A generalization of the Bollobás set pairs inequality

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Abstract

The Bollobás set pairs inequality is a fundamental result in extremal set theory with many applications. In this paper, for $n \geqslant k \geqslant t \geqslant 2$, we consider a collection of k families $\mathcal{A}_i : 1 \leqslant i \leqslant k$ where $\mathcal{A}_i = \{A_{i,j} \subset [n] : j \in [n]\}$ so that $A_{1,i_1} \cap \cdots \cap A_{k,i_k} \neq \varnothing$ if and only if there are at least t distinct indices $i_1, i_2, \ldots, i_k$. Via a natural connection to a hypergraph covering problem, we give bounds on the maximum size $\beta_{k,t}(n)$ of the families with ground set $[n]$.

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1 Introduction

A central topic of study in extremal set theory is the maximum size of a family of subsets of an n-element set subject to restrictions on their intersections. Classical theorems in the area are discussed in Bollobás [2]. In this paper, we generalize one such theorem, known as the Bollobás set pairs inequality or two families theorem [3]:

Theorem 1. (Bollobás) *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \varnothing$ if and only if $i, j \in [m]$ are distinct. Then*

$$\sum_{i=1}^{m} \binom{|A_i \cup B_i|}{|A_i|}^{-1} \leqslant 1. \tag{1}$$

For convenience, we refer to a pair of families $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 1 as a *Bollobás set pair*. The inequality above is tight, as we may take the pairs (A_i, B_i) to be distinct partitions of a set of size $a+b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leqslant i \leqslant \binom{a+b}{a}$.

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The latter inequality was proved for $a = 2$ by Erdős, Hajnal and Moon [5], and in general has a number of different proofs [11, 12, 14, 17, 18]. A geometric version was proved by Lovász [17, 18], who showed that if $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ are respectively a-dimensional and b-dimensional subspaces of a linear space and $\dim(A_i \cap B_j) = 0$ if and only if $i, j \in [m]$ are distinct, then $m \leqslant \binom{a+b}{a}$.

1.1 Main Theorem

Theorem 1 has been generalized in a number of different directions in the literature [6, 9, 13, 16, 21, 24]. In this paper, we give a generalization of Theorem 1 from the case of two families to $k \geqslant 3$ families of sets with conditions on the k-wise intersections. For $2 \leqslant t \leqslant k$, a *Bollobás (k,t)-tuple* is a sequence $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ of set families $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ where $\bigcap_{j=1}^{k} A_{j,i_j} \neq \varnothing$ if and only if at least t of the indices $i_1, i_2, \ldots, i_k$ are distinct. We refer to m as the *size* of the Bollobás (k,t)-tuple. Let $[m]_{(t)}$ denote the set of sequences of t distinct elements of $[m]$ and fix a surjection $\phi : [k] \to [t]$. For $\sigma \in [m]_{(t-1)}$, set $\sigma(t) = \sigma(1)$ and define $A_{1,\sigma}(\phi) = \bigcap_{j:\phi(j)=1} A_{j,\sigma(1)}$ and, for $2 \leqslant j \leqslant t$, we define

$$A_{j,\sigma}(\phi) = \bigcap_{h:\phi(h)=j} A_{h,\sigma(j)} \setminus \bigcup_{h=1}^{j-1} A_{h,\sigma}(\phi).$$

Using this notation, we generalize (1) as follows:

Theorem 2. *Let $k \geqslant t \geqslant 2$ and $m \geqslant t$, let $\phi : [k] \to [t]$ be a surjection, and let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ be a Bollobás (k,t)-tuple of size m. Then*

$$\sum_{\sigma \in [m]_{(t-1)}} \binom{|A_{1,\sigma}(\phi) \cup A_{2,\sigma}(\phi) \cup \cdots \cup A_{t,\sigma}(\phi)|}{|A_{1,\sigma}(\phi)| \ \ |A_{2,\sigma}(\phi)| \ \ \cdots \ \ |A_{t,\sigma}(\phi)|}^{-1} \leqslant 1. \tag{2}$$

We show in Section 2.1 that this inequality is tight for all $k \geqslant t = 2$, but do not have an example to show that this inequality is tight for any $t > 2$.

For $n \geqslant k \geqslant t \geqslant 2$, let $\beta_{k,t}(n)$ denote the maximum m such that there exists a Bollobás (k,t)-tuple of size m consisting of subsets of $[n]$. Then (1) gives $\beta_{2,2}(n) \leqslant \binom{n}{\lfloor n/2 \rfloor}$ which is tight for all $n \geqslant 2$. Letting $H(q) = -q \log_2 q - (1-q)\log_2(1-q)$ denote the standard binary entropy function, we prove the following theorem:

Theorem 3. *For $k \geqslant 3$ and large enough n,*

$$\frac{1}{k} \leqslant \frac{\log_2 \beta_{k,2}(n)}{n} \leqslant H\left(\frac{1}{k}\right) \leqslant \frac{\log_2(ke)}{k}. \tag{3}$$

For $k \geqslant t \geqslant 3$ and large enough n,

$$\frac{\log_2 e}{\binom{k}{t-1}(t+1)t^{t-1}} \leqslant \frac{\log_2 \beta_{k,t}(n)}{n} \leqslant \frac{2}{\binom{k}{t-1}(t-1)^{t-3}}. \tag{4}$$

This determines $\log_2 \beta_{k,2}(n)$ up to a factor of order $\log_2 k$ and $\log_2 \beta_{k,t}(n)$ up to a factor of order t^3. We leave it as an open problem to determine the asymptotic value of $(\log_2 \beta_{k,t}(n))/n$ as $n \to \infty$ for any $k \geqslant 3$ and $t \geqslant 2$. A natural source for lower bounds on $\beta_{k,t}(n)$ comes from the probabilistic method – see the random constructions in Section 3.1 which establish the lower bounds in Theorem 3. To prove Theorem 3, we use a natural connection to hypergraph covering problems.

1.2 Covering hypergraphs

Theorem 1 has a wide variety of applications, from saturation problems [3, 19] to covering problems for graphs [11, 20], complexity of 0-1 matrices [23], geometric problems [1], counting cross-intersecting families [7], crosscuts and transversals of hypergraphs [24, 25, 26], hypergraph entropy [15, 22], and perfect hashing [8, 10]. In this section, we give an application of our main results to hypergraph covering problems. For a k-uniform hypergraph H, let $f(H)$ denote the minimum number of complete k-partite k-uniform hypergraphs whose union is H. In the case of graph covering, a simple connection to the Bollobás set pairs inequality (1) may be described as follows. Let $K_{n,n} \setminus M$ denote the complement of a perfect matching $M = \{x_i y_i : 1 \leqslant i \leqslant n\}$ in the complete bipartite graph $K_{n,n}$ with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. If $H_1, H_2, \ldots, H_m$ are complete bipartite graphs in a minimum covering of $K_{n,n} \setminus M$, then let $A_i = \{j : x_i \in V(H_j)\}$ and $B_i = \{j : y_i \in V(H_j)\}$. Setting $\mathcal{A} = \{A_i\}_{i \in [m]}$ and $\mathcal{B} = \{B_i\}_{i \in [m]}$, it is straightforward to check that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair, and Theorem 1 applies to give

$$f(K_{n,n} \backslash M) = \min\{m : \binom{m}{\lceil m/2 \rceil} \geqslant n\}. \tag{5}$$

In a similar way, Theorem 2 applies to covering complete k-partite k-uniform hypergraphs. Let $K_{n,n,\ldots,n}$ denote the complete k-partite k-uniform hypergraph with parts $X_i = \{x_{ij} : j \in [n]\}$ for $i \in [k]$. Let $H_{k,t}(n)$ denote the subhypergraph consisting of hyperedges $\{x_{1,i_1}, x_{2,i_2}, \ldots, x_{k,i_k}\}$ such that at least t of the indices $i_1, i_2, \ldots, i_k$ are distinct, and set $f_{k,t}(n) = f(H_{k,t}(n))$. Then there is a one-to-one correspondence between Bollobás (k,t)-tuples of subsets of $[m]$ and coverings of $H_{k,t}(n)$ with m complete k-partite k-graphs. We let $\beta_{k,t}(m)$ be the maximum size of a Bollobás (k,t)-tuple of subsets of $[m]$, so that

$$f_{k,t}(n) = \min\{m : \beta_{k,t}(m) \geqslant n\}. \tag{6}$$

This correspondence together with Theorem 2 will be exploited to prove

$$f_{k,2}(n) \geqslant \min\{m : \binom{m}{\lceil m/k \rceil} \geqslant n\} \tag{7}$$

which is partly an analog of (5). More generally, we prove the following theorem:

Theorem 4. *For $k \geqslant 3$ and large enough n,*

$$\frac{k}{\log_2(ke)} \leqslant \frac{1}{H(\frac{1}{k})} \leqslant \frac{f_{k,2}(n)}{\log_2 n} \leqslant k. \tag{8}$$

For $k \geqslant t \geqslant 3$ and large enough n,

$$\binom{k}{t-1}\frac{(t-1)^{t-3}}{2} \leqslant \frac{f_{k,t}(n)}{\log_2 n} \leqslant \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}. \tag{9}$$

The bounds on $\beta_{k,t}(n)$ in Theorem (3) follow immediately from this theorem and (6). Equation (9) gives the order of magnitude for each $t \geqslant 3$ as $k \to \infty$, but for $t = 2$, Equation (8) has a gap of order $\log_2 k$. From (7), we obtain $\beta_{k,2}(n) \leqslant \binom{n}{\lfloor n/k \rfloor}$. It is perhaps unsurprising that the asymptotic value of $f_{k,t}(n)/\log_2 n$ as $n \to \infty$ is not known for any $k > 2$, since a limiting value of $f(K_n^k)/\log_2 n$ is not known for any $k > 2$ – see Körner and Marston [15] and Guruswami and Riazanov [10].

1.3 Organization and notation

Given a subset $A \subset [n]$, let $A^c := [n] \setminus A$ be the complement of A in $[n]$. For positive integers $k \leqslant n$, let $(n)_{(k)} = (n)(n-1)\cdots(n-k+1)$ denote the falling factorial. This paper is organized as follows. In Section 2, we prove Theorem 2. In Section 2.1, we construct a Bollobás $(k,2)$-tuple which achieves equality in Theorem 2 and in Section 2.2, we construct a Bollobás $(k,2)$-tuple which gives the lower bound in Equation (3). The upper bound on $f_{k,t}(n)$ in Theorem 4 comes from a probabilistic construction in Section 3.1, and the proof of the lower bound on $f_{k,t}(n)$ is given in Section 3.3; we prove (7) in Section 3.2.

2 Proof of Theorem 2

Given a Bollobás set (k,t)-tuple $(\mathcal{A}_1,\ldots,\mathcal{A}_k)$ with $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ and a surjection $\phi : [k] \to [t]$, consider $\mathcal{A}_\ell(\phi) : 1 \leqslant \ell \leqslant t$ where $\mathcal{A}_\ell(\phi) = \{A_{\ell,i}(\phi) : 1 \leqslant i \leqslant m\}$ and

$$A_{\ell,i}(\phi) = \bigcap_{h:\phi(h)=\ell} A_{h,i}.$$

It follows that $(\mathcal{A}_1(\phi),\ldots,\mathcal{A}_t(\phi))$ is a Bollobás set (t,t)-tuple and hence it suffices to prove Theorem 2 in the case where $t = k$. In this setting, surjections $\phi : [k] \to [k]$ simply permute the k families and as such we suppress the notation of ϕ for the remainder of this section. One of the proofs of Theorem 1, given a Bollobás set pair, defines a collection of chains $\mathcal{C}_i$ for $i \in [m]$ and shows that these chains are necessarily disjoint. Similarly, given a Bollobás set (k,k)-tuple, we will define a collection of chains $\mathcal{C}_\sigma$ for every ordered collection σ of $(k-1)$ distinct indices of $[m]$ and show these chains are pairwise disjoint.

Let $(\mathcal{A}_1,\ldots,\mathcal{A}_k)$ with $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ be a Bollobás set (k,k)-tuple, and set

$$X = \bigcup_{i=1}^{m}(A_{1,i} \cup A_{2,i} \cup \cdots \cup A_{k,i})$$

with $|X| = n$. For $\sigma \in [m]_{(k-1)}$, define a subset $\mathscr{C}_\sigma$ of permutations $\pi : X \to [n]$ by

$$\mathscr{C}_\sigma := \left\{ \pi : X \to [n] : \max_{x \in A_{1,\sigma}} \pi(x) < \min_{y \in A_{2,\sigma}} \pi(y) \leqslant \max_{y \in A_{2,\sigma}} \pi(y) < \cdots < \min_{z \in A_{k,\sigma}} \pi(z) \right\}.$$

Letting $U_\sigma := A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}$, elementary counting methods give

$$|\mathscr{C}_\sigma| = \binom{n}{|U_\sigma|} |A_{1,\sigma}|! \cdots |A_{k,\sigma}|!(n - |U_\sigma|)! = n! \cdot \binom{|U_\sigma|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|}^{-1}. \tag{10}$$

We will now prove a lemma which states that $\{\mathscr{C}_\sigma\}_{\sigma \in [m]_{(k-1)}}$ forms a disjoint collection of a permutations. The general proof only works for $k \geqslant 4$, so we first consider $k = 3$.

Lemma 5. *If $\sigma_1, \sigma_2 \in [m]_{(2)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \varnothing$.*

Proof. Seeking a contradiction, suppose there exists $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. After relabeling, it suffices to consider the following five cases.

(1) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 4\}$ (2) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 3\}$
(3) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{1, 3\}$ (4) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{2, 3\}$
(5) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{3, 1\}$.

In case (1), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$ and thus $\pi \in \mathscr{C}_{\sigma_2}$ yields

$$\max_{x \in A_{1,1}} \pi(x) \leqslant \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y).$$

Then as $A_{1,1} \cap A_{2,4} \cap A_{3,2} \neq \varnothing$, there exists $w \in A_{1,1} \cap A_{2,4} \cap A_{3,2}$. It follows that $w \notin A_{1,2}$ since if $w \in A_{1,2}$, then $w \in A_{1,2} \cap A_{2,4} \cap A_{3,2} \neq \varnothing$; a contradiction. But this yields a contradiction as

$$\pi(w) \leqslant \max_{x \in A_{1,1}} \pi(x) \leqslant \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y) \leqslant \pi(w).$$

In case (2), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$ and we recover a similar contradiction as case (1) by noting that there exists $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ with $w \notin A_{1,2}$.

In case (3) we may assume $\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\}$ and $\pi \in \mathscr{C}_{1,3}$ yields $\max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$. Thus

$$\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$$

and there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,3}$. It follows that $\pi(w) < \pi(w)$, a contradiction.

In case (4), if $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$, then using $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ and noting $w \notin A_{1,2}$, we get a contradiction. Thus, we may assume otherwise and $\pi \in \mathcal{C}_{1,2}$ gives

$$\max_{x \in A_{1,2}} \pi(x) < \max_{x \in A_{1,1}} \pi(x) < \min_{z \in A_{3,1} \setminus (A_{1,1} \cup A_{2,2})} \pi(z).$$

This is a contradiction as there exists $w \in A_{1,2} \cap A_{2,3} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,2}$. In case (5), if $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,3}\}$, then we may proceed as in the latter part of case (4) using $w \in A_{1,1} \cap A_{2,2} \cap A_{3,3}$ and $w \notin A_{2,1}$ and $w \notin A_{1,3}$ to get a contradiction. Otherwise, proceeding as in case (1) and noting there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$, but $w \notin A_{1,1}$ yields a contradiction. $\square$

A similar argument yields the analog of Lemma 5 to the case where $k \geqslant 4$.

Lemma 6. *Let $k \geqslant 4$. If $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \varnothing$.*

Proof. Since $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, there exists minimal $h \in [k-1]$ so that $\sigma_1(h) \neq \sigma_2(h)$. Seeking a contradiction, suppose there exists a $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. Without loss of generality,

$$\max\{\pi(x) : x \in A_{h,\sigma_1}\} \leqslant \max\{\pi(x) : x \in A_{h,\sigma_2}\} < \min\{\pi(z) : z \in A_{k,\sigma_2}\}.$$

Now, consider a bijection $\tau : [k-1] \setminus \{h\} \to [k-1] \setminus \{1\}$ which has no fixed points. As in Lemma 5, we want to show that there exists a $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ and consider two separate cases.

First, suppose that $\sigma_1(h) \notin \sigma_2([k-1])$. As $|\{\sigma_1(h), \sigma_2(1), \ldots, \sigma_2(k-1)\}| = k$, there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap \bigcap_{l \in [k-1] \setminus \{h\}} A_{l,\sigma_2(\tau(l))}. \tag{11}$$

Next, suppose that $\sigma_1(h) = \sigma_2(x)$ for some x. We now claim that $x \neq 1$. If $h = 1$, then this is trivial. If $h > 1$, then $\sigma_1(1) = \sigma_2(1)$, so $\sigma_1(h) \neq \sigma_2(1)$ since $\sigma_1(h) \neq \sigma_1(1)$. For τ as above, there exists $y \in [k-1] \setminus \{h\}$ so that $\tau(y) = x$. Taking γ distinct from $\{\sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}$, $|\{\sigma_1(h), \gamma, \sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}| = k$ and hence there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{y,h\}} A_{l,\sigma_2(\tau(l))}. \tag{12}$$

By construction, $w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)}$. Suppose there exists a $t \in [k-1] \setminus \{h\}$ so that $w \in A_{t,\sigma_2(t)}$. As τ has no fixed points, replacing the set in the k-wise intersection corresponding to $\mathcal{A}_t$ with $A_{t,\sigma_2(t)}$ in either (11) or (12), w is an element of this new k-wise intersection with $(k-1)$ distinct indices; a contradiction. If $w \in A_{h,\sigma_2(h)}$, then we may similarly replace $A_{h,\sigma_1(h)}$ with $A_{h,\sigma_2(h)}$ in the k-wise intersection in either (11) or (12) to get a contradiction. Thus, $w \notin A_{1,\sigma_2(1)} \cup \cdots \cup A_{k-1,\sigma_2(k-1)}$ and hence $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ so that $\pi(w) < \pi(w)$; a contradiction. $\square$

Using Equation (10), Lemma 5, and Lemma 6, we are now able to prove Theorem 2 in the case where $t = k$. There are $n!$ total permutations, and Lemma 5 and Lemma 6 yield that each of which appears in at most one of the sets $\mathscr{C}_\sigma$ for $\sigma \in [m]_{(k-1)}$. Hence, using $|\mathscr{C}_\sigma|$ in Equation (10),

$$\sum_{\sigma \in [m]_{(k-1)}} |\mathscr{C}_\sigma| = \sum_{\sigma \in [m]_{(k-1)}} n! \cdot \binom{|A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|}^{-1} \leqslant n!$$

and thus the result follows by dividing through by $n!$.

2.1 Sharpness of Theorem 2

We give a simple construction establishing the sharpness of Theorem 2 for $k \geqslant t = 2$. Let $n \geqslant 4k$ and using addition modulo n, define $A_{1,i} = \{i\}^c$, $A_{j,i} = \{i-(j-1), i+(j-1)\}^c$ for $j \in [2, k-1]$, and $A_{k,i} = \{i-k+2, i-k+3, \dots, i+k-2\}$. Letting $\mathcal{A}_j = \{A_{j,i}\}_{i\in[n]}$ for all $j \in [k]$, we will show $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple. Since $|A_{1,i}| = n-1$ and $|A_{2,i} \cap \cdots \cap A_{k,i}| = 1$, Theorem 2 with $t = 2$ and surjection $\phi : [k] \to [2]$ with $\phi(1) = 1$ and $\phi(i) = 2$ for $i \neq 1$ gives

$$1 \;\geqslant\; \sum_{i=1}^{n} \binom{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|}^{-1} \;=\; \sum_{i=1}^{n} \frac{1}{n} \;=\; 1.$$

By construction, for all $i \in [n]$, $A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k,i} = \varnothing$. It thus suffices to show these are the only empty k-wise intersections. To this end, for $\boldsymbol{i} = (i_1, \dots, i_{k-1})$, define

$$A(\boldsymbol{i}) := A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}}.$$

Lemma 7. *Let $\boldsymbol{i} = (i_1, \dots, i_{k-1})$. If $A(\boldsymbol{i})^c = A_{k,i_k}$, then $i_1 = \cdots = i_k$.*

Proof. We proceed by induction on k where the result is trivial when $k = 2$. In the case where $k > 2$, $i_{k-1} - k + 2 = i_k + x$ for some x such that $-(k-2) \leqslant x \leqslant (k-2)$ and thus $i_{k-1} + (k-2) = i_{k-1} - (k-2) + (2k-4) = i_k + x + (2k-4)$.

Next, there is a y such that $-(k-2) \leqslant y \leqslant (k-2)$ with $i_{k-1} + (k-2) = i_k + y$, and since $n \geqslant 4k$, $x + 2k - 4 = y$ with equality over $\mathbb{Z}$ and moreover $i_{k-1} + (k-2) = i_k + (k-2)$ over $\mathbb{Z}$ and hence $i_k = i_{k-1}$. Removing these elements from each set, the result then follows by induction. $\square$

If $A_{1,i_1} \cap \cdots \cap A_{k,i_k} = \varnothing$, then as $A(\boldsymbol{i}) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots A_{k-1,i_{k-1}}$,

$$\varnothing = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = A(\boldsymbol{i}) \cap A_{k,i_k}.$$

The result follows by noting $|A(\boldsymbol{i})| \geqslant n - (2k-3)$, $|A_{k,i_k}| = 2k-3$, and using Lemma 7.

2.2 An Explicit Construction

Let $k \geqslant 3$. An explicit construction of a Bollobás $(k,2)$-tuple $(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k)$ where $|\mathcal{A}_i| = 2^n$ and each $\mathcal{A}_i$ consists of subsets of X for $|X| = kn$ may be described as follows. Let $I_j := \{x_{j,1}, x_{j,2}, \dots, x_{j,k}\}$ and consider $X = I_1 \sqcup \cdots \sqcup I_n$. Now, for each $f : [n] \to [2]$ and $j \in [k]$, define

$$A_{j,f} := \{x_{1,f(1)+j-1},\ \dots\ , x_{n,f(n)+j-1}\}^c$$

where we work modulo k within the subscripts of I_j. It is straightforward to check that $(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple. This establishes the lower bound on $\beta_{k,2}(n)$ in Equation (3) and hence the upper bound on $f_{k,2}(n)$ in Equation (8).

3 Proof of Theorem 4

3.1 Upper bound on $f_{k,t}(n)$

We wish to find a covering of $H_{k,t}(n)$ with complete k-partite k-graphs and assume the parts of $H_{k,t}(n)$ are $X_1, X_2, \ldots, X_k$. For each subset T of $[k]$ of size t, consider the uniformly random coloring $\chi_T : [n] \to T$. Given such a χ_T, let $Y_i \subset X_i$ be the vertices of color i for $i \in T$; that is $Y_i := \{x_{ij} : \chi(j) = i\}$ and $Y_i = X_i$ for $i \notin T$. Denote by $H(T,\chi)$ the (random) complete k-partite hypergraph with parts $Y_1, Y_2, \ldots, Y_k$, and note that $H(T,\chi) \subset H_{k,t}(n)$. We place each $H(T,\chi)$ a total of N times independently and randomly where

$$N = \Big\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} \Big\rfloor$$

and produce $\binom{k}{t}N$ random subgraphs $H(T,\chi)$. For a set partition π of $[k]$, let $|\pi|$ denote the number of parts in the partition and index the parts by $[|\pi|]$. Given a set partition $\pi = (P_1, P_2, \ldots, P_s)$, let

$$f(\pi, t) = \sum_{T \in [s]^{(t)}} \prod_{i \in T} |P_i|.$$

If U is the number of edges of $H_{k,t}(n)$ not in any of these subgraphs, then

$$\mathbb{E}(U) \leqslant \sum_{|\pi| \geqslant t} n^{|\pi|}(1 - t^{-t})^{Nf(\pi,t)} = \sum_{t \leqslant s \leqslant k} n^s \sum_{|\pi|=s} (1 - t^{-t})^{Nf(\pi,t)}. \tag{13}$$

For sufficiently large n, we claim that $\mathbb{E}(U) < 1$, which implies there exists a covering of $H_{k,t}(n)$ with at most $\binom{k}{t}N$ complete k-partite k-graphs, as required. The following technical lemma states that f is a decreasing function in the set partition lattice, and that $f(\pi, t)$ increases when we merge all but one element of a smaller part of π with a larger part of π:

Lemma 8. *Let $k \geqslant s \geqslant t \geqslant 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$.*

- (i) *If π' is a refinement of π with $|\pi'| = s+1$, then $f(\pi, t) \leqslant f(\pi', t)$.*
- (ii) *If $|P_1| \geqslant |P_2| \geqslant 2$ and $a \in P_2$, and π' is the partition $(P_1', P_2', \ldots, P_s')$ of $[k]$ with $P_1' = P_1 \cup P_2 \setminus \{a\}$ and $P_2' = \{a\}$ and with $P_i' = P_i$ for $3 \leqslant i \leqslant s$, then $f(\pi', t) \leqslant f(\pi, t)$.*

The proof of Lemma 8 part (i) is in Appendix A and the proof of (ii) is similar to the proof of (i). By Lemma 8, a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ consists of one part of size $k - s + 1$ and $s - 1$ singleton parts and hence

$$\min\{f(\pi, t) : |\pi| = s\} = (k - s + 1)\binom{s-1}{t-1} + \binom{s-1}{t}. \tag{14}$$

In what follows, we denote a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ by π_s.

For n large enough, and all s where $t \leqslant s \leqslant k$, we will show

$$\frac{\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1-t^{-t})^{Nf(\pi,t)}} \geqslant n^{s-t}.$$

Replacing the numerator with its largest term and each term in denominator with its largest term,

$$\frac{\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1-t^{-t})^{Nf(\pi,t)}} \geqslant \frac{(1-t^{-t})^{Nf(\pi_t,t)}}{S(k,s)(1-t^{-t})^{Nf(\pi_s,t)}} = \frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))}$$

where $S(k,s)$ is the Stirling number of the second kind. Taking $n \geqslant S(k,s)$, we will show in Appendix B that

$$\frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))} \geqslant n^{s-t}. \tag{15}$$

Therefore, the index $s=t$ maximizes the right hand side of Equation (13), and hence

$$\mathbb{E}[U] \leqslant (k-t+1)(n^t)\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)} < (k-t+1)n^tS(k,t)(1-t^{-t})^{N(k-t+1)} < 1$$

for our choice of N provided $n \geqslant kS(k,t)$. Thus,

$$f_{k,t}(n) \leqslant \binom{k}{t}\frac{(t+1)t^t\log_2 n}{(k-t+1)\log_2 e} = \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}\log_2 n.$$

3.2 Lower bound on $f_{k,2}(n)$

In this section, we show

$$f_{k,2}(n) \geqslant \min\{m : \binom{m}{\lceil m/k\rceil} \geqslant n\}. \tag{16}$$

Let $\{H_1, H_2, \ldots, H_m\}$ be a covering of $H_{k,2}(n)$ with $m = f_{k,2}(n)$ complete k-partite k-graphs. We recall $H_{k,2}(n) = K_{n,n,\ldots,n}\backslash M$, where M is a perfect matching of $K_{n,n,\ldots,n}$. For $i \in [k]$ and $j \in [n]$, define $A_{i,j} = \{H_r : x_{ij} \in V(H_r)\}$ and $\mathcal{A}_i = \{A_{i,j} : 1 \leqslant j \leqslant n\}$. As in (6), $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple of size n. For convenience, for each $i \in [k]$, let $\phi_i : [k] \to [2]$ be so that $\phi_i^{-1}(1) = \{i\}$. Taking the sum of inequality from Theorem 2 with $t=2$ over all $i \in [k]$,

$$\sum_{i=1}^{k}\sum_{j=1}^{n}\binom{|A_{1,j}(\phi_i) \cup A_{2,j}(\phi_i)|}{|A_{1,j}(\phi_i)|}^{-1} \leqslant k. \tag{17}$$

We use this inequality to give a lower bound on $f_{k,2}(n) = m$. First we observe

$$\sum_{r=1}^{m}|V(H_r)| = \sum_{j=1}^{n}\sum_{i=1}^{k}|A_{i,j}| = \sum_{j=1}^{n}\sum_{i=1}^{k}|A_{1,j}(\phi_i)|. \tag{18}$$

Let ∂H denote the set of $(k-1)$-tuples of vertices contained in some edge of a hypergraph H. Then

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{2,j}(\phi_i)|. \tag{19}$$

Putting the above identities together,

$$\sum_{r=1}^{m} |V(H_r)| + \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|). \tag{20}$$

We note $|\partial H_r \cap \partial M| \leqslant |V(H_r)|/(k-1)$, and therefore

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| \leqslant \frac{1}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{21}$$

It follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) \leqslant \frac{k}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{22}$$

Subject to the linear inequalities (18) and (22), the left side of (17) is minimized when $kn|A_{1,j}(\phi_i)| = \sum_{r=1}^{m} |V(H_r)|$ and $kn(|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) = (k-1)|A_{1,j}(\phi_i)|$. Since $|V(H_r)| \leqslant (k-1)n$ for all $r \in [m]$, (17) implies $\binom{m}{\lceil m/k \rceil} \geqslant n$, which gives (16). $\square$

3.3 Lower bound on $f_{k,k}(n)$

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,k}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,k}(n))$. Given a k-partite k-graph H, consider its 2-shadow $\partial_2(H) = \{R \subset V(H) : |R| = k-2, R \subset e \text{ for some } e \in H\}$. Let $\partial_2(\mathcal{H}) = \bigcup_{i=1}^{m} \partial_2(H_i)$.

Given $R \in \partial_2(\mathcal{H})$ and $H_i \in \mathcal{H}$, let $H_i(R) := \{e \in \binom{V(H_i)}{2} : e \cup R \in H_i\}$ be the possibly empty link graph of the edge R in the hypergraph H_i and let $V(H_i(R))$ be the set of vertices in the link graph. Observe that double counting yields

$$\sum_{R \in \partial_2(\mathcal{H})} \left(\sum_{i=1}^{m} |V(H_i(R))| \right) = \sum_{i=1}^{m} \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right). \tag{23}$$

An optimization argument yields $|\partial_2(H_i)|$ is maximized when the parts of H_i are of equal or nearly equal maximal size. Since $|V(H_i(R))| \leqslant 2(n-k+2)$, the right hand side of Equation (23) is bounded above by

$$\sum_{i=1}^{m} \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right) \leqslant m \cdot \binom{k}{2} \cdot \left(\frac{n}{k}\right)^{k-2} \cdot 2(n-k+2). \tag{24}$$

For a lower bound on the left hand side of Equation (23), fix $R \in \partial_2(\mathcal{H})$ and without loss of generality suppose that $R = \{x_{1,1}, \ldots, x_{k-2,k-2}\}$. Let $Y = [k-1, n]$. Let $K_{Y,Y}$ be the complete bipartite graph with two distinct copies of Y and $\mathcal{M} = \{(x_{k-1,i}, x_{k,i} : i \in Y\}$ be a perfect matching in $K_{Y,Y}$. Then, $\{H_1(R), \ldots, H_m(R)\}$ forms a biclique cover of $K_{Y,Y} \setminus \mathcal{M}$. Applying the convexity result of Tarjan [23, Lemma 5],

$$\sum_{i=1}^{m} |V(H_i(R))| \geqslant (n-k+2)\log_2(n-k+2).$$

Noting that $|\partial_2(\mathcal{H})| = \binom{k}{2}(n)_{(k-2)}$, the left hand side of Equation (23) is bounded below by

$$\sum_{R \in \partial_2(\mathcal{H})} \left(\sum_{i=1}^{m} |V(H_i(R))| \right) \geqslant \binom{k}{2} (n)_{(k-2)} (n-k+2) \log_2(n-k+2). \tag{25}$$

Comparing the bounds from Equation (24) and Equation (25),

$$m \geqslant \frac{(n)_{(k-2)} \log_2(n-k+2)}{2\left(\frac{n}{k}\right)^{k-2}} \geqslant \frac{k^{k-2}}{2} \log_2 n$$

provided that n is large enough.

For $t \geqslant 3$ and $t < k$, the lower bound on $f_{k,t}(n)$ in Theorem 4 is obtained from the lower bounds on $f_{t-1,t-1}(n-1)$ as follows: Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,t}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,t}(n))$. Given $T \in \binom{[k]}{k-t+1}$, define $H_T \subset H_{k,t}(n)$ by

$$H_T := \{\{x_{1,i_1}, \ldots, x_{k,i_k}\} \in H_{k,t}(n) : i_j = 1 \;\forall\; j \in T\}.$$

It follows that at least $f_{t-1,t-1}(n-1)$ of the complete k-partite k-graphs in $\mathcal{H}$ are needed to cover H_T. Moreover, for distinct $T, T' \in \binom{[k]}{k-t+1}$, the corresponding complete k-partite k-graphs from $\mathcal{H}$ are necessarily pairwise disjoint and hence

$$f_{k,t}(n) \geqslant \binom{k}{k-t+1} f_{t-1,t-1}(n-1) \geqslant \binom{k}{t-1} \frac{(t-1)^{t-3}}{2} \log_2 n$$

provided that n is large enough.

4 Concluding remarks

• Our main theorem, Theorem 2 is tight for $t = 2$ and $k \geqslant 2$, as shown in Section 2.1. It would be interesting to generalize this example to $2 < t \leqslant k$ to determine whether Theorem 2 is tight in general. The first open case is $t = k = 3$.

• A particular case of the Bollobás set pairs inequality occurs when every set in $\mathcal{A}$ has size a and every set in $\mathcal{B}$ has size b, and one obtains the tight bound $|\mathcal{A}| \leqslant \binom{a+b}{b}$. The generalization to Bollobás (k,t)-tuples for $k \geqslant 3$ is equally interesting but wide open, as are potential generalizations to vector spaces – see Lovász [17, 18].

• Orlin [20] proved that the clique cover number $cc(K_n \backslash M)$ of a complete graph K_n minus a perfect matching M is precisely $\min\{m : 2\binom{m-1}{\lfloor m/2 \rfloor} \geqslant n\}$. Theorem 4 yields lower bounds on the clique cover number of the complement of a perfect matching M in the complete k-uniform hypergraph K_n^k:

Corollary 9. *Let $K_n^k \setminus M$ be the complement of a perfect matching in K_n^k. Then*

$$cc(K_n^k \setminus M) \geqslant \frac{\log_2 \frac{n}{k}}{H(\frac{1}{k})} \geqslant \frac{k \log_2 \frac{n}{k}}{\log_2(ke)}.$$

• It would be interesting to prove an analog of Equation (16) for $t \geqslant 3$. That is,

$$f_{k,t}(n) \geqslant \min\{m : \binom{m}{\alpha_1, \ldots, \alpha_t} \geqslant n_{(t-1)}\}$$

for some optimal $\alpha_1, \ldots, \alpha_t$. The difficulty here lies in determining effective bounds on $|A_{i,\sigma}(\phi)|$.

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A Proof of Lemma 8(i)

Let $k \geqslant s \geqslant t \geqslant 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$. In this section, we will show that if π' is a refinement of π with $|\pi'| = s+1$, then $f(\pi,t) \leqslant f(\pi',t)$.

Proof. Let $\pi = P_1|P_2|\cdots|P_s$ and without loss of generality, $\pi' = P_x|P_y|P_2|\cdots|P_s$. Setting $\mathcal{T}(\overline{1}) = \{T \in [s]^{(t)} : 1 \notin T\}$ and $\mathcal{T}'(\overline{x},\overline{y}) = \{T \in \{x,y,2,\ldots,s\}^{(t)} : x,y \notin T\}$, it follows that

$$\sum_{T\in\mathcal{T}(\overline{1})} \prod_{i\in T} |P_i| = \sum_{T\in\mathcal{T}'(\overline{x},\overline{y})} \prod_{i\in T} |P_i|.$$

Now, letting $\mathcal{T}(1) = \{T \in [s]^{(t)} : 1 \in T\}$ and $\mathcal{T}'(x,\overline{y}) = \{T \in \{x,y,2,\ldots,s\}^{(t)} : x \in T, y \notin T\}$ and $\mathcal{T}'(\overline{x},y) = \{T \in \{x,y,2,\ldots,s\}^{(t)} : x \notin T, y \in T\}$, we see that

$$\sum_{T\in\mathcal{T}(1)} \prod_{i\in T} |P_i| = \sum_{T\in\mathcal{T}'(\overline{x},y)} \prod_{i\in T} |P_i| + \sum_{T\in\mathcal{T}'(x,\overline{y})} \prod_{i\in T} |P_i|$$

since $|P_1| = |P_x| + |P_y|$. Thus letting $\mathcal{T}'(x,y) = \{T \in \{x,y,2,\ldots,s\}^{(t)} : x \in T, y \in T\}$,

$$f(\pi',t) - f(\pi,t) = \sum_{T\in\mathcal{T}'(x,y)} \prod_{i\in T} |P'_i|$$

and in particular $f(\pi,t) \leqslant f(\pi',t)$. □

B Proof of Equation (15)

Let $S(k,s)$ be the Stirling number of the second kind and $f(\pi)$ be as in Section 3. In this section we will show

$$\frac{1}{S(k,s)} (1 - t^{-t})^{N(f(\pi_t,t) - f(\pi_s,t))} \geqslant n^{s-t}.$$

Proof. First, we recall that

$$N = \left\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} \right\rfloor \quad \text{and} \quad f(\pi_s,t) = (k-s+1)\binom{s-1}{t-1} + \binom{s-1}{t}.$$

As a result, when $t \leqslant s < k$, a calculation yields that

$$f(\pi_{s+1},t) - f(\pi_s,t) = (k-s)\binom{s-1}{t-2}. \tag{26}$$

Letting $n \geqslant S(k,t)$, after taking $\log_2(\cdot)$ on both sides of (15), it suffices to prove that

$$N \cdot \frac{f(\pi_s,t) - f(\pi_t,t)}{t^t} \left(-t^t \log_2(1 - t^{-t}) \right) \geqslant (s-t+1)\log_2(n). \tag{27}$$

Using the fact that $(1-t^{-t})^{t^t} \leqslant e^{-1}$ and our choice of N, it suffices to show that

$$f(\pi_s,t) - f(\pi_t,t) \geqslant \frac{(s-t+1)(k-t+1)}{t+1}. \tag{28}$$

The inequality in (28) holds for all $k \geqslant s > t \geqslant 3$ by using (26). □