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A GENERALIZATION OF THE BERNSTEIN POLYNOMIALS BASED ON THE q-INTEGERS

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Abstract

This paper is concerned with a generalization of the Bernstein polynomials in which the approximated function is evaluated at points spaced in geometric progression instead of the equal spacing of the original polynomials.

1. Introduction

We begin by recalling that, for any $f \in C[0, 1]$, the Bernstein polynomial of order n is defined by

$$B_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r} x^r(1-x)^{n-r}. \tag{1}$$

These are the polynomials which were introduced (see [2]) by S. N. Bernstein (1880–1968) to give his celebrated constructive proof of the Weierstrass theorem. See also Cheney [3], Davis [4] and Rivlin [15]. In (1) the approximated function f is evaluated at equally spaced intervals. Here we discuss a generalization of the Bernstein polynomials where the approximated function is evaluated at intervals which are in geometric progression. This generalization was proposed in Phillips [12] and further properties of these generalized Bernstein polynomials are discussed in Phillips [11] and [13], Oruç and Phillips [10] and Goodman *et al.* [5]. First we require some preliminary results concerning q-integers. For any fixed real number $q > 0$, we define

$$[i] = \begin{cases} (1-q^i)/(1-q), & q \neq 1, \\ i, & q = 1, \end{cases} \tag{2}$$

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for all non-negative integers i. We refer to $[i]$ as a q-integer and note that $[i]$ is a continuous function of q. In an obvious way we also define a q-factorial,

$$[i]! = \begin{cases} [i]\cdot[i-1]\cdots[1], & i = 1, 2, \dots, \\ 1, & i = 0 \end{cases} \tag{3}$$

and a q-binomial coefficient

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{[k]!}{[r]![k-r]!} \tag{4}$$

for integers $k \geq r \geq 0$. These q-binomial coefficients satisfy the recurrence relations

$$\begin{bmatrix} k+1 \\ r \end{bmatrix} = q^{k-r+1}\begin{bmatrix} k \\ r-1 \end{bmatrix} + \begin{bmatrix} k \\ r \end{bmatrix} \qquad \text{and} \tag{5}$$

$$\begin{bmatrix} k+1 \\ r \end{bmatrix} = \begin{bmatrix} k \\ r-1 \end{bmatrix} + q^r\begin{bmatrix} k \\ r \end{bmatrix}, \tag{6}$$

which both reduce to the Pascal identity for ordinary binomial coefficients when $q = 1$. It follows from the above Pascal identities that the q-binomial coefficient in (4) is a polynomial in q of degree $r(k-r)$. Since they are associated with Gauss, the q-binomial coefficients are also known as Gaussian polynomials (see Andrews [1]). It is easily verified by induction, using either (5) or (6), that

$$(1+x)(1+qx)\cdots(1+q^{k-1}x) = \sum_{r=0}^{k} q^{r(r-1)/2}\begin{bmatrix} k \\ r \end{bmatrix} x^r, \tag{7}$$

which generalizes the binomial expansion, and its inverse

$$(1+x)^{-1}(1+qx)^{-1}\cdots(1+q^{k-1}x)^{-1} = \sum_{r=0}^{\infty}\begin{bmatrix} k+r-1 \\ r \end{bmatrix}(-1)^r x^r. \tag{8}$$

We also need a generalization of the forward difference operator Δ. Let n be a fixed positive integer. For any real function f we define q-differences recursively from

$$\Delta^0 f_i = f_i \qquad \text{for } i = 0, 1, \dots, n \text{ and} \tag{9}$$

$$\Delta^{k+1} f_i = \Delta^k f_{i+1} - q^k \Delta^k f_i \tag{10}$$

for $k = 0, 1, \dots, n-i-1$, where f_i denotes $f([i]/[n])$. If one constructs the Newton divided difference of a function evaluated at points $x_i = [i]/[n]$, one naturally re-discovers these q-differences (see Schoenberg [16] and Lee and Phillips [8]). The

q-differences reduce to ordinary forward differences when $q = 1$ and it is easily verified by induction that

$$\Delta^k f_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}. \tag{11}$$

Koçak and Phillips [7] showed that the kth q-difference of a product can be written in the form

$$\Delta^k (f_i g_i) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} \Delta^{k-r} f_{i+r} \Delta^r g_i. \tag{12}$$

This generalizes the well-known Leibniz rule for the kth ordinary difference of a product.

2. Bernstein polynomials

For each positive integer n, we define

$$B_n(f\,;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \tag{13}$$

where an empty product denotes 1 and, as above, $f_r = f([r]/[n])$. When $q = 1$, we obtain the classical Bernstein polynomial. We observe immediately from (13) that, independently of q,

$$B_n(f\,;0) = f(0), \quad B_n(f\,;1) = f(1), \tag{14}$$

for all functions f and thus $B_n(f\,;x) = f(x)$ for all linear functions f. We now state a generalization of the well-known forward-difference form (see, for example, Davis [4]) of the classical Bernstein polynomial.

THEOREM 1. *The generalized Bernstein polynomial, defined by* (13), *may be expressed in the q-difference form*

$$B_n(f\,;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0\, x^r. \tag{15}$$

This is proved in Phillips [12], where it is also shown that, for $n \geq 0$, 1 and 2 respectively,

$$B_n(1;x) = 1, \quad B_n(x;x) = x \quad \text{and} \quad B_n(x^2;x) = x^2 + \frac{x(1-x)}{[n]}. \tag{16}$$

For the classical Bernstein operator, the uniform convergence of the sequence of polynomials $B_n(f;x)$ to $f \in C[0,1]$ follows as a special case of the Bohman-Korovkin theorem (see Cheney [3] and Lorentz [9]). Convergence is assured by the following two properties:

1. B_n is a monotone operator; and
2. $B_n(f;x)$ converges uniformly to $f \in C[0,1]$ for $f(x) = 1, x$ and x^2.

Recall that if a linear operator L maps an element $f \in C[0,1]$ to $Lf \in C[0,1]$, then L is said to be *monotone* if $f(x) \geq 0$ on $[0,1]$ implies that $Lf(x) \geq 0$ on $[0,1]$. The generalized Bernstein operator defined by (13) is monotone for $0 < q \leq 1$. Yet if $0 < q < 1$ it is clear from (16) and (2) that $B_n(x^2;x)$ does not converge to x^2. In order to obtain convergence for the generalized Bernstein polynomials, it is therefore necessary to let $q = q_n$ in (13), so that q depends on n, and let $q_n \to 1$ from below as $n \to \infty$. The following theorem, which is concerned with convergent sequences of Bernstein polynomials other than the classical case with $q = 1$, is also a special case of the Bohman-Korovkin theorem.

THEOREM 2. *Let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ from below as $n \to \infty$. Then, for any $f \in C[0,1]$, the sequence of generalized Bernstein polynomials defined by*

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q_n^s x), \tag{17}$$

where $f_r = f([r]/[n])$, converges uniformly to $f(x)$ on $[0,1]$.

The above result is discussed in Phillips [12], where there is also a proof of the following generalization of Voronovskaya's theorem. (The latter proof in Phillips [12] closely follows that presented in Davis [4] for the classical Bernstein polynomials.)

THEOREM 3. *Let f be bounded on $[0,1]$ and let x_0 be a point of $[0,1]$ at which $f''(x_0)$ exists. Further, let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ from below as $n \to \infty$. Then the rate of convergence of the sequence of generalized Bernstein polynomials is governed by*

$$\lim_{n\to\infty} [n](B_n(f;x_0) - f(x_0)) = (1/2)\, x_0(1 - x_0) f''(x_0). \tag{18}$$

The Voronovskaya theorem provides an asymptotic estimate of how close $B_n f$ is to f. We now consider an alternative measure of how well $B_n f$ approximates f. Given a function f defined on $[0,1]$, let

$$\omega(\delta) = \sup_{|x_1 - x_2| \leq \delta} |f(x_1) - f(x_2)|, \tag{19}$$

the usual modulus of continuity, where the supremum is taken over all $x_1, x_2 \in [0, 1]$ such that $|x_1 - x_2| \le \delta$. The following result is presented in Phillips [12].

THEOREM 4. *If f is bounded on* $[0, 1]$ *and B_n denotes the generalized Bernstein operator defined by* (13), *then*

$$\| f - B_n f \|_\infty \le (3/2)\, \omega(1/[n]^{1/2}). \tag{20}$$

Rivlin [15] states this theorem for the case where $q = 1$, and his proof is easily extended to the generalized Bernstein operator.

3. Further properties

Algorithm: **for** $r = 0$ **to** n
$f_r^{[0]} := f([r]/[n])$
next r
for $m = 1$ **to** n
  for $r = 0$ **to** $n - m$
  $f_r^{[m]} := (q^r - q^{m-1}x) f_r^{[m-1]} + x f_{r+1}^{[m-1]}$
  next r
next m

For $q = 1$, this is the de Casteljau algorithm for evaluating the classical Bernstein polynomial. See Hoschek and Lasser [6] and Phillips and Taylor [14]. The above algorithm was proposed in Phillips [13], where it is shown that, for $0 \le m \le n$ and $0 \le r \le n - m$, each of the quantities $f_r^{[m]}$ satisfies both

$$f_r^{[m]} = \sum_{t=0}^{m} f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} x^t \prod_{s=0}^{m-t-1} (q^r - q^s x) \tag{21}$$

and the q-difference form

$$f_r^{[m]} = \sum_{s=0}^{m} q^{(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \Delta^s f_r x^s. \tag{22}$$

With $r = 0$ and $m = n$ in (21) or (22), we have $f_0^{[n]} = B_n(f; x)$, which justifies the validity of the above algorithm.

From (5) or (6) it follows by induction, as we remarked above, that the q-binomial coefficient defined by (4) is a polynomial in q of degree $r(k - r)$. We now further observe that this polynomial has non-negative integral coefficients. Thus the q-binomial coefficients are monotonic increasing functions of q and in particular, for $k \ge r \ge 0$,

$$\begin{bmatrix} k \\ r \end{bmatrix} \le \binom{k}{r} \tag{23}$$

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for $0 < q \leq 1$. This property of the q-binomial coefficients is used in Phillips [11], where the following result is proved concerning convergence of the derivatives of the generalized polynomials $B_n(f;x)$ uniformly to $f'(x)$ on [0, 1].

THEOREM 5. *Let $f \in C^1[0, 1]$ and let the sequence (q_n) be chosen so that the sequence (ϵ_n) converges to zero from above faster than $(1/3^n)$, where*

$$\epsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^{n-1}} - 1. \tag{24}$$

Then the sequence of derivatives of the generalized Bernstein polynomials $B_n(f;x)$ converges uniformly on [0, 1] *to $f'(x)$.*

The convergence of the kth derivative of the generalized Bernstein polynomial $B_n(f;x)$ to the kth derivative of $f(x)$, for $k > 1$, can be explored in a similar way.

4. Convexity

We now recall that if a function f is convex on [0, 1] then, for any t_0, t_1 such that $0 \leq t_0 < t_1 \leq 1$ and any λ, $0 < \lambda < 1$,

$$f(\lambda t_0 + (1 - \lambda)t_1) \leq \lambda f(t_0) + (1 - \lambda)f(t_1). \tag{25}$$

This is equivalent to saying that no chord of f lies below the graph of f. With $\lambda = q/(1 + q)$, $t_0 = [m]/[n]$ and $t_1 = [m + 2]/[n]$ in (25), where $0 < q \leq 1$, we see that, if f is convex,

$$f_{m+1} \leq \frac{q}{1+q} f_m + \frac{1}{1+q} f_{m+2}, \tag{26}$$

from which we deduce that

$$f_{m+2} - (1 + q)f_{m+1} + qf_m = \Delta^2 f_m \geq 0. \tag{27}$$

Thus the second q-differences of a convex function are non-negative. (This could also be verified by expressing the second q-difference as a multiple of a second-order divided difference.) It is easily deduced from (25) that, if $\lambda_0, \lambda_1, \ldots, \lambda_n$ are positive real numbers such that $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$ and $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$, then, if f is convex on [0, 1],

$$f\left(\sum_{j=0}^{n} \lambda_j t_j\right) \leq \sum_{j=0}^{n} \lambda_j f(t_j). \tag{28}$$

The following nice relation between a convex function and its Bernstein polynomials follows readily from (28).

THEOREM 6. *If f is convex on* $[0, 1]$ *and* $0 < q \leq 1$ *then, for any* $n \geq 1$,

$$B_n(f;x) \geq f(x), \quad 0 \leq x \leq 1. \tag{29}$$

PROOF. In view of the interpolatory property (14) we need concern ourselves only with $0 < x < 1$. Let us substitute

$$\lambda_j = \lambda_j(x) = \begin{bmatrix} n \\ j \end{bmatrix} x^j \prod_{s=0}^{n-j-1} (1 - q^s x), \quad 0 < x < 1, \tag{30}$$

and $t_j = [j]/[n]$ in (28). For $0 < q \leq 1$ it is clear that the t_j satisfy $0 = t_0 < t_1 < \cdots < t_n = 1$ and that the λ_j given by (30) are positive. Next we observe that the condition $\lambda_0 + \cdots + \lambda_n = 1$ is equivalent to the statement (see (16)) that $B_n(1;x) = 1$. The proof is completed by noting that

$$\sum_{j=0}^{n} \lambda_j(x) t_j = x, \tag{31}$$

which follows (see (16)) from the identity $B_n(x;x) = x$.

It is well known (see Davis [4]) that the classical Bernstein polynomials converge monotonically if the function is convex. The following result of Oruç and Phillips [10] shows that this beautiful monotonicity property extends to the generalized Bernstein polynomials.

THEOREM 7. *Let f be convex on* $[0, 1]$. *Then for any* q, $0 < q \leq 1$,

$$B_{n-1}(f;x) \geq B_n(f;x) \tag{32}$$

for $0 \leq x \leq 1$ *and all* $n \geq 2$. *If* $f \in C[0, 1]$ *the inequality holds strictly for* $0 < x < 1$ *unless f is linear in each of the intervals between consecutive knots* $[r]/[n-1]$, $0 \leq r \leq n-1$, *in which case we have the equality* $B_{n-1}(f;x) = B_n(f;x)$.

To emphasize its dependence on the parameter q, and to allow us to distinguish generalized Bernstein polynomials with different values of this parameter, let us write the generalized Bernstein polynomial as $B_n^q(f;x)$. Using the concept of total positivity, Goodman, Oruç and Phillips [5] have shown for all $n \geq 1$ and $0 < q \leq 1$ that if f is increasing then $B_n^q f$ is increasing, and that if f is convex then $B_n^q f$ is convex. They have also proved the following theorem concerning how the generalized Bernstein polynomials for a convex function vary with the parameter q.

THEOREM 8. *If f is convex on* $[0, 1]$ *then, for* $0 < q \leq r \leq 1$,

$$B_n^r(f;x) \leq B_n^q(f;x), \quad 0 \leq x \leq 1. \tag{33}$$

Thus the generalized Bernstein polynomials for a convex function are not only monotonic in n, the degree, but are also monotonic in the parameter q, for $0 < q \leq 1$.

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