of an elementary monomial $K[\boldsymbol{x}]$-derivation D on $K[\boldsymbol{x}][\boldsymbol{y}]$ of more general form. Let $D(y_i) = \boldsymbol{x}^{\delta_i}$ for each $i = 1, \ldots, n$. Here, we denote by $\boldsymbol{x}^a$ the monomial $x_1^{a_1} \cdots x_m^{a_m}$ for $a = (a_1, \ldots, a_m) \in \mathbf{Z}^m$. Similarly, we denote by $\boldsymbol{y}^b$ the monomial $y_1^{b_1} \cdots y_n^{b_n}$ for $b = (b_1, \ldots, b_n) \in \mathbf{Z}^n$. Put $\varepsilon_{i,j} = \delta_i - \delta_j$ for i, j, and for $k = 1, \ldots, m$, let $\varepsilon_{i,j}^k$ and δ_i^k be the k-th components of $\varepsilon_{i,j}$ and δ_i, respectively.

In Sections 1 and 2, we deal with the case where $n \geq 4$, $m \geq n - 1$ and $\varepsilon_{i,j}^i > 0$ for any $1 \leq i \leq n - 1$, $1 \leq j \leq n$ with $i \neq j$. The derivation $D_{t,m}$ satisfies this condition with

$\varepsilon^i_{i,j} = t+1$ if $j \neq m+1$, and $\varepsilon^i_{i,j} = 1$ otherwise. We define

$$\eta = \frac{\varepsilon^1_{1,n}}{\min\{\varepsilon^1_{1,j} \mid j = 2, \ldots, n-1\}}, \tag{1.3}$$

and

$$\eta_{k,i} = \eta \min\{\max\{\varepsilon^i_{1,k}, \varepsilon^i_{2,k}\}, 0\} \tag{1.4}$$

for $i = 2, \ldots, n-1$ and $k = 3, \ldots, n-1$. For each $k = 3, \ldots, n-1$, we set $\mathcal{L}_{k,n-2}$ to be the system of linear inequalities

$$\begin{cases} u_1 + \cdots + u_{n-2} = 1 \\ u_1 \geq \eta,\ u_i \geq 0\ (i = 2, \ldots, n-2) \\ \displaystyle\sum_{j=1}^{n-2} \min\{\varepsilon^i_{n,1}, \varepsilon^i_{n,j+1}\} u_j + \eta_{k,i} \geq 0\ (i = 2, \ldots, n-1) \end{cases} \tag{1.5}$$

in the $n-2$ variables $u_1, \ldots, u_{n-2}$.

Here is our main result.

THEOREM 1.3. *Assume that $n \geq 4$, $m \geq n-1$ and $\varepsilon^i_{i,j} > 0$ for any $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$. If the system $\mathcal{L}_{k,n-2}$ of linear inequalities has a solution in $\boldsymbol{R}^{n-2}$ for each $k = 3, \ldots, n-1$, then $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is not finitely generated over K.*

By this theorem, we get the following simple criterion for $n = 4$.

THEOREM 1.4. *Assume that $m \geq 3$, $n = 4$ and $\varepsilon^i_{i,j} > 0$ for any $1 \leq i \leq 3$, $1 \leq j \leq 4$ with $i \neq j$. If*

$$\frac{\varepsilon^1_{1,4}}{\min\{\varepsilon^1_{1,2}, \varepsilon^1_{1,3}\}} + \frac{\varepsilon^2_{2,4}}{\min\{\varepsilon^2_{2,3}, \varepsilon^2_{2,1}\}} + \frac{\varepsilon^3_{3,4}}{\min\{\varepsilon^3_{3,1}, \varepsilon^3_{3,2}\}} \leq 1, \tag{1.6}$$

then $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is not finitely generated over K.

The examples of Roberts are included as special cases of this theorem for $m = 3$. In case $(m, n) = (3, 4)$, there exist 2450001 derivations on $K[\boldsymbol{x}][\boldsymbol{y}]$ which satisfy (1.6) and $\gcd\{\boldsymbol{x}^{\delta_1}, \boldsymbol{x}^{\delta_2}, \boldsymbol{x}^{\delta_3}, \boldsymbol{x}^{\delta_4}\} = 1$ even if we impose the restriction $\delta^k_i \leq 10$ for all i, k.

In the following corollary, the case where $m \geq 4$ and $t = 1$ is new, while the case $m \geq 3$ and $t \geq 2$ was proved in [6].

COROLLARY 1.5. *Assume that $n = m+1$. If $m \geq 3$ and $t \geq 2$, or $m \geq 4$ and $t = 1$, then the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of the $K[\boldsymbol{x}]$-derivation $D_{t,m}$ is not finitely generated over K.*

We will prove Theorems 1.3, 1.4 and Corollary 1.5 in Section 2.

We remark that, if $t = 0$, then the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of $D_{t,m}$ is finitely generated for any m by Weitzenböck's theorem (cf. [12, Chapter IV]). In fact, it is isomorphic to a polynomial ring in $2m$ variables over K by the remark after Lemma 4.2 below. If $m \leq 2$, then $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ is also isomorphic to a polynomial ring in $2m$ variables over K for any $t \geq 0$

by [5, Theorem 3.1]. For $(t, m) = (1, 3)$, Kurano [7] showed that $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ is generated by nine elements over $K[\boldsymbol{x}]$.

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2. Construction of invariants. In this section, we prove Theorem 1.3, and show Theorem 1.4 and Corollary 1.5 as its consequences. Throughout this section, we assume that $n \geq 4$, $m \geq n-1$ and that D satisfies $\varepsilon^i_{i,j} > 0$ for any $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$. We denote $K[\boldsymbol{x}, x_n^{-1}, \dots, x_m^{-1}][\boldsymbol{y}] = K[\boldsymbol{x}][\boldsymbol{y}] \otimes_{K[x_n,\dots,x_m]} K[x_n, \dots, x_m, x_n^{-1}, \dots, x_m^{-1}]$. Note that D is uniquely extended to a $K[\boldsymbol{x}]$-derivation on each $K[\boldsymbol{x}]$-subalgebra of $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$.

Theorem 1.3 follows from the following two lemmas.

LEMMA 2.1. *If a monomial of the form $\boldsymbol{x}^a y_n^l$ with $l > 0$ appears in an element of $K[\boldsymbol{x}][\boldsymbol{y}]^D$, then at least one of the first $n-1$ components of $a \in (\mathbf{Z}_{\geq 0})^m$ is positive.*

PROOF. Suppose to the contrary that there appears in $f \in K[\boldsymbol{x}][\boldsymbol{y}]^D$ a monomial $\boldsymbol{x}^a y_n^l$ with the first $n-1$ components of a zero with nonzero coefficient. Then, the monomial $\boldsymbol{x}^a \boldsymbol{x}^{\delta_n} y_n^{l-1}$ appears in $D(f)$. Since $D(f) = 0$, its coefficient in $D(f)$ is zero. Hence, $\boldsymbol{x}^a \boldsymbol{x}^{\delta_n} y_n^{l-1}$ appears as a monomial in $D(\boldsymbol{x}^{a'} \boldsymbol{y}^{b'})$ for some monomial $\boldsymbol{x}^{a'} \boldsymbol{y}^{b'} \neq \boldsymbol{x}^a y_n^l$ of f. Such $\boldsymbol{x}^{a'} \boldsymbol{y}^{b'}$ must be equal to $\boldsymbol{x}^a \boldsymbol{x}^{\varepsilon_{n,i}} y_i y_n^{l-1}$ for some $i < n$. Since $\varepsilon^i_{n,i} < 0$ for $i < n$, we have $\boldsymbol{x}^{a'} \boldsymbol{y}^{b'} \notin K[\boldsymbol{x}][\boldsymbol{y}]$. This contradicts $f \in K[\boldsymbol{x}][\boldsymbol{y}]$. Thus, at least one of the first $n-1$ components of $a \in (\mathbf{Z}_{\geq 0})^m$ is positive. □

The lemma below asserts the existence of an infinite system of invariants.

LEMMA 2.2. *Under the assumption in Theorem* 1.3, *there exists a positive integer α such that a Laurent polynomial of the form*

$$x_1^\alpha y_n^l + (\text{terms of lower degree in } y_n) \tag{2.1}$$

belongs to $K[\boldsymbol{x}, x_n^{-1}, \dots, x_m^{-1}][\boldsymbol{y}]^D$ for each $l > 0$.

First, we show Theorem 1.3 by assuming these lemmas. Suppose that $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by a finite number of elements $g_1, \dots, g_p$. Then, by Lemma 2.1, there exists $r > 0$ such that each monomial appearing in g_i of the form $x_1^\beta \boldsymbol{x}^b y_n^l$ with $l > 0$ and the first $n-1$ components of b zero satisfies $l/\beta < r$ for every i. Since every element of $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is written as a sum of products of $g_1, \dots, g_p$, a monomial appearing in an element of $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is a product of monomials contained in $g_1, \dots, g_p$. Hence, any monomial appearing in an element of $K[\boldsymbol{x}][\boldsymbol{y}]^D$ of the form $x_1^\beta \boldsymbol{x}^b y_n^l$ with $l > 0$ and the first $n-1$ components of b zero also satisfies $l/\beta < r$. By Lemma 2.2, there appears in some $f \in K[\boldsymbol{x}, x_n^{-1}, \dots, x_m^{-1}][\boldsymbol{y}]^D$ a monomial $x_1^\alpha y_n^l$ with $l/\alpha > r$. Since $\boldsymbol{x}^a f$ is in $K[\boldsymbol{x}][\boldsymbol{y}]^D$ for some $a \in (\mathbf{Z}_{\geq 0})^m$ whose first $n-1$ components are zero, we are led to a contradiction. Thus, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is not finitely generated.

Let us denote by $K[\boldsymbol{y}]_l$ the K-vector subspace of $K[\boldsymbol{y}] = K[y_1, \dots, y_n]$ of homogeneous l-forms in $y_1, \dots, y_n$. For each $f = \sum_{b \in \boldsymbol{Z}^n} \lambda_b \boldsymbol{y}^b \in K[\boldsymbol{y}]$, we define the *support* $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) = \{b \in \boldsymbol{Z}^n \mid \lambda_b \neq 0\}. \tag{2.2}$$

For each $a \in \boldsymbol{Z}^m$, we define the K-linear map $\tau_{\boldsymbol{x}^a} : K[\boldsymbol{y}] \to K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$ by $\tau_{\boldsymbol{x}^a}(\boldsymbol{y}^b) = \boldsymbol{x}^{a'}\boldsymbol{y}^b$. Here, $b = (b_1, \dots, b_n)$ and $a' = a + \sum_{j=1}^n b_j \varepsilon_{n,j}$. We define an elementary K-derivation E on $K[\boldsymbol{y}]$ by

$$E = \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial y_n}. \tag{2.3}$$

Then, it follows that $D(\tau_{\boldsymbol{x}^a}(f)) = \boldsymbol{x}^{\delta_n}\tau_{\boldsymbol{x}^a}(E(f))$ for each $a \in \boldsymbol{Z}^m$ and $f \in K[\boldsymbol{y}]$. We set

$$B = K[y_2 - y_1, y_3 - y_1, \dots, y_n - y_1]. \tag{2.4}$$

Then, $\tau_{\boldsymbol{x}^a}(B) \subset K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]^D$ for $a \in \boldsymbol{Z}^m$. Actually, $D(\tau_{\boldsymbol{x}^a}(f)) = \boldsymbol{x}^{\delta_n}\tau_{\boldsymbol{x}^a}(E(f)) = 0$ for $f \in B$, since $E(f) = 0$. We define $\boldsymbol{R}$-linear maps $l_i : \boldsymbol{R}^n \to \boldsymbol{R}$ by

$$l_1((b_1, \dots, b_n)) = \varepsilon_{n,1}^1 b_1 + \min\{\varepsilon_{n,j}^1 \mid j = 2, \dots, n-1\} \sum_{j=2}^{n-1} b_j \tag{2.5}$$

and

$$l_i((b_1, \dots, b_n)) = \sum_{j=1}^{n-1} \min\{\varepsilon_{n,1}^i, \varepsilon_{n,j}^i\} b_j \tag{2.6}$$

for $i = 2, \dots, n-1$. We put $B_l = B \cap K[\boldsymbol{y}]_l$ for each $l \in \boldsymbol{Z}_{\geq 0}$.

We reduce Lemma 2.2 to the following lemma.

LEMMA 2.3. *Under the assumption in Theorem* 1.3, *there exists a positive integer* α *such that, for each positive integer* l, *we may find* $f \in B_l$ *such that* $(0, \dots, 0, l) \in \mathrm{supp}(f)$ *and every* $b \in \mathrm{supp}(f)$ *satisfies* $l_1(b) + \alpha \geq 0$ *and* $l_i(b) \geq 0$ *for* $i = 2, \dots, n-1$.

Lemma 2.2 is proved by this lemma as follows. As we mentioned above, $\tau_{x_1^\alpha}(f)$ is in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]^D$. It has the form of (2.1). We show that it is in $K[\boldsymbol{x}, x_n^{-1}, \dots, x_m^{-1}][\boldsymbol{y}]$. By definition, every monomial appearing in $\tau_{x_1^\alpha}(f)$ is written as $x_1^\alpha \boldsymbol{x}^{a'}\boldsymbol{y}^b$, where $b = (b_1, \dots, b_n) \in \mathrm{supp}(f)$ and $a' = \sum_{j=1}^n b_j \varepsilon_{n,j}$. By assumption, we have

$$\sum_{j=1}^n b_j \varepsilon_{n,j}^1 + \alpha \geq l_1(b) + \alpha \geq 0$$

and

$$\sum_{j=1}^n b_j \varepsilon_{n,j}^i \geq l_i(b) \geq 0$$

for $i = 2, \dots, n-1$. Hence, $x_1^{\alpha} \boldsymbol{x}^{a'} \boldsymbol{y}^b$ does not have negative power in $x_1, \dots, x_{n-1}$. Thus, $\tau_{x_1^{\alpha}}(f)$ is in $K[\boldsymbol{x}, x_n^{-1}, \dots, x_m^{-1}][\boldsymbol{y}]^D$. This proves Lemma 2.2.

Let P_D be the set of $b = (b_1, \dots, b_n) \in (\boldsymbol{R}_{\geq 0})^n$ with

$$b_1 = b_n = 0, \quad b_2 + \cdots + b_{n-1} = 1\,, \quad l_i(b) \geq 0 \quad (i = 2, \dots, n-1)\,. \tag{2.7}$$

Here, we denote by $\boldsymbol{R}_{\geq 0}$ the set of nonnegative real numbers. For each $b = (b_1, \dots, b_{n-2}) \in \boldsymbol{R}^{n-2}$, we set $\iota(b) = (0, b_1, \dots, b_{n-2}, 0)$. Note that, if $b \in (\boldsymbol{R}_{\geq 0})^{n-2}$ is a solution of $\mathcal{L}_{k,n-2}$, then $l_i(\iota(b)) + \eta_{k,i} \geq 0$ for $i = 2, \dots, n-1$. This condition is equivalent to the condition that $\iota(b), \iota(b) + \eta(\boldsymbol{e}_k - \boldsymbol{e}_2) \in P_D$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ are the coordinate unit vectors of $\boldsymbol{R}^n$. Indeed, if $\varepsilon_{n,k}^i < \varepsilon_{n,1}^i$, then

$$\begin{aligned} \eta_{k,i} &= \eta \min\{\max\{\varepsilon_{1,k}^i, \varepsilon_{2,k}^i\}, 0\} \\ &= \eta \min\{\varepsilon_{n,k}^i - \min\{\varepsilon_{n,1}^i, \varepsilon_{n,2}^i\}, 0\} \\ &= \eta \min\{\min\{\varepsilon_{n,k}^i, \varepsilon_{n,1}^i\} - \min\{\varepsilon_{n,1}^i, \varepsilon_{n,2}^i\}, 0\} \\ &= \min\{\eta l_i(\boldsymbol{e}_k - \boldsymbol{e}_2), 0\}\,. \end{aligned} \tag{2.8}$$

If $\varepsilon_{n,k}^i \geq \varepsilon_{n,1}^i$, then $\varepsilon_{1,k}^i \geq 0$. The equality $\eta_{k,i} = \min\{\eta l_i(\boldsymbol{e}_k - \boldsymbol{e}_2), 0\}$ also holds in this case, since the right hand sides of the first and the third equality in (2.8) are zero.

For a convex subset $P \subset \boldsymbol{R}^n$, we denote $rP = \{rb \mid b \in P\}$ for $r \in \boldsymbol{R}_{\geq 0}$.

LEMMA 2.4. *Under the assumption in Theorem* 1.3, *there exists* $\alpha' > 0$ *such that, for any* $r > \alpha'$ *and* $u_3, \dots, u_{n-1} \geq 0$ *with* $\sum_{k=3}^{n-1} u_k \leq \eta(r - \alpha')$, *there exist* $p_3, \dots, p_{n-1} \in \boldsymbol{Z}_{\geq 0}$ *such that*

$$r\boldsymbol{e}_2 + \sum_{k=3}^{n-1} (s_k u_k + p_k)(\boldsymbol{e}_k - \boldsymbol{e}_2) \in rP_D \tag{2.9}$$

for any $s_3, \dots, s_{n-1} \in [0, 1]$.

PROOF. Since $\mathcal{L}_{k,n-2}$ has a solution, there exists $\boldsymbol{b}_k \in P_D$ with $\boldsymbol{b}_k + \eta(\boldsymbol{e}_k - \boldsymbol{e}_2) \in P_D$ for each $k = 3, \dots, n-1$. Let P be the convex hull of

$$\{\boldsymbol{b}_k, \boldsymbol{b}_k + \eta(\boldsymbol{e}_k - \boldsymbol{e}_2) \mid k = 3, \dots, n-1\}$$

in $\boldsymbol{R}^n$, and d a positive number such that the d-neighborhood of a point $\boldsymbol{a} \in P$ is contained in P. Here, we consider the Euclidean topology induced from that on the affine subspace $H = \boldsymbol{e}_2 + \sum_{k=3}^{n-1} \boldsymbol{R}(\boldsymbol{e}_k - \boldsymbol{e}_2)$. Then, define $\alpha' = (1/d)\sqrt{(n-2)(n-3)}$. We show that this α' satisfies the desired property.

Take any $r > \alpha'$. Note that it suffices to show (2.9) for $u_3, \dots, u_{n-1} \geq 0$ with $\sum_{k=3}^{n-1} u_k = \eta(r - \alpha')$. We set $u_k' = u_k / (\eta(r - \alpha'))$ for each k. Then,

$$\sum_{k=3}^{n-1} u_k'(\boldsymbol{b}_k + s_k \eta(\boldsymbol{e}_k - \boldsymbol{e}_2)) \in P \tag{2.10}$$

for any $s_3, \dots, s_{n-1} \in [0, 1]$. Actually, since P is convex,

$$\boldsymbol{b}_k + s_k\eta(\boldsymbol{e}_k - \boldsymbol{e}_2) = (1 - s_k)\boldsymbol{b}_k + s_k(\boldsymbol{b}_k + \eta(\boldsymbol{e}_k - \boldsymbol{e}_2))$$

is in P for each k. Since $\sum_{k=3}^{n-1} u'_k = 1$, we get (2.10).

For each $\boldsymbol{q} \in H$, define a map $T_{\boldsymbol{q}} : P \to rH$ by $T_{\boldsymbol{q}}(c) = \alpha'\boldsymbol{q} + (r - \alpha')c$. Since $0 < \alpha' < r$, we have $T_{\boldsymbol{q}}(P) \subset rP$ if $\boldsymbol{q} \in P$. Put $\boldsymbol{b}' = T_{\boldsymbol{a}}(\sum_{k=3}^{n-1} u'_k\boldsymbol{b}_k)$, and choose $p'_k \in \boldsymbol{R}_{\geq 0}$ so that $\boldsymbol{b}' = r\boldsymbol{e}_2 + \sum_{k=3}^{n-1} p'_k(\boldsymbol{e}_k - \boldsymbol{e}_2)$. Then, let p_k be the nonnegative integer we obtain by adding an element in $(-1/2, 1/2]$ to p'_k for each k. Put $\boldsymbol{b} = r\boldsymbol{e}_2 + \sum_{k=3}^{n-1} p_k(\boldsymbol{e}_k - \boldsymbol{e}_2)$ and $\boldsymbol{a}' = \boldsymbol{a} + (\alpha')^{-1}(\boldsymbol{b} - \boldsymbol{b}')$. Then,

$$|\boldsymbol{b} - \boldsymbol{b}'| = \sqrt{\left(\sum_{k=3}^{n-1}(p_k - p'_k)\right)^2 + \sum_{k=3}^{n-1}(p_k - p'_k)^2} \leq \frac{\sqrt{(n-2)(n-3)}}{2}.$$

So, we have

$$|\boldsymbol{a} - \boldsymbol{a}'| = (\alpha')^{-1}|\boldsymbol{b} - \boldsymbol{b}'| \leq d/2.$$

By the choice of $\boldsymbol{a}$, the point $\boldsymbol{a}'$ is in P. Hence, $T_{\boldsymbol{a}'}(P) \subset rP$. Moreover,

$$T_{\boldsymbol{a}'}(c) - T_{\boldsymbol{a}}(c) = \alpha'(\boldsymbol{a}' - \boldsymbol{a}) = \boldsymbol{b} - \boldsymbol{b}'$$

for $c \in P$. Thus, we get

$$(\boldsymbol{b} - \boldsymbol{b}') + T_{\boldsymbol{a}}(P) \subset rP. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}(\boldsymbol{b} - \boldsymbol{b}') + T_{\boldsymbol{a}}\left(\sum_{k=3}^{n-1} u'_k(\boldsymbol{b}_k + s_k\eta(\boldsymbol{e}_k - \boldsymbol{e}_2))\right) &= \boldsymbol{b} + \sum_{k=3}^{n-1} s_k u_k(\boldsymbol{e}_k - \boldsymbol{e}_2) \\ &= r\boldsymbol{e}_2 + \sum_{k=3}^{n-1}(p_k + s_k u_k)(\boldsymbol{e}_k - \boldsymbol{e}_2).\end{aligned}$$

It is in $(\boldsymbol{b} - \boldsymbol{b}') + T_{\boldsymbol{a}}(P)$ for any $s_k \in [0, 1]$ by (2.10). Then, (2.9) follows from (2.11), since rP is contained in rP_D. Therefore, α' satisfies the desired property. □

Now, let us prove Lemma 2.3. First, we show that the assumption that each $\mathcal{L}_{k,n-2}$ has a solution implies that $\varepsilon^i_{n,1} \geq 0$ and $\varepsilon^1_{n,i} > 0$ for $i = 2, \dots, n-1$. Suppose to the contrary that $\varepsilon^i_{n,1} < 0$ for some $2 \leq i \leq n-1$. Then, for any $(u_1, \dots, u_{n-2}) \in (\boldsymbol{R}_{\geq 0})^{n-2}$ with $\sum_{j=1}^{n-2} u_j = 1$, we have

$$\sum_{j=1}^{n-2} \min\{\varepsilon^i_{n,1}, \varepsilon^i_{n,j+1}\}u_j + \eta_{k,i} \leq \varepsilon^i_{n,1} + \eta_{k,i} < 0.$$

This contradicts the assumption that $\mathcal{L}_{k,n-2}$ has a solution. Thus, $\varepsilon^i_{n,1} \geq 0$ for $i = 2, \dots, n-1$. Suppose that $\varepsilon^1_{n,i} \leq 0$ for some $2 \leq i \leq n-1$. Then, it implies that $\eta \geq 1$, since

$$\varepsilon^1_{1,n} - \min\{\varepsilon^1_{1,j} \mid j = 2, \dots, n-1\} = -\min\{\varepsilon^1_{n,j} \mid j = 2, \dots, n-1\} \geq -\varepsilon^1_{n,i} \geq 0.$$

If $\mathcal{L}_{k,n-2}$ has a solution $u = (u_1, \dots, u_{n-2})$, then $\eta = u_1 = 1$ and $u_j = 0$ for $j = 2, \dots, n-2$. For this u, it follows that

$$\sum_{j=1}^{n-2} \min\{\varepsilon^2_{n,1}, \varepsilon^2_{n,j+1}\} u_j + \eta_{k,2} = \min\{\varepsilon^2_{n,1}, \varepsilon^2_{n,2}\} + \eta_{k,2} \le \varepsilon^2_{n,2} < 0\,.$$

This is a contradiction. Thus, $\varepsilon^1_{n,i} > 0$ for $i = 2, \dots, n-1$.

Take $\alpha' > 0$ as in Lemma 2.4, and set α to be an integer greater than or equal to $\alpha' \varepsilon^1_{1,n}$. Let l be an arbitrary positive integer, and $\mathcal{F}$ the set of $f \in B_l$ such that $(0, \dots, 0, l) \in \operatorname{supp}(f)$ and every $b \in \operatorname{supp}(f)$ satisfies $l_i(b) \ge 0$ for $i = 2, \dots, n-1$. Since

$$l_i\,(j\boldsymbol{e}_1 + (l-j)\boldsymbol{e}_n) = j\varepsilon^i_{n,1} \ge 0$$

for $i = 2, \dots, n-1$ and $j = 0, \dots, l$, we have $(y_n - y_1)^l \in \mathcal{F}$. Hence, $\mathcal{F} \ne \emptyset$. We show that there exists $F_0 \in \mathcal{F}$ such that $l_1(b) + \alpha \ge 0$ for each $b \in \operatorname{supp}(F_0)$. Suppose the contrary. Then, for each $f \in \mathcal{F}$, an element $O(f) = (d, e)$ in $\boldsymbol{Z}^2$ is defined by setting d to be the maximum among the n-th components of $b \in \operatorname{supp}(f)$ with $l_1(b) + \alpha < 0$, and e to be the maximum among the first components of $b \in \operatorname{supp}(f)$ whose n-th components are d. We define the total order $\preceq$ on $\boldsymbol{Z}^2$ by $(d_1, e_1) \preceq (d_2, e_2)$ if $d_1 < d_2$ or $d_1 = d_2, e_1 \le e_2$. For $v_1, v_2 \in \boldsymbol{Z}^2$, we denote $v_1 \prec v_2$ if $v_1 \preceq v_2$ and $v_1 \ne v_2$. Choose $F \in \mathcal{F}$ with $O(F) = (d, e)$ such that $(d, e) \preceq O(h)$ for any $h \in \mathcal{F}$, and set $f \in K[y_2, \dots, y_{n-1}]$ to be the coefficient of $y_1^e y_n^d$ in F.

For $b \in \operatorname{supp}(F)$ whose first and n-th components are e and d, respectively, we have

$$\begin{aligned}
l_1(b) + \alpha &= \varepsilon^1_{n,1} e + \min\{\varepsilon^1_{n,j} \mid j = 2, \dots, n-1\}(l-d-e) + \alpha \\
&= \varepsilon^1_{n,1} e + (\varepsilon^1_{n,1} + \min\{\varepsilon^1_{1,j} \mid j = 2, \dots, n-1\})(l-d-e) + \alpha \\
&= \min\{\varepsilon^1_{1,j} \mid j = 2, \dots, n-1\}(l-d-e) - \varepsilon^1_{1,n}(l-d) + \alpha \\
&\ge \min\{\varepsilon^1_{1,j} \mid j = 2, \dots, n-1\}(l-d-e) - \varepsilon^1_{1,n}(l-d-\alpha') \\
&= \min\{\varepsilon^1_{1,j} \mid j = 2, \dots, n-1\}((l-d-e) - \eta(l-d-\alpha'))\,.
\end{aligned} \tag{2.12}$$

Since $\varepsilon^1_{1,j} > 0$ for $j \ne 1$, the right hand side of the third equality in (2.12) is negative by the maximality of e. By the last equality in (2.12) we get

$$l - d - e < \eta(l - d - \alpha')\,. \tag{2.13}$$

LEMMA 2.5. *In the above notation*, $E(f) = 0$.

PROOF. Suppose that $E(f) \ne 0$. Let $\boldsymbol{y}^b$ be a monomial appearing in $E(f)$ with nonzero coefficient. Let λ'_j be the coefficient of $y_j \boldsymbol{y}^b$ in f, and b_j the j-th component of b for each j. Then, the coefficient μ' of $\boldsymbol{y}^b$ in $E(f)$ is written as

$$\mu' = \sum_{j=2}^{n-1} (b_j + 1)\lambda'_j\,.$$

Let λ_j be the coefficient of $y_j \boldsymbol{y}^b (y_1^e y_n^d)$ in F for each j. Then, $\lambda_j = \lambda'_j$ for $j = 2, \dots, n-1$. The coefficient μ of $\boldsymbol{y}^b (y_1^e y_n^d)$ in $E(F)$ is written as

$$\mu = (e+1)\lambda_1 + \sum_{j=2}^{n-1} (b_j + 1)\lambda_j + (d+1)\lambda_n = (e+1)\lambda_1 + \mu' + (d+1)\lambda_n .$$

Since $E(F) = 0$, we have $\mu = 0$. Moreover, $\lambda_1 = 0$ by the maximality of e. Since $\mu' \neq 0$, we have $\lambda_n \neq 0$, that is,

$$b' = b + e\boldsymbol{e}_1 + (d+1)\boldsymbol{e}_n$$

is in $\operatorname{supp}(F)$. Note that $l_1(b' + \boldsymbol{e}_2 - \boldsymbol{e}_n) + \alpha$ is negative, since it is equal to the left hand side of the first equality in (2.12). Hence,

$$\begin{aligned} l_1(b') + \alpha &= l_1(b' + \boldsymbol{e}_2 - \boldsymbol{e}_n) + \alpha + l_1(\boldsymbol{e}_n - \boldsymbol{e}_2) \\ &< l_1(\boldsymbol{e}_n - \boldsymbol{e}_2) = -\min\{\varepsilon^1_{n,j} \mid j = 2, \dots, n-1\} < 0 . \end{aligned}$$

This contradicts the maximality of d. Thus, we get $E(f) = 0$. □

We claim that $K[\boldsymbol{y}]^E \subset B$. This is a special case of Lemma 4.2 which we shall prove later. By Lemma 2.5, this fact implies that f is in B_{l-d-e}.

LEMMA 2.6. *In the above notation, there exists $G \in B_l$ of the form $G = f y_1^e y_n^d + g$, where $g \in K[\boldsymbol{y}]_l$ such that every $b \in \operatorname{supp}(g)$ satisfies the following. $l_i(b) \geq 0$ for $i = 2, \dots, n-1$. If e' and d' are the first and n-th components of b, respectively, then $(d', e') \prec (d, e)$.*

PROOF. Since f is in $B_{l-d-e} \cap K[y_2, \dots, y_{n-1}]$, we have

$$f = \sum_u \lambda_u \prod_{k=3}^{n-1} (y_2 - y_k)^{u_k}$$

for some $\lambda_u \in K$. Here, the sum in the equality above is taken over $u = (u_3, \dots, u_{n-1}) \in (\mathbf{Z}_{\geq 0})^{n-3}$ with $\sum_{k=3}^{n-1} u_k = l - d - e$. By (2.13), we get $\sum_{k=3}^{n-1} u_k < \eta(l - d - \alpha')$ for each u. Hence, there exist $p_3, \dots, p_{n-1} \in \mathbf{Z}_{\geq 0}$ such that

$$(l-d)\boldsymbol{e}_2 + \sum_{k=3}^{n-1} (s_k u_k + p_k)(\boldsymbol{e}_k - \boldsymbol{e}_2) \in (l-d)P_D \tag{2.14}$$

for any $s_3, \dots, s_{n-1} \in [0, 1]$ by Lemma 2.4. We set

$$h'_u = y_2^{e-p} \prod_{k=3}^{n-1} \left((y_2 - y_k)^{u_k} y_k^{p_k} \right) ,$$

where $p = \sum_{k=3}^{n-1} p_k$. Note that each element of $\operatorname{supp}(h'_u)$ is written as the left hand side of (2.14) for some $s_3, \dots, s_{n-1} \in [0, 1]$. So, $\operatorname{supp}(h'_u)$ is contained in $(l-d)P_D$. In particular,

$e - p \geq 0$. We set

$$h_u = (y_1 - y_2)^{e-p} \prod_{k=3}^{n-1} \big((y_2 - y_k)^{u_k} (y_1 - y_k)^{p_k}\big)$$

for each u, and define

$$G = \left(\sum_u \lambda_u h_u\right)(y_n - y_1)^d .$$

Put $g = G - f y_1^e y_n^d$. Then, the first and n-th components e' and d', respectively, of each $b \in \operatorname{supp}(g)$ satisfy $(d', e') \prec (d, e)$. So, we verify that $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$ for each $b \in \operatorname{supp}(g)$. Each element of $\operatorname{supp}(h_u)$ is contained in $c + \sum_{j=2}^{n-1} \boldsymbol{Z}_{\geq 0}(\boldsymbol{e}_1 - \boldsymbol{e}_j)$ for some $c \in (l-d)P_D$. Indeed, h_u is equal to the polynomial obtained from h'_u by substituting $y_1 - y_k$ for y_k for each k, and $\operatorname{supp}(h'_u) \subset (l-d)P_D$. Therefore, we may write each $b \in \operatorname{supp}(g)$ as

$$b = d_1 \boldsymbol{e}_1 + d_2 \boldsymbol{e}_n + c + \sum_{j=2}^{n-1} v_j (\boldsymbol{e}_1 - \boldsymbol{e}_j) ,$$

where $d_1, d_2, v_2, \ldots, v_{n-1} \in \boldsymbol{Z}_{\geq 0}$ and $c \in (l-d)P_D$. Note that $l_i(\boldsymbol{e}_n) = 0$ and $l_i(\boldsymbol{e}_1), l_i(c) \geq 0$ for $i = 2, \ldots, n-1$. Moreover,

$$\begin{aligned} l_i\left(\sum_{j=2}^{n-1} v_j(\boldsymbol{e}_1 - \boldsymbol{e}_j)\right) &= -\sum_{j=2}^{n-1} \min\{\varepsilon^i_{n,1}, \varepsilon^i_{n,j}\} v_j + \min\{\varepsilon^i_{n,1}, \varepsilon^i_{n,1}\} \sum_{j=2}^{n-1} v_j \\ &= \sum_{j=2}^{n-1} (\varepsilon^i_{n,1} - \min\{\varepsilon^i_{n,1}, \varepsilon^i_{n,j}\}) v_j \geq 0 . \end{aligned}$$

Thus, we get $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$. □

We set $H = F - G$. Then, H is in $\mathcal{F}$. Moreover, $O(H) \prec O(F)$ by the definition of H. This contradicts the choice of F. Hence, there exists $F_0 \in \mathcal{F}$ such that $l_1(b) + \alpha \geq 0$ for each $b \in \operatorname{supp}(F_0)$. We have thus proved Lemma 2.3. Therefore, the proof of Theorem 1.3 is completed.

Now, assume that $m \geq 3$ and $n = 4$. Then, we set

$$\xi_i = \xi_i(D) = \frac{\varepsilon^i_{i,4}}{\min\{\varepsilon^i_{i,j}, \varepsilon^i_{i,k}\}} \tag{2.15}$$

for distinct integers $1 \leq i, j, k \leq 3$, and put $\xi(D) = \xi_1(D) + \xi_2(D) + \xi_3(D)$.

We show Theorem 1.4 as a consequence of Theorem 1.3. We verify that $(1 - \xi_2, \xi_2)$ is a solution of $\mathcal{L}_{3,2}$. Note that $\xi_i > 0$ for $i = 1, 2, 3$, $\eta = \xi_1$, $\eta_{3,2} = 0$ and $\eta_{3,3} = -\xi_1 \min\{\varepsilon^3_{3,1}, \varepsilon^3_{3,2}\}$. So, $\xi_2 > 0$. By (1.6), we have $1 - \xi_2 \geq \xi_1 + \xi_3 > \xi_1 = \eta$. Moreover, it follows that

$$\begin{aligned}&\min\{\varepsilon_{4,1}^2,\varepsilon_{4,2}^2\}(1-\xi_2)+\min\{\varepsilon_{4,1}^2,\varepsilon_{4,3}^2\}\xi_2+\eta_{3,2}\\&\quad=\min\{\varepsilon_{4,1}^2,\varepsilon_{4,2}^2\}+(\min\{\varepsilon_{4,1}^2,\varepsilon_{4,3}^2\}-\min\{\varepsilon_{4,1}^2,\varepsilon_{4,2}^2\})\xi_2+\eta_{3,2}\\&\quad=\varepsilon_{4,2}^2+\min\{\varepsilon_{2,1}^2,\varepsilon_{2,3}^2\}\xi_2=0\,,\end{aligned}$$

and

$$\begin{aligned}&\min\{\varepsilon_{4,1}^3,\varepsilon_{4,2}^3\}(1-\xi_2)+\min\{\varepsilon_{4,1}^3,\varepsilon_{4,3}^3\}\xi_2+\eta_{3,3}\\&\quad=\min\{\varepsilon_{4,1}^3,\varepsilon_{4,2}^3\}+(\min\{\varepsilon_{4,1}^3,\varepsilon_{4,3}^3\}-\min\{\varepsilon_{4,1}^3,\varepsilon_{4,2}^3\})\xi_2+\eta_{3,3}\\&\quad=(\varepsilon_{4,3}^3+\min\{\varepsilon_{3,1}^3,\varepsilon_{3,2}^3\})-\min\{\varepsilon_{3,1}^3,\varepsilon_{3,2}^3\}\xi_2+\eta_{3,3}\\&\quad=\min\{\varepsilon_{3,1}^3,\varepsilon_{3,2}^3\}\,(-\xi_3+1-\xi_2-\xi_1)\geq 0\,.\end{aligned}$$

Therefore, $(1-\xi_2,\xi_2)$ is a solution of $\mathcal{L}_{3,2}$. Hence, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is not finitely generated by Theorem 1.3.

Finally, we show Corollary 1.5. As mentioned in Section 1, $\varepsilon_{i,j}^i>0$ for any $i\neq j$, since $\varepsilon_{i,j}^i=t+1$ if $j\neq m+1$, and $\varepsilon_{i,j}^i=1$ otherwise. Assume that $m=3$ and $t\geq 2$. Then, $\xi(D_{t,m})=3/(t+1)\leq 1$. Hence, $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$ is not finitely generated by Theorem 1.4.

Assume that $m\geq 4$ and $t\geq 1$. For $k=3,\dots,m-1$, we define $u_k=(u_k^1,\dots,u_k^{m-1})\in(\boldsymbol{R}_{\geq 0})^{m-1}$ as follows. Set $u_3^3, u_k^j=1/2$ for j,k with $j=1$ or $k=j+2$, and set $u_k^j=0$ otherwise. We show that u_k is a solution of $\mathcal{L}_{k,m-1}$ for each k. Since $m\geq 4$, we have $\sum_{j=1}^{m-1}u_k^j=1$. Since $t\geq 1$, we get $u_k^1=1/2\geq 1/(t+1)=\eta$. Clearly, $u_k^j\geq 0$ for $j=2,\dots,m-1$. For $i=2,\dots,m-1$, it follows that

$$\sum_{j=1}^{m-1}\min\{\varepsilon_{m+1,1}^i,\varepsilon_{m+1,j+1}^i\}u_k^j+\eta_{k,i}=t-(t+1)u_k^{i-1}+\eta_{k,i}\,. \tag{2.16}$$

Note that $\eta_{k,i}=-1$ if $i=k$, and $\eta_{k,i}=0$ otherwise. If $i=k$, then the right hand side of (2.16) is equal to $t-1$, since $u_k^{k-1}=0$. If $i\neq k$, then it is not less than $(t-1)/2$, since $u_k^{i-1}\leq 1/2$ for any i,k. So, it is nonnegative for every i,k. Therefore, u_k is a solution of $\mathcal{L}_{k,m-1}$ for $k=3,\dots,m-2$. By Theorem 1.3, $K[\boldsymbol{x}]^{D_{t,m}}$ is not finitely generated. Thus, we complete the proof of Corollary 1.5.

3. A SAGBI basis for the counterexample of Roberts. In this section, we consider the counterexample of Roberts. Recall that it is obtained as the kernel of the derivation $D_{t,m}$ on $K[\boldsymbol{x}][\boldsymbol{y}]$ for $(m,n)=(3,4)$ and $t\geq 2$ by the result of Deveney and Finston [2]. We determine its initial algebra for some monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$. Consequently, it will turn out that the infinite system of invariants appearing in Roberts' proof of [14, Lemma 3] is a generating set of $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$.

First, we review the notion of an initial algebra and a SAGBI (Subalgebra Analogue to Gröbner Bases for Ideals) basis. Let $\preceq$ be a monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$, i.e., a total order on $\boldsymbol{Z}^m\times\boldsymbol{Z}^n$ such that $a\preceq b$ implies $a+c\preceq b+c$ for any $a,b,c\in\boldsymbol{Z}^m\times\boldsymbol{Z}^n$ and the zero vector is the minimum among $(\mathbf{Z}_{\geq 0})^m \times (\mathbf{Z}_{\geq 0})^n$ for $\preceq$. We denote $a \prec b$ if $a \neq b$ and $a \preceq b$. We sometimes denote $\boldsymbol{x}^a\boldsymbol{y}^b \preceq \boldsymbol{x}^{a'}\boldsymbol{y}^{b'}$ instead of $(a, b) \preceq (a', b')$. For $f \in K[\boldsymbol{x}][\boldsymbol{y}] \setminus \{0\}$, we define the *initial term* $\mathrm{in}_{\preceq}(f)$ of f by $\alpha \boldsymbol{x}^a\boldsymbol{y}^b$. Here, (a, b) is the maximal element of $\mathrm{supp}(f)$ for $\preceq$, and α is the coefficient of $\boldsymbol{x}^a\boldsymbol{y}^b$ in f. Note that the maximum of $\mathrm{supp}(f)$ always exists, since it is a nonempty finite set. If $f = 0$, then we define $\mathrm{in}_{\preceq}(f) = 0$. Then, it follows that

$$\mathrm{in}_{\preceq}(fg) = \mathrm{in}_{\preceq}(f)\,\mathrm{in}_{\preceq}(g) \tag{3.1}$$

for any $f, g \in K[\boldsymbol{x}][\boldsymbol{y}]$. Assume that A is a K-subalgebra of $K[\boldsymbol{x}][\boldsymbol{y}]$. We define the *initial algebra* $\mathrm{in}_{\preceq}(A)$ of A as the K-vector space generated by $\{\mathrm{in}_{\preceq}(f) \mid f \in A\}$. Then, $\mathrm{in}_{\preceq}(A)$ is a K-algebra by (3.1). We say that a generating set $\mathcal{S}$ of A is a *SAGBI basis* if the initial algebra $\mathrm{in}_{\preceq}(A)$ is generated by $\{\mathrm{in}_{\preceq}(f) \mid f \in \mathcal{S}\}$ over K.

The following is a basic property of a SAGBI basis.

LEMMA 3.1 (Robbiano-Sweedler [13, Proposition 1.16]). *Let $\preceq$ be a monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$. Assume that A is a K-subalgebra of $K[\boldsymbol{x}][\boldsymbol{y}]$, and $\mathcal{S}$ is a subset of A. If $\{\mathrm{in}_{\preceq}(f) \mid f \in \mathcal{S}\}$ generates the initial algebra $\mathrm{in}_{\preceq}(A)$ over K, then $\mathcal{S}$ is a SAGBI basis for A. In particular, $\mathcal{S}$ generates A over K.*

For any elementary monomial $K[\boldsymbol{x}]$-derivation D on $K[\boldsymbol{x}][\boldsymbol{y}]$, we set $\varepsilon_{i,j}^{+}$ to be the vector we obtain from $\varepsilon_{i,j}$ by replacing the negative components by zero, and define $L_{i,j} = \boldsymbol{x}^{\varepsilon_{j,i}^{+}} y_i - \boldsymbol{x}^{\varepsilon_{i,j}^{+}} y_j$ for each i, j. Then, $L_{i,j}$ is in $K[\boldsymbol{x}][\boldsymbol{y}]^D$ for i, j.

Now, let us consider the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of $D_{t,m}$ on $K[\boldsymbol{x}][\boldsymbol{y}]$ for $(m, n) = (3, 4)$. Note that the three elements

$$x_1^{t+1}y_2 - x_2^{t+1}y_1\,, \quad x_1^{t+1}y_3 - x_3^{t+1}y_1\,, \quad x_2^{t+1}y_3 - x_3^{t+1}y_2 \tag{3.2}$$

are contained in $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$. Indeed, they are equal to $L_{2,1}$, $L_{3,1}$ and $L_{3,2}$. Moreover, we know the following (see also [6, Lemma 2.1]).

THEOREM 3.2 (Roberts [14, Lemma 3]). *For each $d \in \mathbf{Z}_{\geq 0}$ and $i = 1, 2, 3$, there exists an element of the form $x_i y_4^d +$ (terms of lower degree in y_4) in $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$.*

We take an arbitrary $I_{d,i} \in K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$ of the form in Theorem 3.2 for each (d, i). Note that $I_{0,i} = x_i$ for each i. Let $\preceq_{\mathrm{lex}}$ be the monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$ for $(m, n) = (3, 4)$ which is the lexicographic order with

$$x_1 \prec_{\mathrm{lex}} x_2 \prec_{\mathrm{lex}} x_3 \prec_{\mathrm{lex}} y_1 \prec_{\mathrm{lex}} y_2 \prec_{\mathrm{lex}} y_3 \prec_{\mathrm{lex}} y_4\,. \tag{3.3}$$

Namely, we define $a \preceq_{\mathrm{lex}} b$ if the last nonzero component of $b - a$ is positive for $a, b \in \mathbf{Z}^3 \times \mathbf{Z}^4$, where we regard a, b as elements of $\mathbf{Z}^7$.

The following is the main result of this section.

THEOREM 3.3. *Assume that $t \geq 2$. Then, the initial algebra of $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$ for $\preceq_{\mathrm{lex}}$ is generated by*

$$\{x_1^{t+1}y_2,\ x_1^{t+1}y_3,\ x_2^{t+1}y_3\} \cup \{x_i y_4^d \mid d \in \mathbf{Z}_{\geq 0},\ i = 1, 2, 3\} \tag{3.4}$$

over K. The set

(3.5)

$$\{x_1^{t+1}y_2 - x_2^{t+1}y_1\,,\ x_1^{t+1}y_3 - x_3^{t+1}y_1\,,\ x_2^{t+1}y_3 - x_3^{t+1}y_2\} \cup \{I_{d,i} \mid d \in \boldsymbol{Z}_{\geq 0}, i = 1, 2, 3\}$$

is a SAGBI basis for $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$ *for* $\preceq_{\text{lex}}$. *In particular, it generates* $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$ *over* K.

To analyze $K[\boldsymbol{x}][\boldsymbol{y}]^D$ in greater detail, we define a grading structure on it. Let D be any elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$. We set

$$\Gamma = (\boldsymbol{Z}^m \times \boldsymbol{Z}^n)\Big/\sum_{i=2}^{n} \boldsymbol{Z}(\varepsilon_{i,1}, \boldsymbol{e}_1 - \boldsymbol{e}_i)\,,$$

and $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$ the K-vector space generated by monomials $\boldsymbol{x}^a\boldsymbol{y}^b$ for $(a, b) \in \boldsymbol{Z}^m \times (\boldsymbol{Z}_{\geq 0})^n$ with the image of (a, b) in Γ equal to γ for each $\gamma \in \Gamma$. Then, it defines a Γ-grading on $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$, i.e., $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}] = \bigoplus_{\gamma\in\Gamma} K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$ and $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\mu \subset K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_{\gamma+\mu}$ for any $\gamma, \mu \in \Gamma$. Moreover, it follows that

$$K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]^D = \bigoplus_{\gamma\in\Gamma} K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma^D\,.$$

Here, for a K-subalgebra A of $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$, we set $A_\gamma = A \cap K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$ for each γ. We say that $f \in K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$ is Γ-homogeneous if f is in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$ for some $\gamma \in \Gamma$. This γ is denoted by $\deg_\Gamma(f)$. Note that each $\gamma \in \Gamma$ is expressed as the image of $(a, l\boldsymbol{e}_n)$ for some $a \in \boldsymbol{Z}^m$ and $l \in \boldsymbol{Z}_{\geq 0}$. Then, we have $\tau_{\boldsymbol{x}^a}(K[\boldsymbol{y}]_l) = K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$. Actually, $\tau_{\boldsymbol{x}^a}(\phi(f)) = f$ for $f \in K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma$, where $\phi : K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}] \to K[\boldsymbol{y}]$ is the homomorphism which substitutes one for each x_i. Since $E \circ \phi = \phi \circ D$, we have $\phi(f) \in K[\boldsymbol{y}]_l^E = B_l$ for $f \in K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma^D$. Hence, $\tau_{\boldsymbol{x}^a}(B_l) = K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]_\gamma^D$.

We remark that, for $f \in K[\boldsymbol{y}]$, $r \in \boldsymbol{Z}_{\geq 0}$ and $a \in \boldsymbol{Z}^m$, the condition that $(y_i - y_j)^r$ divides f implies that $L_{i,j}^r$ is a factor of $\tau_{\boldsymbol{x}^a}(f)$ in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$. This is proved as follows. Note that $\tau_{\boldsymbol{x}^a}(f) = \boldsymbol{x}^a\tau_1(f)$ for any $f \in K[\boldsymbol{y}]$, and $\tau_1(y_i - y_j) = \boldsymbol{x}^{\varepsilon_{n,i}-\varepsilon_{j,i}^+}L_{i,j}$ for i, j. Assume that $f = (y_i - y_j)^r f'$ for some $f' \in K[\boldsymbol{y}]$. Then,

$$\tau_{\boldsymbol{x}^a}(f) = \boldsymbol{x}^a\tau_1((y_i - y_j)^r f') = \boldsymbol{x}^a\tau_1(y_i - y_j)^r\tau_1(f') = \boldsymbol{x}^{a+r(\varepsilon_{n,i}-\varepsilon_{j,i}^+)}L_{i,j}^r\tau_1(f')\,,$$

since τ_1 preserves multiplication. Thus, $L_{i,j}^r$ is a factor of $\tau_{\boldsymbol{x}^a}(f)$ in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]$.

Assume that $n = 3$. Then, each $f \in B_l$ is written as

$$f = (y_2 - y_1)^s(y_3 - y_1)^t\sum_{i=0}^{u}\alpha_i(y_2 - y_1)^i(y_3 - y_1)^{u-i}.$$

Here, $s, t, u \in \boldsymbol{Z}_{\geq 0}$ with $s + t + u = l$ and $\alpha_i \in K$ with $\alpha_0, \alpha_u \neq 0$. If $\beta_1, \dots, \beta_u \in \bar{K}$ are the solutions of the equation $\sum_{i=0}^{u}\alpha_i X^i = 0$, then we get

$$f = \alpha_0(y_2 - y_1)^s(y_3 - y_1)^t\prod_{i=1}^{u}(y_2 - \beta_i y_3 + (\beta_i - 1)y_1)\,, \tag{3.6}$$

where $\bar{K}$ is the algebraic closure of K.

PROPOSITION 3.4. *Assume that $n = 3$, and D is any elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$. Then,*

$$\{x_1, \dots, x_m, L_{2,1}, L_{3,1}, L_{3,2}\} \tag{3.7}$$

is a SAGBI basis for $K[\boldsymbol{x}][\boldsymbol{y}]^D$ with respect to any monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$.

PROOF. Let $\preceq$ be any monomial order on $K[\boldsymbol{x}][\boldsymbol{y}]$. By Proposition 3.1, it suffices to show that $\mathrm{in}_{\preceq}(K[\boldsymbol{x}][\boldsymbol{y}]^D)$ is equal to

$$R = K[\boldsymbol{x}][\mathrm{in}_{\preceq}(L_{2,1}), \mathrm{in}_{\preceq}(L_{3,1}), \mathrm{in}_{\preceq}(L_{3,2})]\,.$$

First, we note that, since $\boldsymbol{x}^a \tau_1(y_i - y_j) \in K[\boldsymbol{x}][\boldsymbol{y}]$, its initial term is in R for $a \in \boldsymbol{Z}^m$ and i, j. Indeed, $\boldsymbol{x}^a \tau_1(y_i - y_j) = \boldsymbol{x}^{a+\varepsilon_{3,i}-\varepsilon^+_{j,i}} L_{i,j}$, which is in $K[\boldsymbol{x}][\boldsymbol{y}]$ if and only if $a + \varepsilon_{3,i} - \varepsilon^+_{j,i} \in (\boldsymbol{Z}_{\geq 0})^m$. We show that $\boldsymbol{x}^a \tau_1(g) \in K[\boldsymbol{x}][\boldsymbol{y}]$ implies that $\mathrm{in}_{\preceq}(\boldsymbol{x}^a \tau_1(g)) \in R \otimes_K \bar{K}$ for $a \in \boldsymbol{Z}^m$, where $g = y_2 - y_1 - \beta(y_3 - y_1)$ with $\beta \in \bar{K}$. If β is zero or one, then we are done. Assume that $\beta \neq 0, 1$. Then, there appears in $\boldsymbol{x}^a \tau_1(g)$ each monomial which appears in $\boldsymbol{x}^a(\tau_1(y_i - y_1))$ for $i = 2, 3$. Hence, if $\boldsymbol{x}^a \tau_1(g)$ is in $K[\boldsymbol{x}][\boldsymbol{y}]$, then $\boldsymbol{x}^a \tau_1(y_i - y_1)$ is also in $K[\boldsymbol{x}][\boldsymbol{y}]$ for $i = 2, 3$. Since $\mathrm{in}_{\preceq}(\boldsymbol{x}^a \tau_1(g))$ is equal to $\mathrm{in}_{\preceq}(\boldsymbol{x}^a \tau_1(y_i - y_1))$ for some $i \in \{2, 3\}$ up to scalar multiplication, it is in $R \otimes_K \bar{K}$.

To show $\mathrm{in}_{\preceq}(K[\boldsymbol{x}][\boldsymbol{y}]^D) = R$, it suffices to verify that the initial term $\mathrm{in}_{\preceq}(F)$ of every Γ-homogeneous element $F \in K[\boldsymbol{x}][\boldsymbol{y}]^D \setminus \{0\}$ is in R. Put $f = \phi(F)$. Then, it is in B_l for some $l \in \boldsymbol{Z}_{\geq 0}$. So, f is expressed as in (3.6). Since $\tau_{\boldsymbol{x}^a}(f) = F$ for some $a \in \boldsymbol{Z}^m$, we get

$$F = \tau_{\boldsymbol{x}^a}(f) = \alpha_0 \boldsymbol{x}^a \tau_1(y_2 - y_1)^s \tau_1(y_3 - y_1)^t \prod_{i=1}^{u} \tau_1(y_2 - \beta_i y_3 + (\beta_i - 1) y_1)\,. \tag{3.8}$$

Since F is in $K[\boldsymbol{x}][\boldsymbol{y}]$, there exist $a', a'', a_i \in \boldsymbol{Z}^m$ with $sa' + ta'' + \sum_{i=1}^{u} a_i = a$ such that $\boldsymbol{x}^{a'} \tau_1(y_2 - y_1)$, $\boldsymbol{x}^{a''} \tau_1(y_3 - y_1)$ and $\boldsymbol{x}^{a_i} \tau_1(y_2 - \beta_i y_3 + (\beta_i - 1) y_1)$ are in $K[\boldsymbol{x}][\boldsymbol{y}]$. Hence, their initial terms are in $R \otimes_K \bar{K}$, as noted in the preceding paragraph. This implies that $\mathrm{in}_{\preceq}(F) \in R$ by (3.8) and (3.1). □

In particular, we have the following.

COROLLARY 3.5 (Khoury [5, Corollary 2.2]). *Assume that $n = 3$, and D is any elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$. Then,*

$$K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][L_{2,1}, L_{3,1}, L_{3,2}]\,. \tag{3.9}$$

As we mentioned before Proposition 3.4, each element $f \in B_l$ is factored into the product of l elements in $\bar{K} \otimes_K B_1$. We note that, if r is the maximal integer such that $(y_3 - y_2)^r$ divides f, then the expansion of f involves the monomials $y_1^{l-r} y_2^r$, $y_1^{l-r} y_3^r$ and does not involve $y_1^{l-r'} y_2^{r'}$, $y_1^{l-r'} y_3^{r'}$ for $0 \leq r' \leq r$.

LEMMA 3.6. *Assume that $(m, n) = (3, 3)$ and $\varepsilon^i_{i,j} > 0$ for any $1 \leq i, j \leq 3$ with $i \neq j$. If $\gamma = \deg_\Gamma(L^p_{2,1} L^q_{3,1} L^r_{3,2})$ for $p, q, r \in \boldsymbol{Z}_{\geq 0}$, then $K[\boldsymbol{x}][\boldsymbol{y}]^D_\gamma$ is equal to the one-dimensional K-vector space generated by $L^p_{2,1} L^q_{3,1} L^r_{3,2}$.*

PROOF. Take any $0 \neq F \in K[\boldsymbol{x}][\boldsymbol{y}]_\gamma^D$, and put $f = \phi(F)$. Then, f is in B_l and $\tau_{\boldsymbol{x}^a}(f) = F$, where $l = p+q+r$ and $a = p(\varepsilon_{2,3}+\varepsilon_{1,2}^+)+q\varepsilon_{1,3}^+ + r\varepsilon_{2,3}^+$. If $(y_2-y_1)^p$, $(y_3-y_1)^q$ and $(y_3-y_2)^r$ divide f, then F is in $K(L_{2,1}^p L_{3,1}^q L_{3,2}^r)$. Actually, it implies that $L_{2,1}^p$, $L_{3,1}^q$ and $L_{3,2}^r$ are factors of F. Suppose, say, that the maximal integer r' such that $(y_3-y_2)^{r'}$ divides f is less than r. Then, $y_1^{l-r'}y_2^{r'}$ and $y_1^{l-r'}y_3^{r'}$ appear in f with nonzero coefficient, as mentioned above. Hence, so do $\tau_{\boldsymbol{x}^a}(y_1^{l-r'}y_2^{r'})$ and $\tau_{\boldsymbol{x}^a}(y_1^{l-r'}y_3^{r'})$ in F. By definition, the first component of $\varepsilon_{2,3}^+$ or $\varepsilon_{3,2}^+$ is zero. If that of $\varepsilon_{2,3}^+$ is zero, then the power of x_1 in $\tau_{\boldsymbol{x}^a}(y_1^{l-r'}y_3^{r'})$ is negative. In fact, $\tau_{\boldsymbol{x}^a}(y_1^{l-r'}y_3^{r'}) = \boldsymbol{x}^{a'}y_1^{l-r'}y_3^{r'}$, where

$$a' = a + (l-r')\varepsilon_{3,1} = p\varepsilon_{2,1}^+ + q\varepsilon_{3,1}^+ + r\varepsilon_{2,3}^+ - (r-r')\varepsilon_{1,3}\,.$$

Since the first components of $\varepsilon_{2,1}^+$, $\varepsilon_{3,1}^+$, $\varepsilon_{2,3}^+$ are zero, that of a' is equal to $-(r-r')\varepsilon_{1,3}^1 < 0$. Similarly, the power of x_1 in $\tau_{\boldsymbol{x}^a}(y_1^{l-r'}y_2^{r'})$ is negative if the first component of $\varepsilon_{3,2}^+$ is zero. This is a contradiction. Therefore, F is in $K(L_{2,1}^p L_{3,1}^q L_{3,2}^r)$. □

Assume that $n = 4$. We define a homomorphism $\tilde{l} : \boldsymbol{Z}^4 \to \boldsymbol{Z}$ of additive groups by

$$\tilde{l}((b_1, b_2, b_3, b_4)) = b_2\varepsilon_{1,2}^1 + b_3\varepsilon_{1,3}^1\,. \tag{3.10}$$

LEMMA 3.7. *Assume that* $n = 4$, $\varepsilon_{1,2}^1 \geq \varepsilon_{1,3}^1 > 0$ *and* F *is an element of* B_l *for some* $l \in \boldsymbol{Z}_{\geq 0}$. *If every* $b \in \operatorname{supp}(F)$ *satisfies* $\tilde{l}(b) \geq p$ *for some* $p \in \boldsymbol{Z}_{\geq 0}$, *then* $(y_3 - y_2)^q$ *divides* F *for the minimal* $q \in \boldsymbol{Z}_{\geq 0}$ *with* $p \leq q\varepsilon_{1,3}^1$.

PROOF. Write

$$F = f_0(y_4-y_1)^l + f_1(y_4-y_1)^{l-1} + \cdots + f_l\,,$$

where $f_i \in K[y_2-y_1, y_3-y_1]_i$ for each i. Suppose that $(y_3-y_2)^q$ did not divide F. Then, there exists i such that $(y_3-y_2)^q$ does not divide f_i. Let i be the minimum among such indices i, and q' the maximal integer such that $(y_3-y_2)^{q'}$ divides f_i. Then, f_i involves the monomial $y_1^{i-q'}y_3^{q'}$, as we noted before Lemma 3.6. We set $b = (i-q', 0, q', l-i)$. Then, $\tilde{l}(b) = q'\varepsilon_{1,3}^1 < q\varepsilon_{1,3}^1$. It implies that $\tilde{l}(b) < p$ by the minimality of q. Hence, $b \notin \operatorname{supp}(F)$.

On the other hand, $f_i(y_4-y_1)^{l-i}$ involves $\boldsymbol{y}^b$. If $j > i$, then $f_j(y_4-y_1)^{l-j}$ does not involve $\boldsymbol{y}^b$, since the exponent of y_4 in each monomial of it is less than $l-i$. Suppose that $f_j(y_4-y_1)^{l-j}$ involved $\boldsymbol{y}^b$ for $j < i$. Then, f_j contains $y_1^{j-q'}y_3^{q'}$. Since $q' < q$, this contradicts the assumption that $(y_3-y_2)^q$ divides f_j by the note above. Therefore, $f_j(y_4-y_1)^{l-j}$ does not involve $\boldsymbol{y}^b$ if $j \neq i$. Hence, $b \in \operatorname{supp}(F)$. This is a contradiction. Therefore, $(y_3-y_2)^q$ divides F. □

We remark that, if $F \in K[\boldsymbol{x}][\boldsymbol{y}]^D$ is expressed as

$$F = f_0y_n^l + f_1y_n^{l-1} + \cdots + f_l$$

for $f_i \in K[\boldsymbol{x}][y_1, \ldots, y_{n-1}]$, then $D(f_0) = 0$. Actually, we get

$$0 = D(F) = D(f_0)y_n^l + (\text{terms of lower degree in } y_n)\,.$$

The following is the key proposition.

PROPOSITION 3.8. *Assume that* $(m,n)=(3,4)$ *and* $\varepsilon^i_{i,j}>0$ *for any* $1\le i,j\le 4$ *with* $i\ne j$. *Then, the monomial* $\boldsymbol{x}^a y_2^p y_3^{q+r} y_4^l$ *is not contained in* $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\boldsymbol{x}][\boldsymbol{y}]^D)$ *for any* $p,q,r,l\in \boldsymbol{Z}_{\ge 0}$, *where we set* $a=p\varepsilon^+_{1,2}+q\varepsilon^+_{1,3}+r\varepsilon^+_{2,3}$.

PROOF. Suppose that there existed $F\in K[\boldsymbol{x}][\boldsymbol{y}]^D$ such that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F)=\boldsymbol{x}^a y_2^p y_3^{q+r} y_4^l$. Then, without loss of generality, we may assume that F is Γ-homogeneous. Write

$$F=f_0y_4^l+f_1y_4^{l-1}+\cdots+f_l\,,$$

where $f_i\in K[\boldsymbol{x}][y_1,y_2,y_3]$ for $i=0,\dots,l$. Then, f_0 is in $K[\boldsymbol{x}][y_1,y_2,y_3]^D$, as we remarked above. Moreover, f_0 is Γ-homogeneous and $\deg_\Gamma(f_0)=\deg_\Gamma(L^p_{1,2}L^q_{1,3}L^r_{2,3})$. Hence, f_0 is equal to $L^p_{1,2}L^q_{1,3}L^r_{2,3}$ up to scalar multiplication by Lemma 3.6.

It suffices to show that each of $L^p_{2,1}$, $L^q_{3,1}$ and $L^r_{3,2}$ must be a factor of F in $K[\boldsymbol{x},\boldsymbol{x}^{-1}][\boldsymbol{y}]$. Indeed, it will imply that $F=L^p_{1,2}L^q_{1,3}L^r_{2,3}F'$ for some $F'\in K[\boldsymbol{x},\boldsymbol{x}^{-1}][\boldsymbol{y}]$, since $L_{2,1}$, $L_{3,1}$ and $L_{3,2}$ are pairwise prime. Then, F' is an element in $K[\boldsymbol{x}][\boldsymbol{y}]^D$. However, F' involves the monomial y_4^l. This contradicts Lemma 2.1.

Since the arguments are similar, we only show that $L^r_{3,2}$ is a factor of F. We assume that $\varepsilon^1_{1,2}\ge\varepsilon^1_{1,3}$. The proof is similar for the other case. We set $f=\phi(F)$, and claim that every $b=(b_1,b_2,b_3,b_4)\in\mathrm{supp}(f)$ satisfies $\tilde{l}(b)\ge r\varepsilon^1_{1,3}$. This implies that $(y_3-y_2)^r$ divides f by Lemma 3.7. Hence, $L^r_{3,2}$ is a factor of F in $K[\boldsymbol{x},\boldsymbol{x}^{-1}][\boldsymbol{y}]$, and the proof is completed. By straightforward computation, we may verify that $\deg_\Gamma(F)$ is equal to the image of $(c,(d+l)\boldsymbol{e}_4)$, where $d=p+q+r$ and

$$c=p\varepsilon^+_{2,1}+q\varepsilon^+_{3,1}+r\varepsilon^+_{2,3}+d\varepsilon_{1,4}+r\varepsilon_{3,1}\,.$$

Thus, it follows that $F=\tau_{\boldsymbol{x}^c}(f)$, as mentioned above. Hence, F involves $\tau_{\boldsymbol{x}^c}(\boldsymbol{y}^b)$ for $b\in\mathrm{supp}(f)$. By simple computation, we get $\tau_{\boldsymbol{x}^c}(\boldsymbol{y}^b)=\boldsymbol{x}^d\boldsymbol{y}^b$, where

$$d=p\varepsilon^+_{2,1}+q\varepsilon^+_{3,1}+r\varepsilon^+_{2,3}+(l-b_4)\varepsilon_{4,1}+r\varepsilon_{3,1}+b_2\varepsilon_{1,2}+b_3\varepsilon_{1,3}\,.$$

Note that the first components of $p\varepsilon^+_{2,1},q\varepsilon^+_{3,1},r\varepsilon^+_{2,3}$ are zero and $b_4\le l$. Since $\boldsymbol{x}^d\boldsymbol{y}^b$ is in $K[\boldsymbol{x}][\boldsymbol{y}]$, the first component of d is nonnegative. Thus, we have

$$0\le(l-b_4)\varepsilon^1_{4,1}+r\varepsilon^1_{3,1}+b_2\varepsilon^1_{1,2}+b_3\varepsilon^1_{1,3}=(l-b_4)\varepsilon^1_{4,1}-r\varepsilon^1_{1,3}+\tilde{l}(b)\le -r\varepsilon^1_{1,3}+\tilde{l}(b)\,.$$

Therefore, $\tilde{l}(b)\ge r\varepsilon^1_{1,3}$. □

Now, let us prove Theorem 3.3. By Lemma 3.1, the last statement is a consequence of the first part. So, we will prove the first part.

We set R to be the K-algebra generated by (3.4). Clearly, $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}})$ contains R. For the converse, it suffices to show that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F)$ is in R for any Γ-homogeneous element $F\in K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}}$. The remark before Proposition 3.8 implies that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F)=\mathrm{in}_{\preceq_{\mathrm{lex}}}(F')y_4^l$ for some $F'\in K[\boldsymbol{x}][y_1,y_2,y_3]^{D_{t,3}}$ and $l\in\boldsymbol{Z}_{\ge 0}$. By Proposition 3.4, the set $\{x_1,x_2,x_3,L_{2,1},L_{3,1},$

$L_{3,2}\}$ is a SAGBI basis for $K[\boldsymbol{x}][y_1, y_2, y_3]^{D_{t,3}}$ with respect to any monomial order. In particular,

$$\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\boldsymbol{x}][y_1, y_2, y_3]^{D_{t,3}}) = K[\boldsymbol{x}][x_1^{t+1}y_2, x_1^{t+1}y_3, x_2^{t+1}y_3] .$$

Hence, there exist $a_1, a_2, a_3, p, q, r \in \boldsymbol{Z}_{\geq 0}$ such that

$$\mathrm{in}_{\preceq_{\mathrm{lex}}}(F) = (x_1^{t+1}y_2)^p(x_1^{t+1}y_3)^q(x_2^{t+1}y_3)^r x_1^{a_1}x_2^{a_2}x_3^{a_3}y_4^l .$$

Obviously, $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F)$ is in R if $l = 0$. Assume that $l > 0$. Then, $a_1 + a_2 + a_3 > 0$ by Proposition 3.8. Hence, it is also in R. Therefore, $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,3}})$ is contained in R. This completes the proof of Theorem 3.3.

4. A condition for finite generation. In this section, we investigate a condition for the finite generation of $K[\boldsymbol{x}][\boldsymbol{y}]^D$, where D is an elementary monomial $K[\boldsymbol{x}]$-derivation. The main result of this section is the following.

THEOREM 4.1. *Assume that* $(m, n) = (3, 4)$, *and there exist* $i \neq j$ *and* k *such that* $\varepsilon_{\tau(i),\tau(j)}^{\sigma(k)} \leq 0$ *and* $\sigma(k) = \tau(i)$ *for every pair of permutations* σ *and* τ *on* $\{1, 2, 3\}$ *and* $\{1, 2, 3, 4\}$, *respectively. Then,* $K[\boldsymbol{x}][\boldsymbol{y}]^D$ *is generated by* L_{k_i,l_i} *for* $i = 1, 2, 3, 4$ *over* $K[\boldsymbol{x}]$ *for some integers* $1 \leq k_i, l_i \leq 4$.

First, we look at general properties on the kernel of an elementary monomial $K[\boldsymbol{x}]$-derivation. For each i, j, we set $\tilde{L}_{i,j} = y_i - \boldsymbol{x}^{\varepsilon_{i,j}}y_j$. It is contained in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]^D$. To avoid confusion, we sometimes denote it by $\tilde{L}_{i,j}^D$ to emphasize D.

LEMMA 4.2. *The kernel* $K[\boldsymbol{x}][\boldsymbol{y}]^D$ *is contained in* $K[\boldsymbol{x}][\tilde{L}_{1,j}, \dots, \tilde{L}_{n,j}]$ *for each* j.

PROOF. Take any $F \in K[\boldsymbol{x}][\boldsymbol{y}]^D$, and let f be the polynomial obtained from F by replacing y_j by zero. Then, define an element F' of $K[\boldsymbol{x}][\tilde{L}_{1,j}, \dots, \tilde{L}_{n,j}]$ as the polynomial which we obtain from f by replacing y_k by $\tilde{L}_{k,j}$ for each k. We show that $F = F'$. Suppose that $F \neq F'$. Write

$$F - F' = (\text{terms of higher degree in } y_j) + gy_j^e ,$$

where g is an element of $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}] \setminus \{0\}$ not involving y_j. Since $F - f$ and $F' - f$ are in $K[\boldsymbol{x}, \boldsymbol{x}^{-1}][\boldsymbol{y}]y_j$, we have $e > 0$. However,

$$0 = D(F - F') = (\text{terms of higher degree in } y_j) + eg\boldsymbol{x}^{\delta_j}y_j^{e-1} ,$$

a contradiction, since $eg\boldsymbol{x}^{\delta_j} \neq 0$. Therefore, $F = F'$. □

Assume that $\delta_j = 0$ for some j. Then, $\tilde{L}_{k,j}$ is in $K[\boldsymbol{x}][\boldsymbol{y}]^D$ for each k. By Lemma 4.2, it implies that $K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][\tilde{L}_{1,j}, \dots, \tilde{L}_{n,j}]$. If this is the case, then $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is isomorphic to $K[\boldsymbol{x}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]$ via the homomorphism which substitutes zero for y_j. In particular, the kernel $K[\boldsymbol{x}][\boldsymbol{y}]^{D_{t,m}}$ of the derivation $D_{t,m}$ for $t = 0$ is generated by $\tilde{L}_{1,m+1}, \dots, \tilde{L}_{m,m+1}$ over $K[\boldsymbol{x}]$, and is isomorphic to the polynomial ring in $2m$ variables over K.

Now, we fix $1 \leq i \leq m$ and $1 \leq j \leq n$. Assume that $\varepsilon^i_{k,j} \geq 0$ for every $k = 1, \dots, n$. Then, put $\mu = \min \{\varepsilon^i_{k,j} \mid k \neq j\}$, and set $\boldsymbol{x}^{\varepsilon'_{k,j}} = x_i^{-\mu} \boldsymbol{x}^{\varepsilon_{k,j}}$ for each k. Let D' be an elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$ such that $D'(y_k)/D'(y_j) = \boldsymbol{x}^{\varepsilon'_{k,j}}$ for each k. For $f \in K[\boldsymbol{x}][\boldsymbol{y}]^D$, we define $T_{j,i}(f)$ to be the polynomial obtained from f by replacing y_j by $x_i^{-\mu} y_j$. Then, it follows that

$$T_{j,i}(\tilde{L}^D_{k,j}) = y_k - \boldsymbol{x}^{\varepsilon_{k,j}}(x_i^{-\mu} y_j) = y_k - \boldsymbol{x}^{\varepsilon'_{k,j}} y_j = \tilde{L}^{D'}_{k,j}$$

for each k.

LEMMA 4.3. *Let i, j be integers with $1 \leq i \leq m$ and $1 \leq j \leq n$. If $\varepsilon^i_{k,j} \geq 0$ for every $k = 1, \dots, n$, then $T_{j,i}$ is an injective homomorphism with the image $K[\boldsymbol{x}][\boldsymbol{y}]^{D'}$.*

PROOF. Suppose that $T_{j,i}(f)$ were not in $K[\boldsymbol{x}][\boldsymbol{y}]^{D'}$ for some $f \in K[\boldsymbol{x}][\boldsymbol{y}]^D$. By Lemma 4.2, f is in $K[\boldsymbol{x}][\{\tilde{L}^D_{k,j} \mid k\}]$. Since $T_{j,i}$ sends $\tilde{L}^D_{k,j}$ to $\tilde{L}^{D'}_{k,j}$, we have $T_{j,i}(f) \in K[\boldsymbol{x}][\{\tilde{L}^{D'}_{k,j} \mid k\}]$. In particular, $D'(T_{j,i}(f)) = 0$. Hence, there appears in $T_{j,i}(f)$ a monomial with negative power in some variable. By the definition of $T_{j,i}(f)$, the variable must be x_i. However, $\tilde{L}^{D'}_{k,j}$ does not have negative power in x_i for each k. Hence, such a monomial cannot appear in $T_{j,i}(f)$. This is a contradiction. Thus, $T_{j,i}(f)$ is in $K[\boldsymbol{x}][\boldsymbol{y}]^{D'}$.

Conversely, a homomorphism $K[\boldsymbol{x}][\boldsymbol{y}]^{D'} \to K[\boldsymbol{x}][\boldsymbol{y}]^D$ is defined by the substitution $y_j \mapsto x_i^{\mu} y_j$. Indeed, it sends each $\tilde{L}^{D'}_{k,j}$ to $\tilde{L}^D_{k,j}$. It is the inverse of $T_{j,i} : K[\boldsymbol{x}][\boldsymbol{y}]^D \to K[\boldsymbol{x}][\boldsymbol{y}]^{D'}$. □

We use the following proposition to reduce problems on the kernel of D to a lower dimensional case.

PROPOSITION 4.4. *Let D be any elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$, and $1 \leq j, k \leq m$ distinct integers. For each $1 \leq i \leq m$, we assume that either $\varepsilon^i_{j,k} \geq 0$ or $\varepsilon^i_{l,k} \geq 0$ for all $l \neq j$. Then,*

$$K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]^D[L_{j,k}]\,. \tag{4.1}$$

PROOF. Clearly, the right hand side of (4.1) is contained in the left hand side. We show the converse. Let S be the set of elements of $K[\boldsymbol{x}][\boldsymbol{y}]^D$ not contained in the right hand side of (4.1). Suppose that S were not empty. Take $f \in S$ with the minimal degree in y_j, and write

$$f = g_d(\boldsymbol{x}^{\varepsilon^+_{k,j}} y_j)^d + g_{d-1}(\boldsymbol{x}^{\varepsilon^+_{k,j}} y_j)^{d-1} + \cdots + g_0\,, \tag{4.2}$$

where $g_i \in K[\boldsymbol{x}, \boldsymbol{x}^{-1}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]$ with $g_d \neq 0$. To complete the proof, it suffices to show that g_d is in $K[\boldsymbol{x}][\boldsymbol{y}]^D$. Indeed, it implies that $f - g_d(L_{j,k})^d$ is in S, but the degree of $f - g_d(L_{j,k})^d$ in y_j is less than d. This is a contradiction, and we get $S = \emptyset$.

Similarly to the remark before Proposition 3.8, we have $D(g_d) = 0$. We show that every monomial appearing in g_d does not have negative power in x_i for each i. First, assume that the i-th component of $\varepsilon^+_{k,j}$ is not zero. Then, it is equal to $\varepsilon^i_{k,j} > 0$, and so $\varepsilon^i_{j,k}$ is negative. Hence, $\varepsilon^i_{l,k} \geq 0$ for any $l \neq j$ by assumption. Since $\varepsilon^i_{l,j} = \varepsilon^i_{l,k} + \varepsilon^i_{k,j}$, we have $0 < \varepsilon^i_{k,j} \leq \varepsilon^i_{l,j}$

for $l \neq j$. Thus, the substitution $y_j \mapsto x_i^{-\varepsilon^i_{k,j}} y_j$ sends f to $T_{j,i}(f)$. If there appeared in g_d a monomial $\boldsymbol{x}^a \boldsymbol{y}^b$ with negative power in x_i, then $T_{j,i}(f)$ would have the monomial $\boldsymbol{x}^a \boldsymbol{y}^b y_j^d$. It also has negative power in x_i. This is a contradiction, since $T_{j,i}(f)$ is in $K[\boldsymbol{x}][\boldsymbol{y}]$ by Lemma 4.3. If the i-th component of $\varepsilon^+_{k,j}$ is zero, then the expression (4.2) also implies that no monomial appearing in g_d has negative power in x_i. Therefore, g_d is in $K[\boldsymbol{x}][\boldsymbol{y}]$. □

As a corollary to Proposition 4.4, we have the following.

COROLLARY 4.5 (Khoury [5, Theorem 3.1]). *If $m = 2$, then there exist $1 \leq l \leq n$ and $1 \leq k_j \leq n$ with $k_j \neq j$ for each $j \neq l$ such that*

$$K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][L_{1,k_1}, \dots, L_{l-1,k_{l-1}}, L_{l+1,k_{l+1}}, \dots, L_{n,k_n}]. \tag{4.3}$$

PROOF. We prove this by induction on n. If $n = 1$, then $K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}]$ by Lemma 4.2. Hence, the assertion is true. Assume that $n > 1$. Then, by change of indices if necessary, we may assume that $\delta^1_1 \leq \cdots \leq \delta^1_n$. If there exist $1 \leq k < j \leq n$ such that $\delta^2_k \leq \delta^2_j$, then $\varepsilon^i_{j,k} \geq 0$ for $i = 1, 2$. Hence,

$$K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]^D[L_{j,k}]$$

by Proposition 4.4. Thus, the assertion follows from the induction assumption. Assume that such k, j do not exist, i.e., $\delta^2_n < \cdots < \delta^2_1$. Then, $\varepsilon^2_{l,n-1} > 0$ for any $l \neq n$. Since $\varepsilon^1_{n,n-1} \geq 0$, we have $K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][y_1, \dots, y_{n-1}]^D[L_{n,n-1}]$ by Proposition 4.4. Hence, the assertion follows similarly. □

Let $\phi_1 : K[\boldsymbol{x}][\boldsymbol{y}] \to K[x_2, \dots, x_m][\boldsymbol{y}]$ be the homomorphism which substitutes one for x_1, and D_1 the elementary $K[x_2, \dots, x_m]$-derivation on $K[x_2, \dots, x_m][\boldsymbol{y}]$ defined by $D_1(f) = \phi_1(D(f))$ for each f. Then, D_1 is a monomial derivation. By definition, it follows that $\phi_1 \circ D = D_1 \circ \phi_1$ on $K[\boldsymbol{x}][\boldsymbol{y}]$. Recall the Γ-grading structure on $K[\boldsymbol{x}][\boldsymbol{y}]$ defined in Section 3. Let Γ_1 be the set of the images of $(a, l\boldsymbol{e}_n)$ in Γ for $l \in \mathbf{Z}$ and $a = (a_1, \dots, a_m) \in \mathbf{Z}^m$ with $a_1 = 0$. Then, Γ_1 is a subgroup of Γ, and $\bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma$ is a $K[x_2, \dots, x_n]$-subalgebra of $K[\boldsymbol{x}][\boldsymbol{y}]$.

LEMMA 4.6. *Assume that $\varepsilon^1_{n,j} \geq 0$ for $j = 1, \dots, n$. Then, ϕ_1 induces an isomorphism*

$$\bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]^D_\gamma \to K[x_2, \dots, x_m][\boldsymbol{y}]^{D_1}. \tag{4.4}$$

PROOF. Set $R = \bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma$ and $R' = K[x_2, \dots, x_m][\boldsymbol{y}]$. It suffices to show that ϕ_1 induces an isomorphism $R \to R'$. Indeed, it implies that $\phi_1(R^D) = (R')^{D_1}$, since $\phi_1 \circ D = D_1 \circ \phi_1$.

First, we show the injectivity. Suppose that there existed $f \in R\backslash\{0\}$ such that $\phi_1(f) = 0$. Then, $f = (x_1 - 1)f'$ for some $f' \in K[\boldsymbol{x}][\boldsymbol{y}] \setminus \{0\}$. Let p and q be the maximal and the minimal integers l with $\deg_\Gamma(x_1^l f'') \in \Gamma_1$ for some nonzero Γ-homogeneous component f'' of f', respectively. Clearly, we have $p \geq 1$ or $q \leq 0$. If $p \geq 1$, then $\deg_\Gamma(f'') \notin \Gamma_1$ for a Γ-homogeneous component f'' of f' with $\deg_\Gamma(x_1^p f'') \in \Gamma_1$. However, $-f''$ is a

Γ-homogeneous component of f by the maximality of p. Hence, $-f''$ is in R. This is a contradiction. Similarly, we get a contradiction if $q \leq 0$. Therefore, $\phi_1(f) \neq 0$ for any $f \in R \setminus \{0\}$.

For the surjectivity, it suffices to show that $\phi_1(R)$ contains every monomial in R'. Take any monomial $\boldsymbol{x}^a\boldsymbol{y}^b \in R'$, and put $l = \sum_{j=1}^n b_j\varepsilon_{n,j}^1$, where $b = (b_1, \dots, b_n)$. Then, l is nonnegative, since $\varepsilon_{n,j}^1 \geq 0$ for all j by assumption. Hence, $x_1^l\boldsymbol{x}^a\boldsymbol{y}^b$ is in $K[\boldsymbol{x}][\boldsymbol{y}]$. Note that

$$\deg_\Gamma(x_1^l\boldsymbol{x}^a\boldsymbol{y}^b) = \deg_\Gamma\left(x_1^l\boldsymbol{x}^a\boldsymbol{y}^b\prod_{j=1}^n(\boldsymbol{x}^{\varepsilon_{j,n}}y_j^{-1}y_n)^{b_j}\right) = \deg_\Gamma\left(\boldsymbol{x}^c y_n^{\sum_{j=1}^n b_j}\right),$$

where $c = (l, 0, \dots, 0) + a + \sum_{j=1}^n b_j\varepsilon_{j,n}$. Since the first component of a is zero, that of c is equal to $l + \sum_{j=1}^n b_j\varepsilon_{j,n}^1 = 0$. Thus, $x_1^l\boldsymbol{x}^a\boldsymbol{y}^b$ is in R. Since $\boldsymbol{x}^a\boldsymbol{y}^b = \phi_1(x_1^l\boldsymbol{x}^a\boldsymbol{y}^b)$, the surjectivity is proved. □

LEMMA 4.7. *Assume that $n = 4$ and $\varepsilon_{1,3}^1, \varepsilon_{1,2}^1 > 0, \varepsilon_{1,4}^1 = 0$. Then, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by x_1 and $L_{3,2}$ over $\bigoplus_{\gamma\in\Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma^D$.*

PROOF. Without loss of generality, we may assume that $\varepsilon_{1,3}^1 \geq \varepsilon_{1,2}^1$. It suffices to show that each Γ-homogeneous element $F \in K[\boldsymbol{x}][\boldsymbol{y}]^D$ is written as $F = x_1^pL_{3,2}^qF'$, where $p, q \in \boldsymbol{Z}_{\geq 0}$ and $F' \in K[\boldsymbol{x}][\boldsymbol{y}]_{\gamma'}$ for some $\gamma' \in \Gamma_1$. Indeed, it also implies that $D(F') = 0$, since $0 = D(F) = x_1^pL_{3,2}^qD(F')$.

Assume that $\deg_\Gamma(F)$ is equal to the image of $(a, l\boldsymbol{e}_4)$, where $a = (a_1, \dots, a_m) \in \boldsymbol{Z}^m$ and $l \in \boldsymbol{Z}_{\geq 0}$. We set $f = \phi(F)$. Then, $F = \tau_{\boldsymbol{x}^a}(f)$, as we noted before Proposition 3.4. Take any $b = (b_1, b_2, b_3, b_4) \in \operatorname{supp}(f)$. Then, by straightforward computation, we get $\tau_{\boldsymbol{x}^a}(\boldsymbol{y}^b) = \boldsymbol{x}^c\boldsymbol{y}^b$, where

$$c = a + (l - b_4)\varepsilon_{4,1} + b_2\varepsilon_{1,2} + b_3\varepsilon_{1,3}\,. \tag{4.5}$$

Since $\varepsilon_{4,1}^1 = 0$, the first component of c is equal to $a_1 + \tilde{l}(b)$. On the other hand, we have $\tilde{l}(b) \geq 0$, since $\varepsilon_{1,2}^1, \varepsilon_{1,3}^1 > 0$. Hence, $x_1^{-a_1}\boldsymbol{x}^c\boldsymbol{y}^b$ does not have negative power. Thus, $x_1^{-a_1}F$ is in $K[\boldsymbol{x}][\boldsymbol{y}]$. Clearly, $\deg_\Gamma(x_1^{-a_1}F)$ is in Γ_1. Therefore, if $a_1 \geq 0$, then we are led to the desired expression $F = x_1^{a_1}(x_1^{-a_1}F)$.

Assume that $a_1 < 0$. Let q be the minimal integer such that $q\varepsilon_{1,3}^1 \geq -a_1$. Since the first component of (4.5) is nonnegative, we have $\tilde{l}(b) \geq -a_1$ for every $b \in \operatorname{supp}(f)$. Hence, $(y_3 - y_2)^q$ divides f by Lemma 3.7. It implies that $F = F'L_{3,2}^q$ for some $F' \in K[\boldsymbol{x}][\boldsymbol{y}]^D$. Note that $\deg_\Gamma(L_{3,2}^q)$ is equal to the image of $q(\varepsilon_{2,3}^+ + \varepsilon_{3,4}, \boldsymbol{e}_4)$ in Γ. Hence, $\deg_\Gamma(F')$ is equal to that of $(a', (l - q)\boldsymbol{e}_4)$, where

$$a' = a - q(\varepsilon_{2,3}^+ + \varepsilon_{3,4}) = a + q\varepsilon_{1,3} - q(\varepsilon_{2,3}^+ + \varepsilon_{1,4})\,.$$

Since the first components of $\varepsilon_{2,3}^+$ and $\varepsilon_{1,4}$ are zero, that of a' is equal to $a_1 + q\varepsilon_{1,3}^1$. By the choice of q, this is nonnegative. Hence, we have $F' = x_1^pF''$ for some $p \in \boldsymbol{Z}_{\geq 0}$ and $F'' \in K[\boldsymbol{x}][\boldsymbol{y}]_{\gamma'}$ with $\gamma' \in \Gamma_1$, as we showed in the preceding paragraph. Therefore, we get a desired expression. □

Now, let us prove Theorem 4.1. Note that the assumption fails if and only if we can exchange the rows and columns of the matrix $(\delta_i^j)_{i,j}$ so that δ_i^i is the maximum among the components of the i-th column for each i. Under the assumption, we are reduced to one of the following two cases by such operations:

(i) $\delta_i^1 \leq \delta_1^1$ and $\delta_i^2 \leq \delta_1^2$ for $i = 1, 2, 3, 4$.

(ii) $\delta_i^1 < \delta_1^1 = \delta_4^1$ for $i = 2, 3$.

In fact, if we are not reduced to (ii), then there exists $1 \leq k_j \leq 4$ for each $j = 1, 2, 3$ such that $\delta_i^j < \delta_{k_j}^j$ for any $i \neq k_j$. If further we were not reduced to (i), then $k_j \neq k_l$ for any $j \neq l$. In this case, we can exchange the rows of $(\delta_i^j)_{i,j}$ so that $k_j = j$ for $j = 1, 2, 3$. This implies that $\delta_i^j < \delta_i^i$ for any $i \neq j$.

First, consider the case (i). By exchanging the row vectors δ_2, δ_3 and δ_4 of $(\delta_i^j)_{i,j}$ if necessary, we may assume that $\delta_4^3 \leq \delta_j^3$, that is, $\varepsilon_{j,4}^3 \geq 0$ for $j = 2, 3, 4$. Since $\delta_4^1 \leq \delta_1^1$ and $\delta_4^2 \leq \delta_1^2$ by assumption, we have $\varepsilon_{1,4}^1, \varepsilon_{1,4}^2 \geq 0$. Hence, $K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][y_1, y_2, y_3]^D[L_{4,1}]$ by Proposition 4.4. Therefore, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by $L_{2,1}, L_{3,1}, L_{3,2}$ and $L_{4,1}$ over $K[\boldsymbol{x}]$ by Corollary 3.5.

Now, consider the case (ii). Since $\varepsilon_{2,1}^1, \varepsilon_{3,1}^1 < 0$ and $\varepsilon_{4,1}^1 = 0$ follow from the condition, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by $x_1, L_{3,2}^D$ over $\bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma^D$ by Lemma 4.7. By Lemma 4.6, $\bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma^D$ is isomorphic to $K[x_2, x_3][\boldsymbol{y}]^{D'}$ via ϕ_1, since $\varepsilon_{4,j}^1 \geq 0$ for any j. Then, by Corollary 4.5, there exist $1 \leq l \leq 4$, and $1 \leq k_i \leq 4$ with $k_i \neq i$ for $i \in \{1, 2, 3, 4\} \setminus \{l\}$ such that $K[x_2, x_3][\boldsymbol{y}]^{D'}$ is generated by $L_{k_i,i}^{D'}$ for $i \in \{1, 2, 3, 4\} \setminus \{l\}$ over $K[x_2, x_3]$. Since $\phi_1(L_{i,j}^D) = L_{i,j}^{D'}$ for i, j, the $K[x_2, x_3]$-algebra $\bigoplus_{\gamma \in \Gamma_1} K[\boldsymbol{x}][\boldsymbol{y}]_\gamma^D$ is generated by $L_{k_i,i}^D$ for $i \in \{1, 2, 3, 4\} \setminus \{l\}$. Therefore, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by $L_{3,2}^D$ and $L_{k_i,i}^D$ for $i \in \{1, 2, 3, 4\} \setminus \{l\}$ over $K[\boldsymbol{x}]$. This completes the proof of Theorem 4.1.

Let D be any elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$ for $(m, n) = (3, 4)$. By Theorems 1.4 and 4.1, we settled the problem of finite generation of $K[\boldsymbol{x}][\boldsymbol{y}]^D$ except in the case $\varepsilon_{i,j}^i > 0$ for any $i \neq j$ and $\xi(D) > 1$.

CONJECTURE 4.8. *Assume that* $(m, n) = (3, 4)$, *and* $\varepsilon_{i,j}^i > 0$ *for any* $i \neq j$. *If* $\xi(D) > 1$, *then* $K[\boldsymbol{x}][\boldsymbol{y}]^D$ *is finitely generated.*

Note that the conjecture is true if there exist distinct $r, s \in \{1, 2, 3\}$ such that $\xi_r(D) \geq 1$ and $\xi_s(D) \geq 1$. We show this for $(r, s) = (2, 3)$. The conditions $\xi_2(D) \geq 1$ and $\xi_3(D) \geq 1$ imply, respectively, that $\varepsilon_{3,4}^2 \geq 0$ or $\varepsilon_{1,4}^2 \geq 0$, and $\varepsilon_{1,4}^3 \geq 0$ or $\varepsilon_{2,4}^3 \geq 0$. Furthermore, we have $\varepsilon_{1,4}^1 > 0$, $\varepsilon_{2,4}^2 > 0$ and $\varepsilon_{3,4}^3 > 0$ by assumption. Hence, for each $i = 1, 2, 3$, we have $\varepsilon_{1,4}^i \geq 0$ or $\varepsilon_{l,4}^i \geq 0$ for $l = 2, 3, 4$. Thus, $K[\boldsymbol{x}][\boldsymbol{y}]^D = K[\boldsymbol{x}][y_2, y_3, y_4]^D[L_{4,1}]$ by Proposition 4.4. Therefore, $K[\boldsymbol{x}][\boldsymbol{y}]^D$ is generated by $L_{3,2}, L_{4,1}, L_{4,2}$ and $L_{4,3}$ over $K[\boldsymbol{x}]$ by Corollary 3.5.

There exists an example of an elementary monomial $K[\boldsymbol{x}]$-derivation on $K[\boldsymbol{x}][\boldsymbol{y}]$ for $(m, n) = (3, 4)$ whose kernel is finitely generated, and $\xi_i(D) < 1$ for $i = 1, 2, 3$. Kurano [7] showed that the kernel of $D_{1,3}$ is finitely generated. In fact, he showed that it is generated by

$x_1, x_2, x_3, L_{i,j}$ for $(i, j) \in \mathbf{Z} \times \mathbf{Z}$ with $1 \leq j < i \leq 4$ and

$$x_i y_4^2 - 2x_j x_k y_i y_4 + x_i x_k^2 y_i y_j + x_i x_j^2 y_i y_k - x_i^3 y_j y_k \tag{4.6}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ over K. Moreover, [7, Lemma 3.2] implies that the set of these polynomials is a SAGBI basis for the lexicographic order $\preceq_{\rm lex}$ with (3.3). For this derivation, we have $\xi_i(D_{1,3}) = 1/2$ for $i = 1, 2, 3$.

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