Commun. math. Phys. 43, 225—254 (1975)

A Generalization of the Classical Moment Problem on *-Algebras with Applications to Relativistic Quantum Theory. I.

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Received January 13, 1975

Abstract. A (non-commutative) generalization of the classical moment problem is formulated on arbitrary *-algebras with units. This is used to produce a C^*-algebra associated with the space of test functions for quantum fields. This C^*-algebra plays a role in theories of bounded localized observables in Hilbert space which is similar to that of the space of test functions in quantum field theories (namely it is represented in Hilbert space). The case of local quantum fields which satisfy a slight generalization of the growth condition is investigated.

1. Introduction and Notations

This paper deals with a sort of non-commutative generalization of measure theory and of the classical moment problem on arbitrary *-algebras. The connexion between the classical moment problem and the hermitian representations (in the sense of Powers [1]) of the algebras of polynomials is well known. The generalization given here has some similar connexions with hermitian representations of *-algebras.

In the usual one dimensional classical moment problem [2, 3], one starts with a sequence of numbers S_n $(n \geqq 0)$ and a closed subset S of $\mathbb{R}$ and one asks the following question: Is there a positive measure μ supported by S such that

$$S_n = \int t^n d\mu(t), \quad \text{for any integer} \quad n \geqq 0?$$

Remembering that there is a bijection $(S_n) \mapsto \phi_{(S_n)}$ from $\mathbb{C}^{\mathbb{N}}$ on the set of all the linear forms on the *-algebra $\mathbb{C}[X]$ of complex polynomials with respect to an indeterminate X,

$$\phi_{(S_n)}(\Sigma a_n X^n) = \Sigma a_n S_n, \quad \forall \Sigma a_n X^n \in \mathbb{C}[X].$$

The classical moment problem may be put in the following form. Let ϕ be a linear form on $\mathbb{C}[X]$ and S be a closed subset of $\mathbb{R}$, is there a positive measure μ on S such that,

$$\phi(P(X)) = \int P(t)\, d\mu(t), \quad \forall P(X) \in \mathbb{C}[X]?$$

In the generalization given in this paper: $\mathbb{C}[X]$ is replaced by an arbitrary *-algebra with unit $\mathfrak{A}$, ϕ is a linear form on $\mathfrak{A}$, μ is replaced by a positive linear

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form on an adapted C^*-algebra (associated with $\mathfrak{A}$) and we give a substitute for the above integration formula. Of course a main part of this paper deals with the construction (and the description) of various auxiliary spaces and algebras needed for the formulation of the problem. This part is possibly of some interest in itself since it describes a sort of non-commutative functional analysis. Roughly speaking, these spaces are generated by an abstract formulation of continuous functions of hermitian elements of $\mathfrak{A}$.

The first step needed to understand the construction is to remark that all the concepts which enter in the formulation of the classical moment problem may be constructed in a purely algebraic way from the algebra of the polynomials $\mathbb{C}[X]$. Indeed, the Stone-Weierstrass theorem implies that the polynomials are dense in the C^*-algebra of all continuous functions on a compact subset of $\mathbb{R}$ (identified with the set of all characters on $\mathbb{C}[X]$). On the other hand, the $\sup(|\,.\,|)$ over compact subsets of $\mathbb{R}$ are exactly all the C^*-semi-norms on the *-algebra $\mathbb{C}[X]$. It follows, for instance, that the *-algebra of all continuous functions on $\mathbb{R}$ equipped with the topology of compact convergence may be identified with the completion of $\mathbb{C}[X]$ for the topology (locally convex) generated by all its C^*-semi-norms. If we are interested in the continuous functions on some closed real subset S, it is sufficient to select the corresponding directed set of C^*-semi-norms. Then, it is not very hard to construct positive continuous functions, polynomially bounded continuous functions, continuous functions vanishing at infinity, positive linear forms on polynomially bounded continuous functions (= rapidly decreasing positive measures), etc. … . It is the direct generalization of this algebraic construction that we describe in this paper. Then, we use it to produce a C^*-algebra associated with the space of test functions of quantum field theory which is, roughly speaking, generated by an "abstract formulation" of continuous functions vanishing at infinity of "general field variables"

In Section 2 we study the completions of *-algebras for locally convex topologies generated by C^*-semi-norms. Besides the fact that we want to generalize the classical moment problem with given supports, there is another reason to consider sets of C^*-semi-norms (instead of all C^*-semi-norms). Namely that, in a forthcoming paper we shall be interested in topological *-algebras, and then it is natural to restrict our attention on continuous C^*-semi-norms.

In Section 3, we introduce the notions of positivity (strong positivity) which are relevant for the problem and we describe an extension theorem for the linear forms which satisfy these strong positivity conditions. Finally, it is pointed out that there are *-algebras where no non-trivial strongly positive linear form exists at all. However, any *-algebra with unit is a quotient of a tensor algebra over an appropriate involutive space and it is proved in Section 4 that any positive linear form on a tensor algebra over an involutive space is strongly positive so one may always "lift" the problem to an appropriate tensor algebra.

In Section 5, we identify the notions introduced in the Sections 2 and 3 in the case when our *-algebra is a polynomials algebra and we discuss some aspects of the classical moment problem. In connexion with this, we prove that any positive linear form on the algebra of all polynomially bounded continuous functions on $\mathbb{R}^n$ is a rapidly decreasing measure. This result is certainly a classical one; we give the proof of it just because we will use it very currently in what follows.

In Section 6, we describe the functional calculus on the completions of *-algebras for locally convex topologies generated by C^*-semi-norms. We define the corresponding associated C^*-algebras (they are generated by the continuous functions vanishing at infinity of the hermitian elements of our *-algebras).

In Section 7, we give the formulation of our generalization of the classical moment problem. It is worth noticing here that this formulation has been suggested to us by the spectal theorem for self-adjoint operators in Hilbert space.

In the end of Section 7, and in Section 8, we begin the study of this moment problem.

In Section 9, we introduce a C^*-algebra associated with the test functions for quantum fields; we call it the quasi-localizable C^*-algebra. For any bounded open subset $\mathcal{O}$ in space-time, we consider the C^*-subalgebra associated with the test functions with supports in $\mathcal{O}$ and we define the localizable algebra as the *-algebra generated by these C^*-algebras (its norm-closure is the quasi-localizable C^*-algebra). We show that there is a group-homomorphism from the group of diffeomorphisms of space-time into the group of automorphisms of the localizable algebra and that the corresponding automorphisms permute the C^*-algebras associated with the bounded open subsets of space-time. The case of a local quantum (neutral scalar) field with a quasi-analytic vacuum is investigated. We show that the result of Borchers and Zimmermann remains true for this slight extension of the growth condition [4]. Moreover, (and this is the important point) in this case we show that the moment problem for the Wightman functional is determined on the localizable algebra and that the corresponding representation of the localizable algebra generates the local rings associated to the field.

Let us say a few words on our notations. In this paper, an involutive vector space is a complex vector space equipped with an anti-linear involution. A *-algebra is a complex associative algebra $\mathfrak{A}$ equipped with an antilinear involution $x \mapsto x^*$ such that $(xy)^* = y^* x^*$ for any $x, y \in \mathfrak{A}$. Such an algebra is, in a natural way, a preordered vector space [5] with positive cone $\mathfrak{A}^+ =$ convex hull of the set $\{x^* x \mid x \in \mathfrak{A}\}$. Therefore, a positive linear form ϕ on $\mathfrak{A}$ is defined to be a linear form on $\mathfrak{A}$ such that $\phi(x^* x) \geqq 0$ for any element x of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a *-algebra with a unit, $\mathbb{1}$, and let ϕ be a positive linear form on $\mathfrak{A}$. The set $\mathfrak{J}_\phi = \{x \mid x \in \mathfrak{A}$ and $\phi(x^* x) = 0\}$ is a left ideal in $\mathfrak{A}$. The positive sesquilinear form $(x, y) \mapsto \phi(x^* y)$[1] pass to the quotient from $\mathfrak{A}$ to $\mathfrak{A}/\mathfrak{J}_\phi$ and we denote the corresponding Hausdorff prehilbertian space by D_ϕ. Let $\mathfrak{H}_\phi$ be the Hilbert space obtained by completion of D_ϕ and let Ψ_ϕ be the canonical mapping of $\mathfrak{A}$ in $\mathfrak{H}_\phi$ (on $D_\phi \subset \mathfrak{H}_\phi$). Let π_ϕ be the cyclic *-representation [1] of $\mathfrak{A}$ in $\mathfrak{H}_\phi$ with dense domain D_ϕ and cyclic vector $\Omega_\phi = \Psi_\phi(\mathbb{1})$ defined by: $\pi_\phi(x)\,\Psi_\phi(y) = \Psi_\phi(xy)$.

Then $(\pi_\phi, \mathfrak{H}_\phi, D_\phi, \Omega_\phi)$ is unique up to unitary equivalence under the conditions:

$$(\Omega_\phi \mid \pi_\phi(x)\Omega_\phi) = \phi(x) \ \ \forall x \in \mathfrak{A}, \qquad D_\phi = \pi_\phi(\mathfrak{A})\Omega_\phi = \mathrm{dom}(\pi_\phi)\,.$$

We refer to this construction as Gelfand-Naimark-Segal construction (G.N.S. construction), [6].

We use the Hahn-Banach theorem in the following form (see Dunford and Schwartz, p. 62, [7]).

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[1] We use the physicist convention for sesquilinear form (linearity in the right variable!).

Let the realfunction p on the real vector space E satisfy

$$p(x+y) \leqq p(x)+p(y), \quad \forall x, y \in E$$
$$p(\varrho x)=\varrho p(x), \quad \forall x \in E \quad \text{and} \quad \forall \varrho \geqq 0 .$$

Let ϕ be a (real) linear form on a subspace M of E with $\phi(x) \leqq p(x)$, $\forall x \in M$. Then there is a (real) linear form $\tilde{\phi}$ on E for which $\tilde{\phi}(x)=\phi(x)$, $\forall x \in M$; $\tilde{\phi}(x) \leqq p(x)$, $\forall x \in E$.

Let $\mathfrak{A}$ be a *-algebra with a unit. A C^*-semi-norm on $\mathfrak{A}$ is a semi-norm p for which $p(xy) \leqq p(x)\, p(y)$, $\forall x, y \in \mathfrak{A}$; $p(x^* x)=p(x)^2$, $\forall x \in \mathfrak{A}$; $p(\mathbb{1})=1$.

2. Topologies Generated by C^*-Semi-Norms on *-Algebras

Let $\mathfrak{A}$ be a *-algebra with a unit, $\mathbb{1} \in \mathfrak{A}$, and let Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$ such that the locally convex topology on $\mathfrak{A}$ generated by Γ, $\mathscr{T}_\Gamma$, is a Hausdorff topology (in other words: $p(x)=0$, $\forall p \in \Gamma \Rightarrow x=0$).

The involution of $\mathfrak{A}$, $x \mapsto x^*$, is $\mathscr{T}_\Gamma$-continuous and the product of $\mathfrak{A}$ is jointly $\mathscr{T}_\Gamma$-continuous [continuous from $(\mathfrak{A}, \mathscr{T}_\Gamma) \times (\mathfrak{A}, \mathscr{T}_\Gamma)$ into $(\mathfrak{A}, \mathscr{T}_\Gamma)$]. It follows that the completion of $\mathfrak{A}$ for $\mathscr{T}_\Gamma$ is canonically a locally convex *-algebra with continuous involution and jointly continuous product; this complete topological *-algebra will be denoted by $\mathscr{A}(\mathfrak{A}, \Gamma)$ or simply by $\mathscr{A}$ when no confusion arises.

Let p_0 be a C^*-semi-norm on $\mathfrak{A}$; p_0 induces a norm on $\mathfrak{A}/p_0^{-1}(0)$ and the Banach space obtained by completion is canonically a C^*-algebra which will be denoted by $\mathfrak{B}_{p_0}$. If p_0 is $\mathscr{T}_\Gamma$-continuous then there is a unique continuous *-homomorphism π_{p_0} from $\mathscr{A}$ into $\mathfrak{B}_{p_0}$ which is an extension of the canonical projection from $\mathfrak{A}$ on $\mathfrak{A}/p_0^{-1}(0)$ (π_{p_0} is in fact surjective as we shall see below). Since Γ is directed, for any $\mathscr{T}_\Gamma$-continuous C^*-semi-norm p_0, there is a positive constant K and a C^*-semi-norm $p \in \Gamma$ such that: $p_0(x) \leqq K p(x)$, $\forall x \in \mathfrak{A}$. The canonical mapping, $\pi_{p_0, p}: \mathfrak{B}_p \to \mathfrak{B}_{p_0}$, is a *-homomorphism with dense image. It follows (since $\mathfrak{B}_p$ and $\mathfrak{B}_{p_0}$ are C^*-algebras) that $\pi_{p_0 p}$ is norm-decreasing and surjective: $\|\pi_{p_0 p}(x)\| \leqq \|x\|$ and $\pi_{p_0 p}(\mathfrak{B}_p)=\mathfrak{B}_{p_0}$. This implies that we have: $p_0(x) \leqq p(x)$, $\forall x \in \mathfrak{A}$.

The mapping $x \mapsto (\pi_p(x))_{p \in \Gamma}$ from $\mathscr{A}$ into the locally convex *-algebra $\prod_{p \in \Gamma} \mathfrak{B}_p$ is a topological *-isomorphism from $\mathscr{A}$ on its image (here and in the following lemma $\mathfrak{B}_p$ denotes the underlying normable space of $\mathfrak{B}_p$). We may identify $\mathscr{A}$ and its image under this mapping; with this convention we have:

Lemma 1. *$\mathscr{A}$ is the projective limit of the family $(\mathfrak{B}_p)_{p \in \Gamma}$ with respect to the mappings $\pi_{p,p'}$ ($p \in \Gamma$, $p' \in \Gamma$ and $p \leqq p'$). In other words, $\mathscr{A}$ is the closed subspace of the topological product $\prod_{p \in \Gamma} \mathfrak{B}_p$ defined by:*

$$\varprojlim_{\Gamma}(\mathfrak{B}_p)=\{(x_p)_{p \in \Gamma} \mid x_p \in \mathfrak{B}_p,\ x_p=\pi_{p,p'}(x_{p'})\ p \leqq p' \text{ in } \Gamma\} .$$

Proof. The proof of this lemma is standard [5].

Let us first remark that we have:

$$(1) \qquad \begin{cases} \pi_{p_1}=\pi_{p_1,p_2} \circ \pi_{p_2}, & \forall p_1, p_2 \in \Gamma \quad \text{with} \quad p_1 \leqq p_2 \\ \pi_{p_1,p_3}=\pi_{p_1,p_2} \circ \pi_{p_2,p_3}, & \forall p_1, p_2, p_3 \in \Gamma \quad \text{with} \quad p_1 \leqq p_2 \leqq p_3 . \end{cases}$$

It follows that we have: $\mathscr{A} \subset \varprojlim_{\Gamma}(\mathfrak{B}_p)$.

Conversely, let $(x_p)_{p\in\Gamma}$ be an element of $\varprojlim_{\Gamma}(\mathfrak{B}_p)$. Consider the family $(S_{p,\varepsilon})_{p\in\Gamma,\varepsilon>0}$ of subset in $\mathscr{A}$ defined by:

$$S_{p,\varepsilon}=\pi_p^{-1}(\{y\,|\,y\in\mathfrak{B}_p \quad\text{and}\quad \|y-x_p\|\leqq\varepsilon\}).$$

These subsets are not empty since $\pi_p(\mathscr{A})$ is dense in $\mathfrak{B}_p$, $\forall p\in\Gamma$. We have

$$S_{p_1,\varepsilon_1}\cap S_{p_2,\varepsilon_2}\supset S_{p,\varepsilon},\quad \forall p\geqq p_1,p_2,\quad \forall\varepsilon\leqq\varepsilon_1,\varepsilon_2.$$

Furthermore, we have: $\forall p\in\Gamma$, $\forall\varepsilon>0$, $S_{p,\varepsilon/2}-S_{p,\varepsilon/2}\subset V_{p,\varepsilon}=\{x\,|\,x\in\mathscr{A}$ and $p(x)\leqq\varepsilon\}$. It follows that $(S_{p,\varepsilon})$ is a Cauchy filter base in $\mathscr{A}$ which converges to an $x\in\mathscr{A}$ since $\mathscr{A}$ is complete. By construction we have $\pi_p(x)=x_p$, $\forall p\in\Gamma$, so $x=(x_p)_{p\in\Gamma}\in\mathscr{A}$. □

Remark 1. It follows from above that the set $\tilde{\Gamma}$ of all $\mathscr{T}_\Gamma$-continuous C^*-seminorms on $\mathfrak{A}$ is directed and that in all the discussion we may replace Γ by $\tilde{\Gamma}$ or by any cofinal subset of $\tilde{\Gamma}$. In particular we have: $\mathscr{A}(\mathfrak{A},\Gamma)=\mathscr{A}(\mathfrak{A},\Gamma')$ for any cofinal subset Γ' in $\tilde{\Gamma}$.

Lemma 2. *Let x be an element of $\mathscr{A}$; then x has an inverse in $\mathscr{A}$ if and only if, for any $p\in\Gamma$, $\pi_p(x)$ has an inverse in $\mathfrak{B}_p$.*

Proof. Suppose that $x\in\mathscr{A}$ has an inverse $x^{-1}\in\mathscr{A}$; then $\pi_p(x^{-1})$ is clearly an inverse for $\pi_p(x)\in\mathfrak{B}_p$.

Conversely, suppose that $x\in\mathscr{A}$ is such that $\pi_p(x)$ has an inverse $\pi_p(x)^{-1}\in\mathfrak{B}_p$, $\forall p\in\Gamma$; then $(\pi_p(x)^{-1})_{p\in\Gamma}$ is an inverse of x in $\prod_{p\in\Gamma}\mathfrak{B}_p$. On the other hand, if $p_1,p_2\in\Gamma$ satisfy $p_1\leqq p_2$, we have: [by (1) and since π_{p_1,p_2} is a homomorphism]

$$\pi_{p_1,p_2}(\pi_{p_2}(x)^{-1})\cdot\pi_{p_1}(x)=\pi_{p_1,p_2}(\mathbb{1})=\mathbb{1},$$
$$\pi_{p_1}(x)\cdot\pi_{p_1,p_2}(\pi_{p_2}(x)^{-1})=\pi_{p_1,p_2}(\mathbb{1})=\mathbb{1},$$

and therefore: $\pi_{p_1,p_2}(\pi_{p_2}(x)^{-1})=\pi_{p_1}(x)^{-1}$. Applying Lemma 1, it follows that $x^{-1}=(\pi_p(x)^{-1})_{p\in\Gamma}\in\mathscr{A}$. □

It follows from this lemma that if $x\in\mathscr{A}$, the spectrum of x in $\mathscr{A}$, $\mathrm{Sp}(x)$, is the union of the spectrums of $\pi_p(x)$ in $\mathfrak{B}_p$, $\mathrm{Sp}(\pi_p(x))$, when p runs over Γ

$$\mathrm{Sp}(x)=\bigcup_{p\in\Gamma}\mathrm{Sp}(\pi_p(x)),\qquad \forall x\in\mathscr{A}. \tag{2}$$

In particular, if $h\in\mathscr{A}$ is hermitian ($h^*=h$), its spectrum is real: $\mathrm{Sp}(h)\subset\mathbb{R}$, $\forall h\in\mathscr{A}^h$.

Lemma 3. *Let h be an element of $\mathscr{A}$. The following conditions are equivalent:*

a) $h=x^2$ with $x=x^*\in\mathscr{A}$,

b) $h=x^*x$ with $x\in\mathscr{A}$,

c) $\mathrm{Sp}(h)\subset\mathbb{R}^+=\{\varrho\,|\,\varrho\in\mathbb{R}$ and $\varrho\geqq 0\}$,

d) $\pi_p(h)\in\mathfrak{B}_p^+$, $\forall p\in\Gamma$, where $\mathfrak{B}_p^+$ denotes the set of all the positive elements in the C^*-algebra $\mathfrak{B}_p$, [8].

Moreover, the set $\mathscr{A}^+$ of all the $h\in\mathscr{A}$ satisfying these conditions is a closed convex cone in $\mathscr{A}$ and we have: $\mathscr{A}^+\cap(-\mathscr{A}^+)=\{0\}$.

Proof. a)⇒b) is obvious; b)⇒d) since π_p is a *-homomorphism; c)⇒d) follows from (2).

Suppose that d) is satisfied. Then, $\forall p \in \Gamma$, there is a unique $\sqrt{\pi_p(h)} \in \mathfrak{B}_p^+$ such that $(\sqrt{\pi_p(h)})^2 = \pi_p(h)$, [9]. If we have $p_1, p_2 \in \Gamma$ with $p_1 \leqq p_2$, then we have: $\pi_{p_1,p_2}(\sqrt{\pi_{p_2}(h)}) \in \mathfrak{B}_{p_1}^+$ and $(\pi_{p_1,p_2}(\sqrt{\pi_{p_2}(h)}))^2 = \pi_{p_1,p_2}(\pi_{p_2}(h)) = \pi_{p_1}(h)$, so we have: $\pi_{p_1,p_2}(\sqrt{\pi_{p_2}(h)}) = \sqrt{\pi_{p_1}(h)}$, $\forall p_1 \leqq p_2 \in \Gamma$. This implies, by Lemma 1, that

$$\sqrt{h} = (\sqrt{\pi_p(h)})_{p \in \Gamma}$$

is an element of $\mathscr{A}$ and we have a) with $x = \sqrt{h}$.

$\mathfrak{B}_p^+$ is a closed convex cone in $\mathfrak{B}_p$ with $\mathfrak{B}_p^+ \cap (-\mathfrak{B}_p^+) = \{0\}$, $\forall p \in \Gamma$. It follows that intersection of $\prod_{p \in \Gamma} \mathfrak{B}_p^+$ with the closed subspace $\mathscr{A} \subset \prod_{p \in \Gamma} \mathfrak{B}_p$ is a closed convex proper cone in $\mathscr{A}$; by d), this cone is just $\mathscr{A}^+$. This completes the proof of Lemma 3. □

Notice that we have $\mathscr{A}^h = \mathscr{A}^+ - \mathscr{A}^+$ as for any *-algebra with a unit $(h = (\mathbb{1} + \frac{1}{4}h)^2 - (\mathbb{1} - \frac{1}{4}h)^2)$. Remembering that if $\mathfrak{B}$ is a C^*-algebra with positive cone $\mathfrak{B}^+$, for any hermitian $h \in \mathfrak{B}$, there is a unique decomposition $h = h^+ - h^-$ with $h^\pm \in \mathfrak{B}^+$ and $h^+ \cdot h^- = 0$. If furthermore $h \in \mathfrak{B}^+$ $(h = h^+)$, then for an arbitrary positive integer n, there is a unique $\sqrt[n]{h} \in \mathfrak{B}^+$ such that $(\sqrt[n]{h})^n = h$; $\sqrt[n]{h}$ is also denoted by $h^{1/n}$, [9]. Using this result, Lemma 1 and the fact that π_{p_1,p_2} are *-homomorphism, it is not difficult to prove the following lemma.

Lemma 4. a) *Let h be an arbitrary hermitian element of $\mathscr{A}$. Then there is a unique decomposition $h = h^+ - h^-$, with $h^+ \in \mathscr{A}^+$, $h^- \in \mathscr{A}^+$ and $h^+ \cdot h^- = 0$.*

b) *Let h be an element of $\mathscr{A}^+$ and let n be a positive integer. Then there is a unique element of $\mathscr{A}^+$, $\sqrt[n]{h}$, such that $(\sqrt[n]{h})^n = h$.*

$\sqrt[n]{h}$ will be also denoted by $h^{1/n}$ (for $h \in \mathscr{A}^+$). The proof is left to the reader [proceed as in the proof of Lemma 2, or as in the proof d)⇒a) in Lemma 3].

These results will be partially generalized in Section 6 (functional calculus).

Lemma 5. *Let $\mathfrak{B}_\infty$ (or $\mathfrak{B}_\infty(\mathfrak{A}, \Gamma)$) be the set of all $x \in \mathscr{A}$ such that $p \mapsto p(x) = \|\pi_p(x)\|$ is bounded on Γ. Then $\mathfrak{B}_\infty$ is a *-subalgebra of $\mathscr{A}$ with $\mathbb{1} \in \mathfrak{B}_\infty$ and it is a C^*-algebra for the norm $x \mapsto \|x\| = \sup_{p \in \Gamma}(\|\pi_p(x)\|)$.*

Let x be an arbitrary element of $\mathscr{A}$. Then, for any strictly positive number ε, $\mathbb{1} + \varepsilon x^* x$ has an inverse in $\mathscr{A}$ and $x_\varepsilon = x \cdot (\mathbb{1} + \varepsilon x^* x)^{-1}$ is in $\mathfrak{B}_\infty$. Furthermore we have in $\mathscr{A}$:

$$\lim_{\varepsilon \to 0}(x_\varepsilon) = x .$$

Proof. $\mathfrak{B}_\infty$ is the intersection of $\mathscr{A}$ with the C^*-product of the family $(\mathfrak{B}_p)_{p \in \Gamma}$ of C^*-algebras in $\prod_{p \in \Gamma} \mathfrak{B}_p$.

Let x be an element of $\mathscr{A}$. $\mathbb{1} + \varepsilon x^* x$ has an inverse in $\mathscr{A}$, by Lemma 2. We have $\|\pi_p(x_\varepsilon)\|^2 = \|\pi_p(x_\varepsilon^* x_\varepsilon)\| = \|\pi_p(x^* x \cdot [\mathbb{1} + \varepsilon x^* x]^{-2})\|$, $\forall p \in \Gamma$. So we have: $\|\pi_p(x_\varepsilon)\|^2 \leqq \sup_{t \geqq 0} \left\{\frac{t}{(1+\varepsilon t)^2}\right\} = \frac{1}{4\varepsilon}$. It follows that $x_\varepsilon \in \mathfrak{B}_\infty$, $\left(\|x_\varepsilon\| \leqq \frac{1}{2\sqrt{\varepsilon}}\right)$. On the other hand we have: $\forall p \in \Gamma$,

$$\|\pi_p(x) - \pi_p(x_\varepsilon)\| = \|\pi_p(\varepsilon \cdot x x^* x \cdot [\mathbb{1} + \varepsilon x^* x]^{-1})\| \leqq \varepsilon \|\pi_p(x)\|^2 \cdot \|\pi_p(x_\varepsilon)\| .$$

It follows that $\lim_{\varepsilon \to 0}[p(x - x_\varepsilon)] = 0$, $\forall p \in \Gamma$; this means $\lim_{\varepsilon \to 0} x_\varepsilon = x$ in $\mathscr{A}$. □

It follows that $\mathfrak{B}_\infty$ is dense in $\mathscr{A}$ and therefore, for any $p \in \Gamma$, $\pi_p(\mathfrak{B}_\infty)$ is dense in $\mathfrak{B}_p$. This implies $\pi_p(\mathfrak{B}_\infty) = \mathfrak{B}_p$ (since π_p is a *-homomorphism and $\mathfrak{B}_\infty$ and $\mathfrak{B}_p$ are C^*-algebras) for any $p \in \Gamma$. So π_p is surjective.

3. Strong Positivity for Linear Forms

Let $\mathfrak{A}$ and Γ be as in Section 2, and let us use the same notations ($\mathscr{A}$, $\mathscr{T}_\Gamma$ etc. ...). Let $\mathfrak{A}^h$, (resp. $\mathscr{A}^h$), be the set of all the hermitian elements of $\mathfrak{A}$, (resp. $\mathscr{A}$). $\mathfrak{A}^h$, (resp. $\mathscr{A}^h$), is a real linear subspace in $\mathfrak{A}$, (resp. $\mathscr{A}$), and $\mathfrak{A}^h = \mathfrak{A}^+ - \mathfrak{A}^+$, where $\mathfrak{A}^+$ is the convex cone generated by $\{x^* x \mid x \in \mathfrak{A}\}$.

Let us define $\mathfrak{M}^h(\mathfrak{A}, \Gamma) \subset \mathscr{A}^h$ to be

(3) $$\mathfrak{M}^h(\mathfrak{A}, \Gamma) = \{x \mid x \in \mathscr{A}^h \quad \text{and} \quad \exists x_1, x_2 \in \mathfrak{A} \quad \text{with} \quad x_1 - x, x - x_2 \in \mathscr{A}^+\}$$

this is a (real) linear subspace of $\mathscr{A}^h$ containing $\mathfrak{A}^h$ which will simply be denoted by $\mathfrak{M}^h$ (when no confusion arises). The complex subspace of $\mathscr{A}$ generated by $\mathfrak{M}^h$ will be denoted by $\mathfrak{M}(\mathfrak{A}, \Gamma)$, or simply by $\mathfrak{M}$; $\mathfrak{M} = \mathfrak{M}^h + i\mathfrak{M}^h \subset \mathscr{A}$. We have:

$$\mathfrak{A} \subset \mathfrak{M} \subset \mathscr{A}, \qquad \mathfrak{A}^h \subset \mathfrak{M}^h \subset \mathscr{A}^h .$$

Finally let $\mathfrak{M}^+$ be the cone $\mathfrak{M} \cap \mathscr{A}^+$. $\mathfrak{M}$ is an ordered vector space with positive cone $\mathfrak{M}^+$, and,

(4) $$\mathfrak{M}^+ = \{x \mid x \in \mathscr{A}^+ \quad \text{and} \quad \exists y \in \mathfrak{A} \quad \text{with} \quad y - x \in \mathscr{A}^+\} .$$

Lemma 6. *We have*

a) $x^* \cdot \mathfrak{M}^+ \cdot x \subset \mathfrak{M}^+$, $\forall x \in \mathfrak{A}$,

b) $x^* \cdot \mathfrak{M} \cdot y \subset \mathfrak{M}$, $\forall x, y \in \mathfrak{A}$,

c) $\mathfrak{M}^h = \mathfrak{M}^+ - \mathfrak{M}^+$

d) $\mathfrak{B}_\infty \subset \mathfrak{M}$.

Proof. $y - x_0 \in \mathscr{A}^+$, $x_0 \in \mathscr{A}^+$ and $y \in \mathfrak{A} \Rightarrow x^*(y - x_0)\, x \in \mathscr{A}^+$, $x^* x_0 x \in \mathscr{A}^+$ and $x^* y x \in \mathfrak{A}$, $\forall x \in \mathfrak{A}$. This proves a).

We have: $\mathfrak{M}^+ - \mathfrak{M}^+ \subset \mathfrak{M}^h$. Conversely, let x be an arbitrary element of $\mathfrak{M}^h$; then there are two elements of $\mathfrak{A}$, x_1, and x_2, such that $x_1 - x \in \mathscr{A}^+$ and $x - x_2 \in \mathscr{A}^+$. x_1 and x_2 are hermitian elements of $\mathfrak{A}$, so $[\mathbb{1} \pm \frac{1}{8}(x_1 + x_2)]^2$ are in $\mathfrak{A}^+ \subset \mathfrak{M}^+$. We have: $x = y_1 - y_2$, where, $y_1 = \frac{1}{2}(x - x_2) + [\mathbb{1} + \frac{1}{8}(x_1 + x_2)]^2 \in \mathfrak{M}^+$, and, $y_2 = \frac{1}{2}(x_1 - x) + [\mathbb{1} - \frac{1}{8}(x_1 + x_2)]^2 \in \mathfrak{M}^+$. This proves c).

b) is a consequence of a), c), and of the following polarization identity:

(5) $$2x^* z y = (x + y)^* z (x + y) - i(x + iy)^* z (x + iy) + (i - 1)\,[x^* z x + y^* z y] ,$$

for any x, y and z in $\mathscr{A}$.

Finally d) follows from the fact that we have: $\varrho\mathbb{1} - x \in \mathfrak{B}_\infty^+$, $\forall x = x^* \in \mathfrak{B}_\infty$ and $\forall \varrho > \|x\|$. □

Let us remark that the closure for the topology $\mathscr{T}_\Gamma$ of the positive cone $\mathfrak{A}^+$ of $\mathfrak{A}$ is equal to $\mathfrak{A} \cap \mathscr{A}^+$ and is also the closure of $\{x^* x \mid x \in \mathfrak{A}\}$.

Definition 1-A. Let ϕ be a linear form on $\mathfrak{A}$. Then, ϕ will be called a *Γ-strongly positive* linear form on $\mathfrak{A}$ if it is positive on the closure $\overline{\mathfrak{A}^+}^{\mathscr{T}_\Gamma}$ of $\mathfrak{A}^+$ for the topology $\mathscr{T}_\Gamma$ (in $\mathfrak{A}$).

Then, we have the following theorem.

Theorem 1. *Let* ϕ *be a linear form on* $\mathfrak{A}$. *Then the following conditions are equivalent:*

a) ϕ *is* Γ*-strongly positive* $(\phi(\mathfrak{A}\cap\mathscr{A}^+)\subset\mathbb{R}^+)$,

b) *There is a linear form* $\tilde{\phi}$ *on* $\mathfrak{M}$ *which is positive on* $\mathfrak{M}^+$ *and such that we have:* $\tilde{\phi}(x)=\phi(x)$, $\forall x\in\mathfrak{A}$ *(*$\tilde{\phi}$ *is an extension of* ϕ*).*

Proof. b)$\Rightarrow$a) since we have: $\overline{\mathfrak{A}^+}^{\mathscr{T}_\Gamma}=\mathfrak{A}\cap\mathscr{A}^+=\mathfrak{A}\cap\mathfrak{M}^+$. Conversely suppose that ϕ is positive on $\mathfrak{A}\cap\mathfrak{M}^+$ and define the following real function on the real vector space $\mathfrak{M}^h$:

$$x\mapsto p(x)=\inf_{\substack{y\in\mathfrak{A}\\ y-x\in\mathscr{A}^+}}\{\phi(y)\}\qquad(x\in\mathfrak{M}^h)\,. \tag{6}$$

This function satisfies:

$$p(x_1+x_2)\leqq p(x_1)+p(x_2),\qquad \forall x_1,x_2\in\mathfrak{M}^h,$$
$$p(\varrho x)=\varrho p(x),\qquad \forall x\in\mathfrak{M}^h\quad\text{and}\quad\forall\varrho\in\mathbb{R}^+\,.$$

Furthermore we have: $\phi(x)=p(x)$, $\forall x\in\mathfrak{A}^h$. The Hahn-Banach theorem implies that there is a real linear form ϕ_1 on the real vector space $\mathfrak{M}^h$ satisfying $\phi_1(x)=\phi(x)$, $\forall x\in\mathfrak{A}^h$, and $\phi_1(x)\leqq p(x)$, $\forall x\in\mathfrak{M}^h$. The latter inequality implies that ϕ_1 is positive on $\mathfrak{M}^+$. The the following equality defines a positive linear form $\tilde{\phi}$ on $\mathfrak{M}$ which extends ϕ: $\tilde{\phi}(x)=\phi_1\left(\frac{x+x^*}{2}\right)+i\phi_1\left(\frac{x-x^*}{2i}\right)$, $\forall x\in\mathfrak{M}$. □

Any linear form on $\mathscr{A}$ defines by restriction a linear form on $\mathfrak{M}$ and a linear form on $\mathfrak{A}$. Moreover these restrictions are injective on the topological dual $\mathscr{A}'$ of $\mathscr{A}$ since $\mathfrak{A}$ is dense in $\mathscr{A}$. It follows that $\mathscr{A}'$ may be identified with a linear subspace of the algebraic dual $\mathfrak{M}^*$ of $\mathfrak{M}$ or with a linear subspace of the algebraic dual $\mathfrak{A}^*$ of $\mathfrak{A}$.

Proposition 1. *Let* $\mathfrak{M}^*_\sigma$ *(resp.* $\mathfrak{A}^*_\sigma$*), denote the algebraic dual space of* $\mathfrak{M}$ *(resp.* $\mathfrak{A}$*), equipped with the weak topology* $\sigma(\mathfrak{M}^*,\mathfrak{M})$ *(resp.* $\sigma(\mathfrak{A}^*,\mathfrak{A})$*).*

a) *The set of all the linear forms on* $\mathfrak{M}$ *which are positive on* $\mathfrak{M}^+$ *is the closure in* $\mathfrak{M}^*_\sigma$ *of the convex cone of all continuous positive linear forms on* $\mathscr{A}$ *(restricted to* $\mathfrak{M}$*).*

b) *Let* K *be a set of* Γ*-strongly positive linear forms on* $\mathfrak{A}$ *which is closed and bounded in* $\mathfrak{A}^*_\sigma$ *(*$\Leftrightarrow$*compact). Then the set* $\tilde{K}$ *of all the positive linear forms on* $\mathfrak{M}$ *which have restrictions to* $\mathfrak{A}$ *in* K *is a compact subset of* $\mathfrak{M}^*_\sigma$. *If* K *is convex, then* $\tilde{K}$ *is also convex.*

Proof. a) $\mathfrak{M}^+=\mathfrak{M}\cap\mathscr{A}^+$ is the set $\{x\,|\,x\in\mathfrak{M}$ and $\phi(x)\geqq0$, $\forall\phi\in\mathscr{A}'$ with ϕ positive$\}$. Therefore, the set of all positive linear forms on $\mathfrak{M}$ is the bipolar of the set of the restrictions to $\mathfrak{M}$ of positive continuous linear forms on $\mathscr{A}$. This last set is a convex cone in $\mathfrak{M}^*$. This implies a).

b) The restriction to $\mathfrak{A}$ is a continuous linear mapping of $\mathfrak{M}^*_\sigma$ into $\mathfrak{A}^*_\sigma$, so $\tilde{K}$ is closed. $\tilde{K}$ is obviously convex if K is convex. In the algebraic dual E^* of a vector space E, any weakly closed bounded subset is compact (E^*_σ is a closed subspace of $\mathbb{C}^E$), so it remains to prove that $\tilde{K}$ is bounded in $\mathfrak{M}^*_\sigma$. By positivity, linearity and by Lemma 6c), it is sufficient to show that for any element x of $\mathfrak{M}^+$ the set $\{\psi(x) \mid \psi \in \tilde{K}\}$ of positive numbers is bounded. But for any $x \in \mathfrak{M}^+$, there is an element x_1 of $\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ with $x_1 - x \in \mathfrak{M}^+$, and, by the hypothesis K bounded in $\mathfrak{A}_\sigma^*$, $\{\psi(x_1)\}$ is bounded and therefore $\sup_{\psi \in \tilde{K}} \{\psi(x)\} \leqq \sup_{\psi \in \tilde{K}} \{\psi(x_1)\} = \sup_{\phi \in K} \{\phi(x_1)\} < \infty$. □

Remark 2. The convex cone of all $\mathcal{T}_\Gamma$-continuous positive linear forms on $\mathfrak{A}$ is identical with the set of restrictions to $\mathfrak{A}$ of all continuous positive linear forms on $\mathcal{A}$; its closure in $\mathfrak{A}_\sigma^*$ is the convex cone of all Γ-strongly positive linear forms on $\mathfrak{A}$. Notice also that any algebraic dual E^* is weakly complete, so the convex cone of all Γ-strongly positive linear forms on $\mathfrak{A}$ is weakly complete. If ϕ is Γ-strongly positive, the set of all positive linear forms $\tilde{\phi}$ on $\mathfrak{M}$ with $\phi = \tilde{\phi} \upharpoonright \mathfrak{A}$ is weakly compact.

In what follows we shall need the generalization of the theory for arbitrary *-algebra with unit, $\mathfrak{A}$, and arbitrary directed set of C^*-semi-norms on $\mathfrak{A}$, Γ. In other words, $\mathcal{T}_\Gamma$ will not be supposed to be a Hausdorff topology on $\mathfrak{A}$. Let us say a few words on this general situation. Notice first that the closure of $\{0\}$ for $\mathcal{T}_\Gamma$ is the *-invariant two-sided ideal $\mathfrak{I}_\Gamma = \bigcap_{p \in \Gamma} p^{-1}(0)$. It follows that $\mathfrak{A}/\mathfrak{I}_\Gamma$ is again a *-algebra with a unit. Any C^*-semi-norm in Γ (and any $\mathcal{T}_\Gamma$-continuous C^*-semi-norm on $\mathfrak{A}$) induces a C^*-semi-norm on $\mathfrak{A}/\mathfrak{I}_\Gamma$. The set of these C^*-semi-norms on $\mathfrak{A}/\mathfrak{I}_\Gamma$ will again be denoted by Γ. This set of C^*-semi-norms on $\mathfrak{A}/\mathfrak{I}_\Gamma$ generates the quotient topology of $\mathcal{T}_\Gamma$ which will again be denoted by $\mathcal{T}_\Gamma$ and is now a Hausdorff topology on $\mathfrak{A}/\mathfrak{I}_\Gamma$. With the notations given above we define the spaces $\mathcal{A}(\mathfrak{A}, \Gamma)$, $\mathfrak{M}(\mathfrak{A}, \Gamma)$, $\mathfrak{B}_\infty(\mathfrak{A}, \Gamma)$, $\mathcal{A}^+(\mathfrak{A}, \Gamma)$ etc. ... by:

$$\mathcal{A}(\mathfrak{A}, \Gamma) = \mathcal{A}(\mathfrak{A}/\mathfrak{I}_\Gamma, \Gamma), \qquad \mathfrak{M}(\mathfrak{A}, \Gamma) = \mathfrak{M}(\mathfrak{A}/\mathfrak{I}_\Gamma, \Gamma), \qquad \text{etc.} \ldots$$

and, we complete Definition 1-A by the following.

Definition 1-B. Let ϕ be a linear form on $\mathfrak{A}$. Then we say that ϕ is a *Γ-strongly positive* linear form on $\mathfrak{A}$ if ϕ is positive on the closure $\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ of $\mathfrak{A}^+$ for the topology $\mathcal{T}_\Gamma$. If furthermore Γ is the set of all C^*-semi-norms on $\mathfrak{A}$, ϕ will simply be called a *strongly positive* linear form on $\mathfrak{A}$.

The following lemma is an obvious consequence of this definition.

Lemma 7. *Let Γ' be a directed set of C^*-semi-norms on $\mathfrak{A}$ such that $\mathcal{T}_{\Gamma'}$ is finer than $\mathcal{T}_\Gamma$. Then any Γ-strongly positive linear form on $\mathfrak{A}$ is Γ'-strongly positive (and so strongly positive). Furthermore any strongly positive linear form on $\mathfrak{A}$ is a positive linear form on $\mathfrak{A}$.*

Remark 3. There are "many" *-algebras on which no non trivial C^*-semi-norm exists: so it may happen that the trivial linear form, $x \mapsto 0$, is the only strongly positive linear form. However, we are going to show, in the next section, that every positive linear form on the tensor algebra $T(E)$ over an involutive vector space E is strongly positive. It is worth noticing that, if E is an involutive vector space with $\dim(E) \geqq 2$, there are positive linear forms on the symmetric algebra $S(E)$ over E which are not strongly positive [in spite of the fact that the C^*-semi-norms on $S(E)$ generate a Hausdorff locally convex topology on $S(E)$]. This latter point is connected with the non solubility of Hamburger's moment problem for positive linear forms on $\mathbb{C}[X_1, X_2]$ ($\approx S(\mathbb{C}^2)$), [3, 10].

Let $\pi_\Gamma : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}_\Gamma$ be the canonical projection of $\mathfrak{A}$ on $\mathfrak{A}/\mathfrak{J}_\Gamma$; π_Γ is a *-homomorphism with $\pi_\Gamma(\mathbb{1}) = \mathbb{1}$. For any Γ-strongly positive linear form ϕ on $\mathfrak{A}/\mathfrak{J}_\Gamma$, $\phi \circ \pi_\Gamma$ is a Γ-strongly positive linear form on $\mathfrak{A}$ and, conversely, any Γ-strongly positive linear form on $\mathfrak{A}$ is of this form for a unique Γ-strongly positive linear form ϕ on $\mathfrak{A}/\mathfrak{J}_\Gamma$. Therefore all the results given above are easily translated in results for the general situation (where $\mathcal{T}_\Gamma$ is not Hausdorff). So, as long as Γ is kept fixed, it is not a restriction to assume that $\mathcal{T}_\Gamma$ is Hausdorff (replace $\mathfrak{A}$ by $\mathfrak{A}/\mathfrak{J}_\Gamma$); We say in this case that Γ separates $\mathfrak{A}$.

4. The Case of Tensor Algebras over Involutive Vector Spaces

Let E be an involutive vector space. There is a unique involution, $x \mapsto x^*$, on the tensor algebra $T(E)$ over E such that it extends the involution of E, it is antilinear and satisfies:

$$(xy)^* = y^* x^*, \qquad \forall x, y \in T(E).$$

Equipped with this involution, $T(E)$ is a *-algebra with a unit. Furthermore any *-algebra with unit is (in a non unique way), a factor algebra of such a tensor algebra by a two-sided *-invariant ideal. Therefore it is important to know what happens when $\mathfrak{A}$ is the tensor algebra over an involutive vector space E.

Let us define, as usual, an involution, $f \mapsto f^*$, in the dual E^* of E by:

$$\langle f^*, x\rangle = \overline{\langle f, x^*\rangle}, \qquad \forall x \in E.$$

Let us recall that if α_1 is a linear mapping from E into a *-algebra with unit $\mathfrak{A}$, such that $\alpha_1(x^*) = \alpha_1(x)^*$, $\forall x \in E$; then there is a unique *-homomorphism α from $T(E)$ into $\mathfrak{A}$ satisfying $\alpha(\mathbb{1}) = \mathbb{1}$ and $\alpha(x) = \alpha_1(x)$, $\forall x \in E$. In particular if $(f_{nm})_{n,m=1,\ldots,N}$ is a finite family of linear forms on E satisfying $f^*_{nm} = f_{mn}$, then there is a unique *-homomorphism, π, from $T(E)$ into the C^*-algebra $M_N(\mathbb{C})$ of all complex $N \times N$ matrices such that $\pi(\mathbb{1}) = \mathbb{1}$ and $\pi(x) = \begin{pmatrix} f_{11}(x) \ldots f_{1N}(x) \\ \vdots \quad \ddots \quad \vdots \\ f_{N1}(x) \ldots f_{NN}(x) \end{pmatrix}$ $\forall x \in E$; π will be called a *matrix representation* and the f_{nm} its *coefficients*. Clearly if π is a matrix representation then $y \mapsto \|\pi(y)\|$ is a C^*-semi-norm on $T(E)$.

Theorem 2. *Let E be an involutive vector space, let E' be a *-invariant subspace of the (algebraic) dual space of E and let $\Gamma_{E'}$ be the set of all C^*-semi-norms on $T(E)$ $y \mapsto \|\pi(y)\|$ where π runs over the matrix representations of $T(E)$ with coefficients in E'. Suppose that E' separates E; then any positive linear form on $T(E)$ is $\Gamma_{E'}$-strongly positive.*

Proof. Let ω be a positive linear form on $T(E)$ and let $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ be the cyclic *-representation π_ω of $T(E)$ in $\mathfrak{H}_\omega$ with cyclic vector Ω_ω associated with ω by G.N.S. construction.

For any finite set $h_1, \ldots, h_M$ of hermitian linearly independent elements in E, choose $f_1 = f_1^*, \ldots, f_M = f_M^*$ in E' such that $\langle f_k, h_l\rangle = \delta_{kl}$, and let P_N be the orthogonal projection on the finite dimensional subspace $\mathfrak{H}_N$ of $\mathfrak{H}_\omega$ spanned by the vectors $\pi_\omega(h_{i_1}) \ldots \pi_\omega(h_{i_K})\Omega_\omega$, with $K \leq N$ and $i_l \in \{1, 2, \ldots, M\}$ $(\forall l \in \{1, 2, \ldots, K\})$.

Define $\pi_1^{h_1,\ldots,h_M;N}(x)$ for $x \in E$ by:

$$\pi_1^{h_1,\ldots,h_M;N}(x) = \sum_{k=1}^{k=M} \langle f_k, x\rangle P_N \pi_\omega(h_k) P_N, \qquad \forall x \in E.$$

$\pi_1^{(h);N}$ is a linear mapping from E into $\mathscr{L}(\mathfrak{H}_\omega)$ satisfying $\pi_1^{(h);N}(x^*) = \pi_1^{(h);N}(x)^*$. It follows that there is a unique *-homomorphism $\pi^{(h);N}$ from $T(E)$ into the C^*-algebra $\mathscr{L}(\mathfrak{H}_\omega)$ such that $\pi^{(h);N}(\mathbb{1}) = \mathbb{1}$ and $\pi^{(h);N}(x) = \pi_1^{(h);N}(x)$, $\forall x \in E$. $\mathfrak{H}_N$ is invariant by $\pi^{(h);N}(T(E))$ and the corresponding representation in $\mathfrak{H}_N$ defines, for any hilbertian basis in $\mathfrak{H}_N$, a matrix representation with coefficients in E'. It follows that the linear form $x \mapsto \omega_{(h);N}(x) = (\Omega_\omega | \pi^{(h);N}(x)\Omega_\omega)$ is a $\Gamma_{E'}$-strongly positive linear form on $T(E)$. Furthermore, $\omega_{(h);N}(x) = \omega(x)$,

$$\forall x = \sum_{K=0}^{K=N} \Sigma_{(i)} \lambda_{i_1 \ldots i_K} h_{i_1} \otimes \cdots \otimes h_{i_K}, \ (\lambda_{(i)} \in \mathbb{C}).$$

So $\forall x \in T(E)$, $\exists (h), N$ as above such that $\omega(x)$ agrees with $\omega_{(h);N}(x)$. This implies that ω is $\Gamma_{E'}$-strongly positive since the cone of all Γ-strongly positive linear forms on a *-algebra $\mathfrak{A}$ is weakly closed in the dual space of $\mathfrak{A}$ (for any directed set, Γ, of C^*-semi-norms on $\mathfrak{A}$). □

Except for the formulation this theorem was proved by Borchers [11] (see the proof of II. 3.8 in that paper).

Corollary 1. *Let E be an involutive vector space. Then any positive linear form on $T(E)$ is strongly positive.*

5. Algebras of Polynomials and the Classical Moment Problem

Let $\mathbb{C}[X_1, \ldots, X_n]$ denote the *-algebra of all complex polynomials with respect to the indeterminates $X_1, \ldots, X_n$. There is a bijection $(x_1, \ldots, x_n) \mapsto \chi_{(x_1,\ldots,x_n)}$ from $\mathbb{R}^n$ onto the set of all characters[2] on $\mathbb{C}[X_1, \ldots, X_n]$ given by:

(7) $\quad \chi_{(x_1,\ldots,x_n)}(P(X_1, \ldots, X_n)) = P(x_1, \ldots, x_n), \ \forall P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n].$

Any C^*-semi-norm on $\mathbb{C}[X_1, \ldots, X_n]$ is of the form

$$(8) \qquad \begin{cases} P(X_1, \ldots, X_n) \mapsto p_B(P(X_1, \ldots, X_n)) = \sup\limits_{(x_1,\ldots,x_n)\in B} (|P(x_1, \ldots, x_n)|), \\ \text{for some closed bounded subset } B \text{ of } \mathbb{R}^n. \end{cases}$$

Furthermore $B \mapsto p_B$ is an order preserving bijection from the directed set of all compact subsets of $\mathbb{R}^n$ (ordered by inclusion) onto the directed set of all C^*-semi-norms on $\mathbb{C}[X_1, \ldots, X_n]$; (remind that Stone-Weierstrass theorem implies that polynomials are dense in the C^*-algebra $\mathscr{C}(K)$ of all continuous functions on K for any compact $K \subset \mathbb{R}^n$, and that, on the other hand, for any compact $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus K$, there is a continuous function from $\mathbb{R}^n$ into $[0, 1]$, f, with $f(x) = 1$ and $f(K) = \{0\}$). We have: $B = \{(x_1, \ldots, x_n) \mid |P(x_1, \ldots, x_n)| \leq p_B(P(X_1, \ldots, X_n)),$ $\forall P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]\}$, for any compact subset B of $\mathbb{R}^n$. The compact B

[2] In this paper, a character on a *-algebra with unit $\mathfrak{A}$ is a *-homomorphism (of *-algebras with units) from $\mathfrak{A}$ into $\mathbb{C}$.

will be called the *support* of the C^*-semi-norm p_B and, more generally, for any directed set Γ of C^*-semi-norms on $\mathbb{C}[X_1, \ldots, X_n]$ we define the *support of* Γ to be the closed set:

(9) $\operatorname{Supp}(\Gamma) = \overline{\bigcup_{p_B \in \Gamma} B}$ = closure of the union of the supports of the elements of Γ.

Let B be an arbitrary bounded subset of $\mathbb{R}^n$, then we have:

$$p_B(P(X_1, \ldots, X_n)) = \sup_{x \in B}(|P(x)|) = \sup_{x \in \bar{B}}(|P(x)|) = p_{\bar{B}}(P(X_1, \ldots, X_n)).$$

(If S is a subset of $\mathbb{R}^n$, $\bar{S}$ denotes its closure in $\mathbb{R}^n$.)

We want now to identify the various things defined in Sections 2 and 3 in the cases where $\mathfrak{A} = \mathbb{C}[X_1, \ldots, X_n]$ and $\Gamma = \Gamma_S^{(\sigma)}$ or $\Gamma = \Gamma_S^{(\beta)}$, where $S \subset \mathbb{R}^n$ and $\Gamma_S^{(\sigma)}$, $\Gamma_S^{(\beta)}$ are defined by:

(10)
$$\begin{cases} \Gamma_S^{(\sigma)} = \{p_B \mid B \text{ runs over the finite subsets of } S\}, \\ \Gamma_S^{(\beta)} = \{p_B \mid B \text{ runs over the bounded subsets of } S\}. \end{cases}$$

It is not difficult to show that we have:

(11)
$$\begin{cases} \mathscr{A}(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\sigma)}) = \mathbb{C}^S \text{ equipped with the product topology}, \\ \mathfrak{M}(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\sigma)}) = \text{all complex polynomially bounded functions on } S, \\ \mathscr{A}^+(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\sigma)}) = (\mathbb{R}^+)^S, \\ \mathfrak{B}_\infty(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\sigma)}) = C^*\text{-algebra of all bounded complex functions on } S \text{ (equipped with the sup norm)}. \end{cases}$$

In order to solve the case $\Gamma = \Gamma_S^{(\beta)}$, let us remark that all the constructions given in Sections 2 and 3 do only depend on the set $\tilde{\Gamma}$ of $\mathscr{T}_\Gamma$-continuous C^*-semi-norms; Therefore, we may replace Γ by any other directed set of C^*-semi-norms which is cofinal with respect to $\tilde{\Gamma}$ ($\Leftrightarrow$ generates $\mathscr{T}_\Gamma$; see Section 2). On the other hand, we have: $\tilde{\Gamma}_S^{(\beta)} = \Gamma_{\bar{S}}^{(\beta)} = \tilde{\Gamma}_{\bar{S}}^{(\tau)}$, where

(10′)
$$\Gamma_{\bar{S}}^{(\tau)} = \{p_B \mid B \text{ runs over the compact subsets of } \bar{S}\}.$$

With this in mind, it is easy to prove that we have:

(12)
$$\begin{cases} \mathscr{A}(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\beta)}) = \mathscr{C}(\bar{S}) \text{ equipped with the topology of compact convergence}. \\ \mathfrak{M}(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\beta)}) = \mathscr{P}(\bar{S}) = \{\text{polynomially bounded continuous functions on } \bar{S}\}, \\ \mathscr{A}^+(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\beta)}) = \mathscr{C}(\bar{S})^+ = \{\text{positive valued continuous functions on } \bar{S}\}. \\ \mathfrak{B}_\infty(\mathbb{C}[X_1, \ldots, X_n], \Gamma_S^{(\beta)}) = \left(C^*\text{-algebra of bounded continuous complex functions on } \bar{S} \text{ equipped with the } \sup_{\bar{S}} \text{ norm}\right), \end{cases}$$

where $\mathscr{C}(\bar{S})$ denotes the set of all complex continuous functions on $\bar{S}$.

Let us remark that for any directed set of C^*-semi-norms on $\mathbb{C}[X_1, \ldots, X_n]$, Γ, there is a subset S of $\mathbb{R}^n$, namely $S = \bigcup_{p \in \Gamma} \mathrm{Supp}(p)$ (so $\bar{S} = \mathrm{Supp}(\Gamma)$), such that

$$\Gamma_S^{(\sigma)} = \tilde{\Gamma}_S^{(\sigma)} \subset \tilde{\Gamma} \subset \tilde{\Gamma}_S^{(\beta)} = \Gamma_{\mathrm{Supp}(\Gamma)}^{(\beta)} . \tag{13}$$

In this sense, the two families $\Gamma_S^{(\sigma)}$ and $\Gamma_{\bar{S}}^{(\beta)}$ are extreme cases. We have: $\mathrm{Supp}(\Gamma_S^{(\sigma)}) = \mathrm{Supp}(\Gamma_{\bar{S}}^{(\beta)}) = \bar{S}$.

It follows from (11) and (12) that a linear form, ϕ, on $\mathbb{C}[X_1, \ldots, X_n]$ is $\Gamma_S^{(\sigma)}$-strongly positive if and only if it is positive on the polynomials P such that $P(S) \subset \mathbb{R}^+$ and that ϕ is $\Gamma_{\bar{S}}^{(\beta)}$-strongly positive if and only if it is positive on the polynomials P such that $P(\bar{S}) \subset \mathbb{R}^+$. However, we have: $P(\bar{S}) \subset \mathbb{R}^+ \Leftrightarrow P(S) \subset \mathbb{R}^+$ since $\mathbb{R}^+$ is closed in $\mathbb{C}$ and since $(x_1, \ldots, x_n) \mapsto P(x_1, \ldots, x_n)$ is continuous on $\mathbb{R}^n$, $\forall P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$. This implies that Γ-strong positivity only depends on $\mathrm{Supp}(\Gamma)$, i.e.

$$\begin{cases} \phi \text{ is } \Gamma\text{-strongly positive} \Leftrightarrow \phi \text{ is } \Gamma'\text{-strongly positive, if } \Gamma \text{ and } \Gamma' \text{ are} \\ \text{such that } \mathrm{Supp}(\Gamma) = \mathrm{Supp}(\Gamma'), \text{ for } \phi \in \mathbb{C}[X_1, \ldots, X_n]^* \ (\simeq \mathbb{C}^{\mathbb{N}^n}) . \end{cases}$$

Let ϕ be a linear form on $\mathbb{C}[X_1, \ldots, X_n]$ and let S be a closed subset of $\mathbb{R}^n$. Then, the *S-moment problem* for ϕ is the following problem:

Is there a positive Radon measure, μ, supported by S and such that

$$\begin{aligned} &\phi(P(X_1, \ldots, X_n)) \\ &= \int_S P(x_1, \ldots, x_n)\, d\mu(x_1, \ldots, x_n), \ \forall P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]? \end{aligned} \tag{14}$$

If the answer is yes, the S-moment problem for ϕ is said to be soluble and μ is called a solution. If μ is the unique solution, then the problem is said to be determined. It is said to be indetermined if there are several solutions. The main solubility criterion is the following Riesz' criterion.

Riesz' Criterion. *The S-moment problem for ϕ is soluble if and only if $\phi(P(X_1, \ldots, X_n)) \geqq 0$ for any $P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) \geqq 0$, $\forall (x_1, \ldots, x_n) \in S$.*

We know (from the above discussion) that this condition is equivalent to $\Gamma_S^{(\beta)}$-strong positivity for ϕ [$\Leftrightarrow \Gamma$-strong positivity for $\mathrm{Supp}(\Gamma) = S$]. Let us now give a brief discussion of the proof of this criterion. Remark first, that the condition is obviously necessary. Secondly, suppose that ϕ is $\Gamma_S^{(\beta)}$-strongly positive, then, it follows from (12) and Theorem 1 that ϕ has an extension $\tilde{\phi}$ to $\mathscr{P}(S)$ which is positive on $\mathscr{P}(S)^+ = \mathscr{P}(S) \cap \mathscr{C}(S)^+$. Let μ be the restriction of $\tilde{\phi}$ to the C^*-algebra $\mathscr{C}_{(0)}(S)$ of all complex continuous functions on S vanishing at infinity, ($\mu = \tilde{\phi} \upharpoonright \mathscr{C}_{(0)}(S)$); μ is a positive linear form on $\mathscr{C}_{(0)}(S)$ with $\mu(1) = \phi(\mathbb{1})$. So μ is a bounded positive Radon measure supported by S, ($\mathrm{Supp}(\mu) \subset S$). The end of the proof of Riesz' criterion follows from the following lemma.

Lemma 8. *Let ψ be a positive linear form on $\mathscr{P}(S)$ (where S is a closed subset of $\mathbb{R}^n$) and let μ be its restriction to $\mathscr{C}_{(0)}(S)$. Then, any $f \in \mathscr{P}(S)$ is μ-integrable and we have:*

$$\psi(f) = \int_S f\, d\mu, \ \forall f \in \mathscr{P}(S) .$$

Proof. Let (χ_k) denote a sequence of continuous function on $\mathbb{R}^n$ with values in $[0, 1]$ and such that $\chi_k(x) = 1$ if $\|x\| \leqq k$ and $\chi_k(x) = 0$ if $\|x\| \geqq k+1$.

For any $f \in \mathscr{P}(S)$, $\chi_k \cdot f$ is μ-integrable and $f(x) = \lim_{k\to\infty} \chi_k(x) \cdot f(x)$, $\forall x \in S$, and $|\chi_k f| \leqq |f|$. On the other hand $|\chi_k f| = \chi_k |f|$ is an increasing sequence of μ-integrable functions with $\sup_k (\chi_k |f|) = |f| \in \mathscr{P}(S)$. It follows that $\psi(\chi_k |f|) = \int \chi_k |f| \, d\mu \leqq \psi(|f|)$, $(\forall k \geqq 0)$, and therefore $|f|$ is μ-integrable and we have (Lebesgue's theorem)

$$\int |f| \, d\mu = \sup_k \int \chi_k |f| \, d\mu \leqq \psi(|f|), \quad \forall f \in \mathscr{P}(S).$$

This implies that f is also μ-integrable and,

$$\int f \, d\mu = \lim_{k\to\infty} \int \chi_k f \, d\mu .$$

We have:

$$|\psi(f) - \int \chi_k f \, d\mu| = |\psi([1-\chi_k] f)| \leqq \psi([1-\chi_k] \, |f|),$$

(by positivity of ψ). And, there is [for each $f \in \mathscr{P}(S)$] a positive A and a positive integer such that $|f(x)| \leqq A(1 + \|x\|^r)$, $\forall x \in S$. It follows that

$$[1-\chi_k(x)] \, |f(x)| \leqq A(1 + \|x\|^r)^2 \sup_{\|x\| \geqq k} \left(\frac{1}{1 + \|x\|^r} \right) = \frac{A(1 + \|x\|^r)^2}{1 + k^r} .$$

So, we have:

$$|\psi(f) - \int \chi_k f \, d\mu| \leqq A\psi([1 + \|x\|^r]^2) \frac{1}{1 + k^r} \to 0 .$$

This implies that

$$\psi(f) = \lim_{k\to\infty} \int \chi_k f \, d\mu = \int f \, d\mu, \quad \forall f \in \mathscr{P}(S). \quad \square$$

We recall that the $\mathbb{R}^n$-moment problem for a linear form on $\mathbb{C}[X_1, \ldots, X_n]$, ϕ, is called *Hamburger's moment* problem for ϕ. It is well-known that if $n = 1$, the Hamburger's moment problem is always soluble for a positive linear form on the $*$-algebra $\mathbb{C}[X]$. But since $\mathbb{C}[X]$ is canonically isomorphic with $T(\mathbb{C})$, this appears as a specific case of Theorem 2 (the most trivial). On the other hand, there are positive linear forms on $\mathbb{C}[X_1, X_2]$ which are not positive on the polynomials which define positive functions on $\mathbb{R}^2$ (in other words there are positive linear forms on $\mathbb{C}[X_1, X_2]$ which are not strongly positive); the Hamburger's moment problem is not soluble in these cases.

An important result which will be used in below is the following:

Carleman's Theorem [12]. *Let ϕ be a positive linear form on the $*$-algebra $\mathbb{C}[X]$ such that*

$$\sum_{n \geqq 1} \phi(X^{2n})^{-\frac{1}{2n}} = \infty ,$$

then, the Hamburger moment problem for ϕ is determined.

6. Functional Calculus and the C^*-Algebras $\mathfrak{B}(\mathfrak{A}, \Gamma)$

Let again $\mathfrak{A}$ be a *-algebra with unit ($\mathbb{1} \in \mathfrak{A}$), Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$ and let us use the previous notations. Let S be a closed subset of $\mathbb{R}$ and let 1_S and Id_S be the real functions on S defined by:

$$(15) \qquad 1_S(x) = 1, \ \forall x \in S \quad \text{and} \quad \mathrm{Id}_S(x) = x, \ \forall x \in S .$$

1_S and Id_S are elements of the *-algebra $\mathscr{P}(S)$ of all continuous polynomially bounded complex functions on S.

Theorem 3. *Let h be a hermitian element of $\mathscr{A}$ and let S be a closed subset of $\mathbb{R}$ which contains* $\mathrm{Sp}(h)$. *Then, there is a unique *-homomorphism, α, from $\mathscr{P}(S)$ into $\mathscr{A}$ such that $\alpha(1_S) = \mathbb{1}$ and $\alpha(\mathrm{Id}_S) = h$. Furthermore α is continuous if $\mathscr{P}(S)$ is equipped with the topology of compact convergence and, if $h \in \mathfrak{A}$, then $\alpha(\mathscr{P}(S)) \subset \mathfrak{M}$.*

Proof. a) There is a unique *-homomorphism, α_0, from the algebra of polynomials $\mathbb{C}[X]$ into $\mathscr{A}$ such that $\alpha_0(1) = \mathbb{1}$ and $\alpha_0(X) = h$ (remind that $\mathbb{C}[X]$ is isomorphic with the tensor algebra over one dimensional involutive space). Let p be an element of Γ, then $p \circ \alpha_0$ is a C^*-semi-norm on $\mathbb{C}[X]$ and the support of $p \circ \alpha_0$ is the spectrum of $\pi_p(h)$ in $\mathfrak{B}_p$ (remembering that if h_0 is a hermitian element of a C^*-algebra with a unit, $\mathbb{1}$, then the C^*-subalgebra generated by $\mathbb{1}$ and h_0 is isomorphic with the C^*-algebra of all continuous functions on the spectrum of h_0). It follows that there is a unique *-homomorphism, $\tilde{\alpha}$, from the *-algebra of all polynomial functions on S $\left(S \supset \mathrm{Sp}(h) = \bigcup_{p \in \Gamma} \mathrm{Sp}(\pi_p(h))\right)$ into $\mathscr{A}$ such that $\tilde{\alpha}(1_S) = 1$, $\tilde{\alpha}(\mathrm{Id}_S) = h$ and that this *-homomorphism is continuous for the topology of compact convergence on S. On the other hand, polynomial functions are dense in the set of continuous functions on S, $\mathscr{C}(S)$, for the topology of compact convergence. Therefore there is a unique continuous *-homomorphism $\bar{\alpha}: \mathscr{C}(S) \to \mathscr{A}$ satisfying $\bar{\alpha}(1_S) = \mathbb{1}$ and $\bar{\alpha}(\mathrm{Id}_S) = h$. The restriction $\alpha = \bar{\alpha} \restriction \mathscr{P}(S)$ has obviously all the properties mentionned in Theorem 3. It remains to show that α is the unique *-homomorphism from $\mathscr{P}(S)$ into $\mathscr{A}$ satisfying $\alpha(1_S) = \mathbb{1}$, $\alpha(\mathrm{Id}_S) = h$.

b)[3] We know that α is unique on polynomial functions on S. It follows that α is also unique on the *-algebra generated by the polynomial functions and the inverses of polynomial functions inversible in $\mathscr{P}(S)$. In particular we have: $\forall \varepsilon > 0$, $\forall \varrho \in \mathbb{R}$

$$\alpha([\mathrm{Id}_S - \varrho 1_S] \cdot [1_S + \varepsilon(\mathrm{Id}_S - \varrho 1_S)^2]^{-1}) = (h - \varrho \mathbb{1}) \cdot [\mathbb{1} + \varepsilon(h - \varrho \mathbb{1})^2]^{-1} ,$$

and these elements are in the C^*-algebra $\mathfrak{B}_\infty$, by Lemma 5. Since *-homomorphisms of C^*-algebras are continuous and since the functions

$$t \mapsto \frac{t - \varrho}{1 + \varepsilon(t - \varrho)^2} \ (t \in S)$$

separate strongly S, [13] [and so generate the C^*-algebra $\mathscr{C}_{(0)}(S)$ by Stone-Weierstrass], it follows that α is unique on the *-algebra generated by the C^*-algebra $\mathscr{C}_{(0)}(S)$ (of continuous functions vanishing at infinity on S) and the polynomial functions. But this *-algebra is just $\mathscr{P}(S)$; indeed $f \in \mathscr{P}(S) \Leftrightarrow f$ continuous

[3] This proof of uniqueness has been suggested by H. Epstein.

and $\exists N, K$ with $|f(t)| \leqq K(1+t^2)^N \ \forall t \in S$, so $t \mapsto \dfrac{f(t)}{(1+t^2)^{N+1}}$ is in $\mathscr{C}_{(0)}(S)$ and its product with the polynomial $(1+t^2)^{N+1}$ is f. $\square$

Lemma 9. *Let h, S, α be as in Theorem 3.*

a) *If f is a bounded continuous function on S, then $\alpha(f)$ is in the C^*-algebra $\mathfrak{B}_\infty$.*

b) *$\alpha(\mathscr{C}_{(0)}(S))$ is the C^*-subalgebra of $\mathfrak{B}_\infty$ generated by the set* $\{(h-\varrho\mathbb{1})\cdot[\mathbb{1}+(h-\varrho\mathbb{1})^2]^{-1} | \varrho \in \mathbb{R}\}$.

Proof. Let p be an arbitrary element in Γ, let $\bar{\alpha}$ be as in the proof of Theorem 3 and let $f \in \mathscr{C}(S)$. Then we have: $\pi_p(\alpha(f)) = f(\pi_p(h))$, $\forall p \in \Gamma$ (in the C^*-algebra $\mathfrak{B}_p$, with the usual notation [8, 9]), and, $\|\pi_p(\alpha(f))\| = \operatorname*{Sup}_{t \in \mathrm{Sp}(\pi_p(h))} (|f(t)|)$. Part a) of the lemma follows from this and from the definition of $\mathfrak{B}_\infty$. Let us consider the functions $t \mapsto \dfrac{t-\varrho}{1+(t-\varrho)^2} = f_\varrho(t)$; $\forall t \in S$, $\exists \varrho \in \mathbb{R}$ with $\dfrac{t-\varrho}{1+(t-\varrho)^2} \neq 0$, and $\forall t_1, t_2 \in S$ with $t_1 \neq t_2$, $\exists \varrho \in \mathbb{R}$ with $\dfrac{t_1-\varrho}{1+(t_1-\varrho)^2} \neq \dfrac{t_2-\varrho}{1+(t_2-\varrho)^2}$. It follows that $\mathscr{C}_{(0)}(S)$ is the C^*-algebra generated by these functions (Stone-Weierstrass). On the other hand, we know that $\alpha(f_\varrho) = (h-\varrho\mathbb{1})\cdot[\mathbb{1}+(h-\varrho\mathbb{1})^2]^{-1}$. This implies b). $\square$

Remark 5. Lemma 4 is not an obvious consequence of Theorem 3.

Definition 2. Let h, S, α be as in Theorem 3 and let $\bar{\alpha}$ be the unique continuous extension of α to $\mathscr{C}(S)$. Then, we define $f(h)$ for

$$f \in \mathscr{C}(S) \quad \text{by:} \quad f(h) = \bar{\alpha}(f), \ (\forall f \in \mathscr{C}(S)).$$

Notice that if (h, S_1, α_1) and (h, S_2, α_2) are as above and if $S_1 \supset S_2$, then, by uniqueness, we have:

$$f_1(h) = (f_1 \restriction S_2)(h), \ \forall f_1 \in \mathscr{C}(S_1),$$

(where $f_1 \restriction S_2$ is the restriction of f_1 to S_2). So, Definition 2 is coherent.

Definition 3. Let $\mathfrak{A}$ be a $*$-algebra with a unit and let Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$.

a) If Γ separates $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{A}, \Gamma)$ is defined to be the C^*-subalgebra of $\mathfrak{B}_\infty(\mathfrak{A}, \Gamma)$ generated by the set $\{h(\mathbb{1}+h^2)^{-1} | h \in \mathfrak{A} \text{ and } h = h^*\}$. If Γ does not separate $\mathfrak{A}$, we define $\mathfrak{B}(\mathfrak{A}, \Gamma)$ to be $\mathfrak{B}(\mathfrak{A}/\mathfrak{I}_\Gamma, \Gamma)$ (see Section 3 for the notations).

b) If Γ is the set of all C^*-semi-norms on $\mathfrak{A}$. Then $\mathfrak{B}(\mathfrak{A}, \Gamma)$ will be called the *C^*-algebra associated with the $*$-algebra* $\mathfrak{A}$ and will simply be denoted by $\mathfrak{B}(\mathfrak{A})$.

c) If $\mathfrak{A}$ is a topological $*$-algebra and if Γ is the set of all continuous C^*-semi-norms on $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{A}, \Gamma)$ will be called the *C^*-algebra associated with the topological $*$-algebra* $\mathfrak{A}$.

Notice that $\mathfrak{B}(\mathfrak{A})$ is associated with $\mathfrak{A}$ equipped with its finest locally convex topology. $\mathfrak{B}(\mathfrak{A}, \Gamma)$ will simply be denoted by $\mathfrak{B}_\Gamma$ (when no confusion arises).

Let $\mathfrak{A}$ be a $*$-algebra with a unit ($\mathbb{1} \in \mathfrak{A}$). If p is a C^*-semi-norm on $\mathfrak{A}$, the C^*-algebra obtained by completion of $\mathfrak{A}/p^{-1}(0)$ for the norm induced by p is denoted by $\mathfrak{B}_p(\mathfrak{A})$ ($=\mathfrak{B}_p$ in Section 2). Let Γ_1 and Γ_2 be two directed sets of C^*-semi-norms on $\mathfrak{A}$ and suppose that the locally convex topology $\mathscr{T}_{\Gamma_1}$ generated by Γ_1 is finer than the locally convex topology $\mathscr{T}_{\Gamma_2}$ generated by Γ_2. Then the identity mapping of $\mathfrak{A}$ is continuous from $\mathfrak{A}$ equipped with $\mathcal{T}_{\Gamma_1}$ on $\mathfrak{A}$ equipped with $\mathcal{T}_{\Gamma_2}$; therefore, it determines a continuous *-homomorphism π_{Γ_2,Γ_1} from $\mathscr{A}(\mathfrak{A},\Gamma_1)$ into $\mathscr{A}(\mathfrak{A},\Gamma_2)$. We have $\pi_{\Gamma_2,\Gamma_1}(\mathfrak{B}_\infty(\mathfrak{A},\Gamma_1))\subset\mathfrak{B}_\infty(\mathfrak{A},\Gamma_2)$, but $\pi_{\Gamma_2,\Gamma_1}(\mathfrak{B}_\infty(\mathfrak{A},\Gamma_1))$ is generally a strict C^*-subalgebra of $\mathfrak{B}_\infty(\mathfrak{A},\Gamma_2)$ (see the examples given in Section 5).

Proposition 2. *Let $\mathfrak{A}$ be a *-algebra with unit, let p be an arbitrary C^*-semi-norm on $\mathfrak{A}$ and let Γ_1 and Γ_2 be two directed sets of C^*-semi-norms on $\mathfrak{A}$ such that $\mathcal{T}_{\Gamma_1}$ is finer than $\mathcal{T}_{\Gamma_2}$. We have:*

a) $\mathfrak{B}(\mathfrak{A},\{p\})=\mathfrak{B}_p(\mathfrak{A})$,

b) $\pi_{\Gamma_2,\Gamma_1}(\mathfrak{B}(\mathfrak{A},\Gamma_1))=\mathfrak{B}(\mathfrak{A},\Gamma_2)$.

Proof. For any hermitian element h of $\mathfrak{A}/p^{-1}(0)$ and for any $\varepsilon>0$, $h_\varepsilon=h(\mathbb{1}+\varepsilon h^2)^{-1}=\frac{1}{\sqrt{\varepsilon}}(\sqrt{\varepsilon}h)(\mathbb{1}+(\sqrt{\varepsilon}h)^2)^{-1}$ is an element of $\mathfrak{B}(\mathfrak{A},\{p\})$; so we have a), by Lemma 5. On the other hand, b) follows easily from Definition 3 and from the fact that any *-homomorphism of C^*-algebras with a dense image is a surjective *-homomorphism. □

So, $\mathfrak{B}(\mathfrak{A},\Gamma)$ is a quotient C^*-algebra of $\mathfrak{B}(\mathfrak{A})$.

In connexion with Definition 3, the following slight lemma is worth noticing.

Lemma 10. *Let $\mathfrak{A}$ be a C^*-algebra with unit. Then the C^*-algebra associated with $\mathfrak{A}$ considered as a *-algebra is identical with the C^*-algebra associated with $\mathfrak{A}$ considered as a topological *-algebra, and both are canonically identical with $\mathfrak{A}$, ($\mathfrak{B}(\mathfrak{A})=\mathfrak{A}$).*

Proof. For any C^*-semi-norm on $\mathfrak{A}$, p, the *-homomorphism π_p must be a norm-decreasing surjective *-homomorphism of $\mathfrak{A}$ on $\mathfrak{B}_p$. Therefore, any C^*-semi-norm on $\mathfrak{A}$ is continuous; this proves the first statement. The other part of Lemma 10 follows from the fact that

$$h=\lim_{\varepsilon\to 0}h(\mathbb{1}+\varepsilon h^2)^{-1},\quad \forall h=h^*\in\mathfrak{A}.\quad \square$$

It follows that any positive linear form on a C^*-algebra with a unit is strongly positive but this does not mean that Γ-strong positivity has no interest in this case. For instance, let $\mathfrak{A}$ be an arbitrary *-algebra with unit and let $\mathscr{A}(\mathfrak{A})$ be $\mathscr{A}(\mathfrak{A}$, {all C^*-semi-norms on $\mathfrak{A}$}), we have: $\mathfrak{B}(\mathfrak{A})\subset\mathscr{A}(\mathfrak{A})$. Let Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$; Γ induces a directed set of C^*-semi-norms on $\mathscr{A}(\mathfrak{A})$ and, by restriction, a directed set of C^*-semi-norms on $\mathfrak{B}(\mathfrak{A})$ which will again be denoted by Γ. Then it is not hard to see that we have:

$$\mathfrak{B}(\mathfrak{A},\Gamma)=\mathfrak{B}(\mathfrak{B}(\mathfrak{A}),\Gamma)\quad\text{canonically}. \tag{16}$$

7. A Generalization of Classical Moment Problem on *-Algebras

Let $\mathfrak{A}$ be a *-algebra with a unit, ($\mathbb{1}$), and let Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$. We suppose that Γ separates $\mathfrak{A}$ (otherwise replace $\mathfrak{A}$ by $\mathfrak{A}/\mathfrak{J}_\Gamma$).

For any hermitian element of $\mathfrak{A}$, h, and for any positive linear form on $\mathfrak{B}(\mathfrak{A},\Gamma)$, ω, there is a unique bounded positive Radon measure on $\mathbb{R}$, $\mu_{h,\omega}$, such that

$$\omega(f(h))=\int f(t)\,d\mu_{h,\omega}(t),\ \forall f\in\mathscr{C}_{(0)}(\mathbb{R}). \tag{17}$$

Morever, $\mu_{h,\omega}$, is supported by the closure (in $\mathbb{R}$), $\overline{\mathrm{Sp}_\Gamma(h)}$, of the spectrum in $\mathscr{A}(\mathfrak{A},\Gamma)$ of h, $\mathrm{Sp}_\Gamma(h)$:

$$(18) \qquad \mathrm{Supp}(\mu_{h,\omega}) \subset \overline{\mathrm{Sp}_\Gamma(h)}\,.$$

Indeed, by uniqueness in Theorem 3, $f \mapsto f(h)$ must factorize as $f \mapsto f \upharpoonright S \mapsto f(h)$ [where $f \mapsto f \upharpoonright S$ is the restriction $\upharpoonright S: \mathscr{P}(\mathbb{R}) \to \mathscr{P}(S)$], $\forall S \subset \mathbb{R}$ closed and such that $\mathrm{Sp}_\Gamma(h) \subset S$.

Definition 4. Let ϕ be a linear form on $\mathfrak{A}$. Then, we define the *$\tilde{m}(\Gamma)$-problem for ϕ* to be the following problem: Is there a positive linear form on $\mathfrak{B}(\mathfrak{A},\Gamma)$, ω, such that

$$\phi(h) = \int t\, d\mu_{h,\omega}(t),\ \forall h = h^* \in \mathfrak{A}\,?$$

If the answer is yes, we say that the problem is *soluble* and that ω is a *solution.*

If Γ is the set of all C^*-semi-norms on $\mathfrak{A}$, the $\tilde{m}(\Gamma)$-problem for ϕ will simply be called the $\tilde{m}$-problem for ϕ.

Let f be an arbitrary element of $\mathscr{C}_{(0)}(\mathbb{R})$; then, for any integer $n \geqq 1$, $t \mapsto f(t^n)$ is also an element of $\mathscr{C}_{(0)}(\mathbb{R})$. So, applying the definition (17) of $\mu_{h,\omega}$, we find:

$$(19) \qquad \omega(f(h^n)) = \int f(t^n)\, d\mu_{h,\omega}(t) = \int f(t)\, d\mu_{h^n,\omega}(t)\,.$$

On the other hand, if ω is a solution of the $\tilde{m}(\Gamma)$-problem for ϕ we have: $\omega(\mathbb{1}) = \|\mu_{h,\omega}\| = \int d\mu_{h,\omega}(t) = \phi(\mathbb{1})$. It follows that if the $\tilde{m}(\Gamma)$-problem for ϕ is soluble and if ω is an arbitrary solution, we have:

$$(20) \qquad \phi(P(h)) = \int P(t)\, d\mu_{h,\omega}(t),\ \forall h = h^* \in \mathfrak{A} \quad \text{and} \quad \forall P(X) \in \mathbb{C}[X]\,.$$

In other words, $\mu_{h,\omega}$ is a solution of the classical moment problem for the linear form on $\mathbb{C}[X]$ defined by: $P(X) \mapsto \phi(P(h))$.
The generalization of Riesz' criterion is given by the following theorem.

Theorem 4. *Let ϕ be a linear form on $\mathfrak{A}$. Then, the $\tilde{m}(\Gamma)$-problem for ϕ is soluble if and only if ϕ is a Γ-strongly positive linear form on $\mathfrak{A}$.*

Proof. Suppose that the $\tilde{m}(\Gamma)$-problem for ϕ is soluble and let ω be a solution. Then, by (18), we have: $\mathrm{Supp}(\mu_{h,\omega}) \subset \mathbb{R}^+$, $\forall h \in \mathfrak{A} \cap \mathscr{A}^+(\mathfrak{A},\Gamma)$ (use Lemma 3); So $\phi(h) \geqq 0$, $\forall h \in \mathfrak{A} \cap \mathscr{A}^+(\mathfrak{A},\Gamma)$ which means that ϕ is a Γ-strongly positive linear form on $\mathfrak{A}$.

Conversely, suppose that ϕ is Γ-strongly positive. Then, by Theorem 1, there is a linear form on $\mathfrak{M}(\mathfrak{A},\Gamma)$, $\tilde{\phi}$, which is positive on $\mathfrak{M}^+(\mathfrak{A},\Gamma) = \mathfrak{M}(\mathfrak{A},\Gamma) \cap \mathscr{A}^+(\mathfrak{A},\Gamma)$ and which is an extension of ϕ. Let ω denotes the restriction of $\tilde{\phi}$ to $\mathfrak{B}(\mathfrak{A},\Gamma)$ and let h be an arbitrary hermitian element of $\mathfrak{A}$. Then, ω is a positive linear form on $\mathfrak{B}(\mathfrak{A},\Gamma)$ and $\mu_{h,\omega}$ is, by definition, the restriction to $\mathscr{C}_{(0)}(\mathbb{R})$ of the positive linear form on $\mathscr{P}(\mathbb{R})$ defined by: $f \mapsto \tilde{\phi}(f(h))$. Therefore, it follows from Lemma 8, that we have: $\tilde{\phi}(f(h)) = \int f(t)\, d\mu_{h,\omega}(t)$, $\forall f \in \mathscr{P}(\mathbb{R})$, $\forall h = h^* \in \mathfrak{A}$. So ω is a solution of the $\tilde{m}(\Gamma)$-problem for ϕ. □

Remark 6. a) Combined with (18) and (20), this theorem means that the $\tilde{m}(\Gamma)$-problem for ϕ is soluble if and only if, for any hermitian element h of $\mathfrak{A}$, the $\mathrm{Sp}_\Gamma(h)$-moment problem for the linear form $P(X) \mapsto \phi(P(h))$ on $\mathbb{C}[X]$ is soluble.

b) It should be clear that any solution of some $\tilde{m}(\Gamma)$-problem is a solution of the $\tilde{m}(\Gamma)$-problem for a unique linear form on $\mathfrak{A}$.

c) In the proof of the Theorem 4, we construct a solution of the $\tilde{m}(\Gamma)$-problem for a Γ-strongly positive linear form by restriction of a positive linear form on $\mathfrak{M}(\mathfrak{A},\Gamma)$. It is not obvious that all the solutions are of this type.

Definition 4'. Let ϕ be a linear form on $\mathfrak{A}$. *The $m(\Gamma)$-problem for ϕ* is the following problem. Is there a positive linear form ω on $\mathfrak{B}(\mathfrak{A},\Gamma)$ such that $b=b^*\in\mathfrak{B}(\mathfrak{A},\Gamma)$ and $h-b\in\mathscr{A}^+(\mathfrak{A},\Gamma)$ with $h\in\mathfrak{A}$ imply $\phi(h)\geqq\omega(b)$?

Theorem 5. *Let ϕ be a linear form on $\mathfrak{A}$ and let ω be a positive linear form on $\mathfrak{B}(\mathfrak{A},\Gamma)$. Then ω is solution of the $m(\Gamma)$-problem for ϕ if and only if there is a positive linear form ψ on the ordered vector space $\mathfrak{M}(\mathfrak{A},\Gamma)$ such that $\phi=\psi\upharpoonright\mathfrak{A}$ and $\omega=\psi\upharpoonright\mathfrak{B}(\mathfrak{A},\Gamma)$. Then we have:*

$$\psi(f(h))=\int f(t)\,d\mu_{h,\omega}(t),\ \forall f\in\mathscr{P}(\mathbb{R})\quad \text{and}\quad \forall h=h^*\in\mathfrak{A}\,.$$

So the $m(\Gamma)$-problem for ϕ is soluble if and only if ϕ is Γ-strongly positive and any solution is a solution of the $\tilde{m}(\Gamma)$-problem for ϕ.

Proof. It is clear (from Definition 4') that if ψ is a positive linear form on $\mathfrak{M}(\mathfrak{A},\Gamma)$, $\psi\upharpoonright\mathfrak{B}(\mathfrak{A},\Gamma)$ is solution of the $m(\Gamma)$-problem for $\psi\upharpoonright\mathfrak{A}$.

Conversely, let ω be a solution of the $m(\Gamma)$-problem for ϕ; it follows from the definition that we have:

$$\omega(b)\leqq p(b)=\inf_{\substack{h=h^*\in\mathfrak{A}\\ h-b\in\mathscr{A}^+}}\{\phi(h)\}\,,\qquad \forall b=b^*\in\mathfrak{B}(\mathfrak{A},\Gamma)\,.$$

Applying again Hahn-Banach theorem (see the proof of Theorem 1), we find that there is a real linear form, ψ_1, on the real subspace $\mathfrak{M}^h$ of $\mathfrak{M}$ which is an extension of ω [restricted to the hermitian elements of $\mathfrak{B}(\mathfrak{A},\Gamma)$] and satisfies:

$$\psi_1(x)\leqq p(x)=\inf_{\substack{h^*=h\in\mathfrak{A}\\ h-x\in\mathscr{A}^+}}\{\phi(h)\}\,.$$

It is not hard to see that $x\mapsto\psi(x)=\psi_1\left(\frac{x+x^*}{2}\right)+i\psi_1\left(\frac{x-x^*}{2i}\right)$ satisfies the conditions of the theorem. □

The last statement in Theorem 5 (which is a direct consequence of lemma 8), implies that ψ is in fact uniquely determined on the subspace of $\mathfrak{M}(\mathfrak{A},\Gamma)$ spanned by $\{f(h)\,|\,h=h^*\in\mathfrak{A},\ f\in\mathscr{P}(\mathbb{R})\}$.

Let $\mathfrak{B}(\mathfrak{A},\Gamma)'_\sigma$ be the topological dual space of the C^*-algebra $\mathfrak{B}(\mathfrak{A},\Gamma)$ equipped with the weak dual topology $\sigma(\mathfrak{B}(\mathfrak{A},\Gamma)',\mathfrak{B}(\mathfrak{A},\Gamma))$. The restriction to $\mathfrak{B}(\mathfrak{A},\Gamma)$ of positive linear forms on $\mathfrak{M}(\mathfrak{A},\Gamma)$ is weakly continuous. Therefore, it follows from Proposition 1, that Theorem 5 has the following corollary.

Corollary 2. *Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$; then, the set $\mathfrak{S}_\phi$ of all the solutions of the $m(\Gamma)$-problem for ϕ is a weakly compact convex subset in $\mathfrak{B}(\mathfrak{A},\Gamma)'$.*

More generally, if K is a set of Γ-strongly positive linear forms on $\mathfrak{A}$ which is closed and bounded in $\mathfrak{A}^*_\sigma$, then the set $\mathfrak{S}_K=\bigcup_{\phi\in K}\mathfrak{S}_\phi$ is compact in $\mathfrak{B}(\mathfrak{A},\Gamma)'_\sigma$ (and convex whenever K is convex).

Let $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ denotes the linear hull [in $\mathscr{A}(\mathfrak{A}, \Gamma)$] of $\mathfrak{B}(\mathfrak{A}, \Gamma) \cdot \mathfrak{A}$. We have:

$$(21) \qquad [\mathfrak{N}_0(\mathfrak{A}, \Gamma)]^* \cdot \mathfrak{N}_0(\mathfrak{A}, \Gamma) \subset \mathfrak{M}(\mathfrak{A}, \Gamma)$$

(where $[\mathfrak{N}_0(\mathfrak{A}, \Gamma)]^*$ is the set $\{x^* | x \in \mathfrak{N}_0(\mathfrak{A}, \Gamma)\}$) since $\mathfrak{B}(\mathfrak{A}, \Gamma)$ is a subset of $\mathfrak{M}(\mathfrak{A}, \Gamma)$ and since $\mathfrak{A} \cdot \mathfrak{M}(\mathfrak{A}, \Gamma) \cdot \mathfrak{A} \subset \mathfrak{M}(\mathfrak{A}, \Gamma)$ (Lemma 6b). It follows that $(x, y) \mapsto \psi(x^* y)$ is a positive sesquilinear form on $\mathfrak{N}_0(\mathfrak{A}, \Gamma) \times \mathfrak{N}_0(\mathfrak{A}, \Gamma)$ for any positive linear form ψ on $\mathfrak{M}(\mathfrak{A}, \Gamma)$. Furthermore, for any element y of $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$, $x \mapsto \psi(y^* x y)$ is a positive linear form on $\mathfrak{B}(\mathfrak{A}, \Gamma)$ so we have:

$$(22) \qquad \psi(y^* x^* x y)^{\frac{1}{2}} \leqq \|x\| \psi(y^* y)^{\frac{1}{2}}, \qquad \forall x \in \mathfrak{B}(\mathfrak{A}, \Gamma), \qquad \forall y \in \mathfrak{N}_0(\mathfrak{A}, \Gamma).$$

On the other hand, for any positive linear form $x \mapsto \psi(x)$ on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ and for any $y \in \mathfrak{A}$, $x \mapsto \psi(y^* x y)$ is again a positive linear form on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ (Lemma 6a).

Let us equip $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ with the locally convex topology generated by the semi-norms $x \mapsto \psi(x^* x)^{\frac{1}{2}}$ where ψ runs over the set of all positive linear forms on $\mathfrak{M}(\mathfrak{A}, \Gamma)$. It follows from the above discussion that $(x, y) \mapsto xy$ is a jointly continuous bilinear mapping of $\mathfrak{B}(\mathfrak{A}, \Gamma) \times \mathfrak{N}_0(\mathfrak{A}, \Gamma)$ into $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ and that, for any $y \in \mathfrak{A}$, $x \mapsto xy$ is a continuous linear mapping of $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ into itself. Let us denote the completion of $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ by $\mathfrak{N}(\mathfrak{A}, \Gamma)$, [remark that $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ is a Hausdorff space]. We have, by continuity, a continuous bilinear mapping of $\mathfrak{B}(\mathfrak{A}, \Gamma) \times \mathfrak{N}(\mathfrak{A}, \Gamma)$ into $\mathfrak{N}(\mathfrak{A}, \Gamma)$ [again denoted by $(x, y) \mapsto xy$ for $x \in \mathfrak{B}(\mathfrak{A}, \Gamma)$, $y \in \mathfrak{N}(\mathfrak{A}, \Gamma)$] and, for any element z of $\mathfrak{A}$, a continuous linear mapping of $\mathfrak{N}(\mathfrak{A}, \Gamma)$ into itself [again denoted by $y \mapsto yz$ for $y \in \mathfrak{N}(\mathfrak{A}, \Gamma)$].

$$(23) \qquad \begin{cases} x_1(x_2 \cdot y) = (x_1 \cdot x_2) y, \ \forall x_1, x_2 \in \mathfrak{B}(\mathfrak{A}, \Gamma), \ \forall y \in \mathfrak{N}(\mathfrak{A}, \Gamma) \\ (y \cdot z_1) z_2 = y \cdot (z_1 z_2), \ \forall z_1, z_2 \in \mathfrak{A}, \ \forall y \in \mathfrak{N}(\mathfrak{A}, \Gamma) \\ y \cdot (\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 y z_1 + \lambda_2 y z_2, \ \forall \lambda_1, \lambda_2 \in \mathbb{C}, \ \forall z_1, z_2 \in \mathfrak{A}, \ \forall y \in \mathfrak{N}(\mathfrak{A}, \Gamma). \end{cases}$$

so $\mathfrak{N}(\mathfrak{A}, \Gamma)$ is a left $\mathfrak{B}(\mathfrak{A}, \Gamma)$-module and a right $\mathfrak{A}$-module. $\mathfrak{B}(\mathfrak{A}, \Gamma)$ and $\mathfrak{A}$ are both subspaces of $\mathfrak{N}(\mathfrak{A}, \Gamma)$.

Lemma 11. *$\mathfrak{B}(\mathfrak{A}, \Gamma)$ is a dense subspace of the space $\mathfrak{N}(\mathfrak{A}, \Gamma)$.*

Proof. Let ψ be an arbitrary positive linear form on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ and let ω be its restriction to $\mathfrak{B}(\mathfrak{A}, \Gamma)$. Then, ω is solution of the $m(\Gamma)$-problem for the restriction of ψ to $\mathfrak{A}$ and we know (Theorem 5) that we have:

$$\psi(f(h)) = \int f(t) d\mu_{h,\omega}(t), \ \forall f \in \mathscr{P}(\mathbb{R}) \quad \text{and} \quad \forall h = h^* \in \mathfrak{A}.$$

Let x be an arbitrary hermitian element of $\mathfrak{A}$ and let x_n be defined by $x_n = x\left(\mathbb{1} + \frac{x \cdot x}{n}\right)^{-1}$ for any integer $n > 0$. Then $x_n \in \mathfrak{B}(\mathfrak{A}, \Gamma)$ and $\psi((x - x_n)(x - x_n))$ $= \int \left(t - \frac{t}{1 + t^2/n}\right)^2 d\mu_{x,\omega}(t) \to 0$. It follows that the set of all bermitian elements of $\mathfrak{A}$ is in the closure of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in $\mathfrak{N}(\mathfrak{A}, \Gamma)$. Therefore $\mathfrak{A}$ is in the closure of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in $\mathfrak{N}(\mathfrak{A}, \Gamma)$ (since this closure is a linear subspace). This implies that $\mathfrak{B}(\mathfrak{A}, \Gamma) \cdot \mathfrak{A}$ is also in the closure of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in $\mathfrak{N}(\mathfrak{A}, \Gamma)$, [since $(x, y) \mapsto xy$ is jointly continuous from $\mathfrak{B}(\mathfrak{A}, \Gamma) \times \mathfrak{N}(\mathfrak{A}, \Gamma)$ into $\mathfrak{N}(\mathfrak{A}, \Gamma)$]. So (again by linearity), $\mathfrak{N}_0(\mathfrak{A}, \Gamma)$ is in the closure of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ and is dense in $\mathfrak{N}(\mathfrak{A}, \Gamma)$ (by construction). It follows that the closure of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in $\mathfrak{N}(\mathfrak{A}, \Gamma)$ is $\mathfrak{N}(\mathfrak{A}, \Gamma)$. □

Remark 7. Let ψ be a positive linear form on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ and let ω be its restriction to $\mathfrak{B}(\mathfrak{A}, \Gamma)$. Lemma 11 implies that the positive sesquilinear form on $\mathfrak{N}_0(\mathfrak{A}, \Gamma) \times \mathfrak{N}_0(\mathfrak{A}, \Gamma)$ defined by $(x, y) \mapsto \psi(x^* y)$ does only depend on ω [and the same is true for its continuous extension to $\mathfrak{N}(\mathfrak{A}, \Gamma) \times \mathfrak{N}(\mathfrak{A}, \Gamma)$]. In other words, Lemma 11 means that $\mathfrak{N}(\mathfrak{A}, \Gamma)$ is the completion of the space $\mathfrak{B}(\mathfrak{A}, \Gamma)$ equiped with the locally convex topology generated by the semi-norms $x \mapsto \omega(x^* x)^{\frac{1}{2}}$ where ω runs over the set of all solutions of $m(\Gamma)$-problems. Notice also that the linear hull of the $f(h)$, for $f \in \mathscr{P}(\mathbb{R})$ and $h = h^* \in \mathfrak{A}$ is canonically a subspace of $\mathfrak{N}(\mathfrak{A}, \Gamma)$.

Proposition 3. *Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$, let ω be a solution of the $m(\Gamma)$-problem for ϕ and let $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ be the cyclic *-representation π_ω of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in the Hilbert space $\mathfrak{H}_\omega$ with cyclic vector Ω_ω associated with ω by* G.N.S. *construction. Then, there is a unique continuous linear mapping of $\mathfrak{N}(\mathfrak{A}, \Gamma)$ into $\mathfrak{H}_\omega$, Ψ_ω, such that $\Psi_\omega(x) = \pi_\omega(x)\Omega_\omega$, $\forall x \in \mathfrak{B}(\mathfrak{A}, \Gamma)$. Furthermore, we have: $\pi_\omega(x)\Psi_\omega(y) = \Psi_\omega(x \cdot y)$, $\forall x \in \mathfrak{B}(\mathfrak{A}, \Gamma)$ and $\forall y \in \mathfrak{N}(\mathfrak{A}, \Gamma)$; $(\Psi_\omega(x) | \Psi_\omega(y)) = \phi(x^* y)$, $\forall x, y \in \mathfrak{A}$. For any fixed element of $\mathfrak{A}$, x, let ω_x be the positive linear form on $\mathfrak{B}(\mathfrak{A}, \Gamma)$ defined by: $\omega_x(y) = (\Psi_\omega(x) | \pi_\omega(y)\Psi_\omega(x))$, $\forall y \in \mathfrak{B}(\mathfrak{A}, \Gamma)$. Then ω_x is solution of the $m(\Gamma)$-problem for ϕ_x, where ϕ_x is the Γ-strongly positive linear form on $\mathfrak{A}$ defined by: $\phi_x(y) = \phi(x^* y x)$, $\forall y \in \mathfrak{A}$, ($\phi_x$ is Γ-strongly positive by Lemma* 6a).

Proof. The mapping $x \mapsto \pi_\omega(x)\Omega_\omega$ from $\mathfrak{B}(\mathfrak{A}, \Gamma)$ into $\mathfrak{H}_\omega$ is clearly continuous for the topology induced [on $\mathfrak{B}(\mathfrak{A}, \Gamma)$] by the topology of $\mathfrak{N}(\mathfrak{A}, \Gamma)$. So, by Lemma 11, there is a unique continuous linear mapping, Ψ_ω, from $\mathfrak{N}(\mathfrak{A}, \Gamma)$ into $\mathfrak{H}_\omega$ such that $\Psi_\omega(x) = \pi_\omega(x)\Omega_\omega$, for any $x \in \mathfrak{B}(\mathfrak{A}, \Gamma)$. Let x be an arbitrary element of $\mathfrak{B}(\mathfrak{A}, \Gamma)$, then, [by (22)] $y \mapsto \pi_\omega(x)\Psi_\omega(y)$ and $y \mapsto \Psi_\omega(x \cdot y)$ are both continuous on $\mathfrak{N}(\mathfrak{A}, \Gamma)$ and they coincide on $\mathfrak{B}(\mathfrak{A}, \Gamma)$; so we have: $\pi_\omega(x)\Psi_\omega(y) = \Psi_\omega(y \cdot x)$, $\forall y \in \mathfrak{N}(\mathfrak{A}, \Gamma)$. Let $x = x^*$ be an arbitrary hermitian element of $\mathfrak{A}$. Then

$x_n = x\left(\mathbb{1} + \frac{x^2}{n}\right)^{-1}$ is in $\mathfrak{B}(\mathfrak{A}, \Gamma)$, $\forall n \geqq 1$, and we have:

$$\phi(x^2) = \int t^2 \, d\mu_{x,\omega}(t)$$

and

$$\|\Psi_\omega(x_n) - \Psi_\omega(x_m)\|^2 = \int \left(\frac{t}{1 + \frac{t^2}{n}} - \frac{t}{1 + \frac{t^2}{m}} \right)^2 d\mu_{x,\omega}(t) \to 0 .$$

It follows that $\phi(x^2) = \|\Psi_\omega(x)\|^2$, $\forall x = x^* \in \mathfrak{A}$, and therefore we have (by polarization):

$$\phi(x^* y) = (\Psi_\omega(x) | \Psi_\omega(y)), \qquad \forall x, y \in \mathfrak{A} .$$

Let ψ be any positive linear form on the ordered space $\mathfrak{M}(\mathfrak{A}, \Gamma)$ such that $\psi(x) = \phi(x)$ if x is in $\mathfrak{A}$ and $\psi(x) = \omega(x)$ if x is in $\mathfrak{B}(\mathfrak{A}, \Gamma)$ (the existence of ψ follows from Theorem 5). Then we have $\psi(x^* y x) = \phi_x(y)$ if x, y are in $\mathfrak{A}$ and $\omega_x(y) = \psi(x^* y x)$ if $x \in \mathfrak{A}$ and $y \in \mathfrak{B}(\mathfrak{A}, \Gamma)$; but $y \mapsto \psi_x(y) = \psi(x^* y x)$ is again a positive linear form on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ (by Lemma 6a), and therefore (proceed as in the proof of Theorem 4) we have:

$$\psi_x(f(h)) = \int f(t)\, d\mu_{h,\omega_x}(t), \qquad \forall f \in \mathscr{P}(\mathbb{R}), \quad \forall h = h^* \in \mathfrak{A}.$$

So ω_x is a solution of the $m(\Gamma)$ problem for ϕ_x. □

8. Subspaces and Determination

Let us keep the hypothesis and notations of last section and let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$. In complete analogy with what is done in the study of the classical moment problem (see the definition of $\bar{\mu}$ and $\underline{\mu}$ in Ref. [2]), we associate to ϕ the following real functionals on $\mathfrak{M}^h(\mathfrak{A}, \Gamma)$:

$$(24) \qquad \begin{cases} x \mapsto \phi^*(x) = \operatorname{Inf}\{\phi(y) \mid y \in \mathfrak{A} \text{ and } y - x \in \mathscr{A}^+(\mathfrak{A}, \Gamma)\} \\ x \mapsto \phi_*(x) = \operatorname{Sup}\{\phi(y) \mid y \in \mathfrak{A} \text{ and } x - y \in \mathscr{A}^+(\mathfrak{A}, \Gamma)\}. \end{cases}$$

We have of course:

$$(25) \qquad \phi^*(x) \geqq \phi_*(x), \qquad \forall x \in \mathfrak{M}^h(\mathfrak{A}, \Gamma).$$

Proposition 4. *Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$ and let h be an arbitrary hermitian element of $\mathfrak{M}(\mathfrak{A}, \Gamma)$. Then, for any real number, r, such that $\phi_*(h) \leqq r \leqq \phi^*(h)$, there is a positive linear form on $\mathfrak{M}(\mathfrak{A}, \Gamma)$, ψ, such that we have: $\phi(x) = \psi(x)$, $\forall x \in \mathfrak{A}$, and $\psi(h) = r$.*

Proof. Consider, on the linear subspace $\mathfrak{L}$ of $\mathfrak{M}$ spanned by $\{x + \lambda h \mid x \in \mathfrak{A}$ and $\lambda \in \mathbb{C}\}$, the linear form ϕ_1 defined by: $\phi_1(x + \lambda h) = \phi(x) + \lambda r$, $\forall x \in \mathfrak{A}$ and $\forall \lambda \in \mathbb{C}$. Let $\mathfrak{L}^h = \mathfrak{A}^h + \mathbb{R}.\, h$ be the real subspace of $\mathfrak{L}$ of all the hermitian elements of $\mathfrak{L}$. Then it is easy to see that ϕ_1 is real-valued on $\mathfrak{L}^h$ and that $\phi^*(x + \varrho h) \geqq \phi(x) + \varrho r = \phi_1(x + \varrho h)$, for any hermitian $x \in \mathfrak{A}$ and for any real number ϱ. On the other hand, we have: $\phi^*(h_1 + h_2) \leqq \phi^*(h_1) + \phi^*(h_2)$, $\forall h_1, h_2 \in \mathfrak{M}^h(\mathfrak{A}, \Gamma)$ and, $\phi^*(\varrho h_1) = \varrho \phi^*(h_1)$, $\forall h_1 \in \mathfrak{M}^h(\mathfrak{A}, \Gamma)$ and $\forall \varrho \geqq 0$. Therefore, it again follows from Hahn-Banach theorem that there is a real linear form on $\mathfrak{M}^h(\mathfrak{A}, \Gamma)$, $\tilde{\phi}_1$, such that $\tilde{\phi}_1 \leqq \phi^*$ and $\tilde{\phi}_1(x) = \phi_1(x)$ for any $x \in \mathfrak{L}^h$. $\tilde{\phi}_1$ is positive on $\mathfrak{M}^+(\mathfrak{A}, \Gamma)$ (since $\tilde{\phi}_1 \leqq \phi^*$), and the linear form ψ on $\mathfrak{M}(\mathfrak{A}, \Gamma)$ defined by: $\psi(x) = \tilde{\phi}_1\left(\frac{x + x^*}{2}\right) + i\tilde{\phi}_1\left(\frac{x - x^*}{2i}\right)$ ($\forall x \in \mathfrak{M}$) satisfies the conditions of Proposition 4. □

Definition 5. Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$ and let V be a *-invariant subspace of $\mathfrak{B}(\mathfrak{A}, \Gamma)$. Suppose that all the solutions of the $m(\Gamma)$-problem for ϕ coincide on V; then we say that the $m(\Gamma)$-problem for ϕ is *determined on V.* If $V = \mathfrak{B}(\mathfrak{A}, \Gamma)$ we simply say that it is *determined.*

As an immediate corollary of Proposition 4, we have the following result.

Corollary 3. *Let ϕ and V be as in Definition 5. Then the $m(\Gamma)$-problem for ϕ is determined on V if and only if we have:*

$$\phi^*(h) = \phi_*(h), \qquad \forall h = h^* \in V.$$

Practically V will be a C^*-subalgebra of $\mathfrak{B}(\mathfrak{A}, \Gamma)$. For instance if the objects of interest form a system of hermitian generators of $\mathfrak{A}$, it is natural to restrict our attention on the C^*-subalgebra generated by the continuous functions vanishing at infinity of these hermitian generators. This will typically be the case in quantum field theory (see the next section).

9. First Application to Quantum Field Theory

A. The Localizable Algebra

Let M be the space $\mathbb{R}^{s+1}$ equipped with the bilinear form

$$(x, y) = x^0 y^0 - \sum_{k=1}^{k=s} x^k y^k = x^0 y^0 - \vec{x} \cdot \vec{y}\,.$$

Let $\mathscr{D} = \mathscr{D}(M)$ be the Schwartz' space of complex C^∞ functions with compact supports on M equipped with its usual topology [14] and let $\mathscr{D}' = \mathscr{D}'(M)$ be the topological dual space of $\mathscr{D}$ that is the space of distributions on M. $\mathscr{D}$ is an involutive vector space with the continuous involution $g \mapsto g^*$ defined by $g^*(x) = \overline{g(x)}$, $(\forall x \in M)$. The tensor algebra $T(\mathscr{D})$ over $\mathscr{D}$ is canonicaly a *-algebra with a unit. Let $\Gamma_{\mathscr{D}'}$ be (as in Section 4) the set of all C^*-semi-norms on $T(\mathscr{D})\, y \mapsto \|\pi(y)\|$ where π runs over the matrix representations of $T(\mathscr{D})$ with coefficients in $\mathscr{D}'$. $\mathscr{D}'$ separates $\mathscr{D}$ so (by Theorem 2), any positive linear form on the *-algebra $T(\mathscr{D})$ is $\Gamma_{\mathscr{D}'}$-strongly positive. Therefore (by Theorem 4), the $m(\Gamma_{\mathscr{D}'})$-problem for a positive linear form on $T(\mathscr{D})$ is always soluble.

Let $\mathscr{F}_b$ be the family of bounded open subsets of M and for any $\mathcal{O} \in \mathscr{F}_b$ let $\mathfrak{B}(\mathcal{O})$ denotes the C^*-subalgebra of $\mathfrak{B}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$ generated by the family $f(h)$ where $f \in \mathscr{C}_{(0)}(\mathbb{R})$ and where h runs over the hermitian elements of $\mathscr{D}$ [considered as a subset of the *-algebra $T(\mathscr{D})$] with supports in $\mathcal{O}$ [$f(h)$ is defined in Section 6, Definition 2]. We have (by construction):

$$\text{(26)} \qquad \mathcal{O}_1, \mathcal{O}_2 \in \mathscr{F}_b \quad \text{and} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{B}(\mathcal{O}_1) \subset \mathfrak{B}(\mathcal{O}_2)$$

and $\mathfrak{B}(\mathcal{O}_1)$ is a C^*-subalgebra of $\mathfrak{B}(\mathcal{O}_2)$.

Definition 6. A) $\bigcup_{\mathcal{O} \in \mathscr{F}_b} \mathfrak{B}(\mathcal{O})$ is canonically a normed *-algebra which will be called the *localizable algebra* and will be denoted by $\mathfrak{B}(M)$. The completion $\hat{\mathfrak{B}}(M)$ of $\mathfrak{B}(M)$ is the C^*-subalgebra of $\mathfrak{B}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$ generated by $\mathfrak{B}(M)$. $\hat{\mathfrak{B}}(M)$ will be called the *quasi-localizable* C^*-algebra.

B) A *-algebra automorphism of $\mathfrak{B}(M)$, α, will be called an automorphism of $\mathfrak{B}(M)$ iff there is a permutation φ_α of $\mathscr{F}_b$ such that, for any $\mathcal{O} \in \mathscr{F}_b$, the restriction of α to $\mathfrak{B}(\mathcal{O})$ is a *-isomorphism of $\mathfrak{B}(\mathcal{O})$ on $\mathfrak{B}(\varphi_\alpha(\mathcal{O}))$.

The set of all these automorphisms of $\mathfrak{B}(M)$ forms a group which will be denoted by $\mathrm{Aut}(\mathfrak{B}(M))$.

It follows that any element of $\mathrm{Aut}(\mathfrak{B}(M))$ is isometric and therefore it has a unique extension to $\hat{\mathfrak{B}}(M)$ which is an automorphism of the C^*-algebra $\hat{\mathfrak{B}}(M)$.

$\mathrm{Aut}(\mathfrak{B}(M))$ will be identified with the corresponding subgroup of the group $\mathrm{Aut}(\hat{\mathfrak{B}}(M))$ of all the automorphisms of the C^*-algebra $\hat{\mathfrak{B}}(M)$.

We denote the group of all diffeomorphisms[4] of M by $\mathrm{Diff}(M)$. Remembering that $\mathrm{Diff}(M)$ operates (to the left) on $\mathscr{D}(M)$ by $h(x) \mapsto h_\varphi(x) = h(\varphi^{-1}(x))$ and that $(h^*)_\varphi = (h_\varphi)^*$, $\forall \varphi \in \mathrm{Diff}(M)$, we have the following:

Proposition 5. *There is a unique group homomorphism* α, *from* $\mathrm{Diff}(M)$ *into* $\mathrm{Aut}(\mathfrak{B}(M))$ *such that* $\alpha(\varphi)[f(h)] = f(h_\varphi)$, $\forall \varphi \in \mathrm{Diff}(M)$, $\forall h = h^* \in \mathscr{D}(M)$ *and* $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$. *Then we have:*

$$\alpha(\varphi)\,\mathfrak{B}(\mathcal{O}) = \mathfrak{B}(\varphi(\mathcal{O})), \qquad \forall \mathcal{O} \in \mathscr{F}_b, \quad \forall \varphi \in \mathrm{Diff}(M).$$

Proof. Since $h \mapsto h_\varphi$ is linear and satisfies $(h^*)_\varphi = (h_\varphi)^*$ there is a unique $*$-homomorphism $\tilde{\alpha}(\varphi)$ of $T(\mathscr{D})$ into itself such that $\tilde{\alpha}(\varphi)[\mathbb{1}] = \mathbb{1}$ and $\tilde{\alpha}(\varphi)[h] = h_\varphi$, $\forall h \in \mathscr{D}$. It is not very hard to see that $\tilde{\alpha}(\mathrm{Id}_M) = \mathrm{Id}_{T(\mathscr{D})}$ and that $\tilde{\alpha}(\varphi_1 \circ \varphi_2) = \tilde{\alpha}(\varphi_1) \circ \tilde{\alpha}(\varphi_2)$, $\forall \varphi_1, \varphi_2 \in \mathrm{Diff}(M)$. Remembering that, for any diffeormorphism φ, the mapping $h \mapsto h_\varphi$ is continuous for the usual topology of $\mathscr{D}$; it follows that if π is a matrix representation of $T(\mathscr{D})$ with coefficients in $\mathscr{D}'$ then the same is true for $\pi \circ \tilde{\alpha}(\varphi)$, $\forall \varphi \in \mathrm{Diff}(M)$. This implies that $\tilde{\alpha}(\varphi)$ is continuous for the topology $\mathscr{T}_{\Gamma_{\mathscr{D}'}}$ (see Section 2) and therefore $\tilde{\alpha}(\varphi)$ has a unique continuous extension $\hat{\alpha}(\varphi)$ which is a $*$-homomorphism of $\mathscr{A}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$ into itself. By uniqueness we have:

$$\hat{\alpha}(\mathrm{Id}_M) = \mathrm{Id}_{\mathscr{A}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})} \quad \text{and} \quad \hat{\alpha}(\varphi_1 \circ \varphi_2) = \hat{\alpha}(\varphi_1) \circ \hat{\alpha}(\varphi_2), \qquad \forall \varphi_1, \varphi_2 \in \mathrm{Diff}(M).$$

Let x be an arbitrary element of $\mathfrak{B}_\infty(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$ and let φ be an element of $\mathrm{Diff}(M)$. Then, for any $p \in \Gamma_{\mathscr{D}'}$, $\|\pi_p(\hat{\alpha}(\varphi)(x))\| \leqq \sup_{p' \in \Gamma_{\mathscr{D}'}} \|\pi_{p'}(x)\| = \|x\|$; So, $\hat{\alpha}(\varphi)(x) \in \mathfrak{B}_\infty(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$.

Let $\alpha(\varphi)$ be the restriction of $\hat{\alpha}(\varphi)$ to $\mathfrak{B}(M)$. Then $\alpha(\varphi)\mathfrak{B}(\mathcal{O}) = \mathfrak{B}(\varphi(\mathcal{O}))$ follows from the definitions, and $\varphi \mapsto \alpha(\varphi)$ satisfies all the conditions stated in the proposition. The uniqueness of α follows from the fact that the $\{f(h)\}$ generate a dense subalgebra of $\mathfrak{B}(M)$ and that an automorphism of $\mathfrak{B}(M)$ is continuous (for the norm topology). $\square$

Remark 8. Up to now, nothing is changed if we replace M by an arbitrary (finite dimensional C^∞) differentiable manifold.

Let a be an element of M and let t_a the corresponding translation ($t_a \in \mathrm{Diff}(M)$). Then we simply write h_a instead of h_{t_a} to denote the function $x \mapsto h(x-a)$ and similarly, $\alpha(t_a)$ will simply be denoted by $\alpha_a (\in \mathrm{Aut}(\mathfrak{B}(M)))$.

Let S_1 and S_2 be two subsets of M. Then we say that S_1 and S_2 are *space-like* separated or that S_1 is *space-like separated with respect to* S_2 iff. $(x_1 - x_2, x_1 - x_2) < 0$ (strict), $\forall x_1 \in S_1$ and $\forall x_2 \in S_2$.

B. Quantum Field with Quasi-analytic Vacuum

Let S be a symmetric operator in a Hilbert space $\mathfrak{H}$ and let $\Omega \in \mathfrak{H}$ be a vector which belongs to $\bigcap_{n \geqq 1} \mathrm{dom}(S^n)$ [$\mathrm{dom}(A)$ denotes the domain of an operator A

[4] By diffeomorphism, we mean here, a bijection which is C^∞ and which has a C^∞ inverse.

acting in $\mathfrak{H}$]. According to Nussbaum [15], we say that Ω is a *quasi-analytic vector for* S iff. Ω is such that

(27) $$\sum_{n=1}^{\infty} \|S^n\Omega\|^{-\frac{1}{n}} = \infty .$$

An analytic vector for S, [16], is, of course, a quasi-analytic vector for S, but the converse proposition is not true. Let us state a theorem of Nussbaum [15] (Theorem 2 in that paper).

Nussbaum's Theorem A. *Let S be a closed symmetric operator in a Hilbert space $\mathfrak{H}$. Then S is self-adjoint if and only if S has a total set of quasi-analytic vectors.*

In other words, this theorem is Nelson's theorem [16], where "analytic vectors" is replaced by "quasi-analytic" "vectors".

Another theorem of Nussbaum (Theorem 5 of Ref. [15]) will be applied in this section in the following form.

Nussbaum's Theorem B. *Let S and T be symmetric operators in a Hilbert space $\mathfrak{H}$ and let D be the set of all vectors in $\mathfrak{H}$ which are quasi-analytic for both S and T and which are in the domain of the operators T^nS^m, S^mT^n, for $n, m = 1, 2, \ldots$ and such that $(T^nS^m - S^mT^n)D = 0$, $\forall n, m \geqq 1$. If D is dense in $\mathfrak{H}$, then S and T are essentially selfadjoint and the spectral resolutions of their closures commute.*

Let A be an operator in the Hilbert space $\mathfrak{H}$ and assume that $\mathrm{dom}(A)$ is dense in $\mathfrak{H}$. Then we denote the adjoint of A by A^+. Let D be a dense subspace of $\mathfrak{H}$ and let $\mathscr{E}^*(D)$ be the set of all operators in $\mathfrak{H}$, A, such that we have:

(28) $$\mathrm{dom}(A) = D, \ \mathrm{dom}(A^+) \supset D \quad \text{and} \quad AD \subset D, \ A^+D \subset D .$$

$\mathscr{E}^*(D)$ is a *-algebra (with unit) if we equip it with the involution defined by

(29) $$A \mapsto A^* = (A^+ \upharpoonright D),$$

where $\upharpoonright D$ means the restriction to D.

Now, we consider a scalar neutral field, $A(h)$, satisfying the usual assumptions [17]. Namely, we have a strongly continuous unitary representation of the translation group in the Hilbert space $\mathfrak{H}$, $M \ni a \mapsto U(a) = \int_M e^{i(p,a)} dE(p)$ (by Stone theorem), a dense subspace D of $\mathfrak{H}$ such that $U(a)D \subset D$, $\forall a \in {}'M$, and a linear mapping $h \mapsto A(h)$ of $\mathscr{D}(M)$ into $\mathscr{E}^*(D)$ such that

a) $A(h^*) = A(h)^*$ in $\mathscr{E}^*(D)$, $\forall h \in \mathscr{D}(M)$,

b) $U(a)A(h)$, $U(a)^{-1} = A(h_a)$, $\forall h \in \mathscr{D}$, where $h_a(x) = h(x-a)$,

c) $\mathrm{Supp}(dE(p)) \subset \bar{V}_+ = \{p \mid p^0 \geqq 0, (p, p) \geqq 0\}$,

d) $\exists \Omega \in D$ with $U(a)\Omega = \Omega$, $\forall a \in M$,

e) D is the linear hull of $A(h_1) \ldots A(h_n)\Omega$, $n \geqq 0$ where the h_k run over $\mathscr{D}(M)$,

f) $(A(h_1)A(h_2) - A(h_2)A(h_1)) = 0$, $\forall h_1, h_2 \in D$ such that $(x-y, x-y) < 0$

$$\forall x \in \mathrm{Supp}(h_1) \quad \text{and} \quad \forall y \in \mathrm{Supp}(h_2) .$$

a) is the hermiticity condition, b) is the translation invariance, c) is the spectrum condition, d) is the existence of a "vacuum" Ω, e) is the cyclicity of the vacuum Ω + a specification of the domain D, f) is the local commutativity.

A is a linear mapping of $\mathscr{D}$ into the *-algebra with unit $\mathscr{E}^*(D)$ such that $A(h^*)=A(h)^*$, $\forall h \in \mathscr{D}$. It follows that (see Section 4) there is a unique *-homomorphism again denoted by A from $T(\mathscr{D})$ into $\mathscr{E}^*(D)$ such that $A(1)=1$ and which extends A [$\mathscr{D}$ is a subspace of the *-algebra $T(\mathscr{D})$]. Then, e) may be written in the form $D=A(T(\mathscr{D}))\Omega$. Furthermore, as well known [6, 17], everything is determined up to a unitary by the knowledge of the positive linear form $\mathfrak{W}$ on $T(\mathscr{D}(M))$ defined by:

$$\text{(30)} \qquad \mathfrak{W}(h_1 \otimes \cdots \otimes h_n) = (\Omega | A(h_1) \dots A(h_n)\Omega), \qquad \forall h_k \in \mathscr{D}(M), \quad \forall n \geqq 0 .$$

Theorem 6. *Let A be a field theory satisfying the assumption* a) *to* f) *(above). Suppose that the vacuum Ω is a quasi-analytic vector for each $A(h)$, $\forall h=h^* \in \mathscr{D}(M)$. Then we have:*

a) *$\forall h=h^* \in \mathscr{D}(M)$, $A(h)$ is essentially self-adjoint and if $h_1=h_1^*$ and $h_2=h_2^*$ are in $\mathscr{D}(M)$ and have space-like separated supports then the spectral resolutions of $\overline{A(h_1)}$ and $\overline{A(h_2)}$ commute.*

b) *The $m(\Gamma_{\mathscr{D}'})$-problem for $\mathfrak{W}$ is determined on the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ ($\subset \mathfrak{B}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$) and we have:*

$$\omega(f_1(h_1) \dots f_n(h_n)) = (\Omega | f_1(\overline{A(h_1)}) \dots f_n(\overline{A(h_n)})\Omega)^{5}, \qquad \forall n \geqq 0$$

$$\forall f_1, \dots, f_n \in \mathscr{C}_{(0)}(\mathbb{R}), \quad \forall h_1 = h^*, \dots, h_n = h_n^* \in \mathscr{D}(M)$$

and where ω is the unique positive linear form on $\hat{\mathfrak{B}}(M)$ obtained by restriction to $\hat{\mathfrak{B}}(M)$ of a solution of the $m(\Gamma_{\mathscr{D}'})$-problem for $\mathfrak{W}$.

c) *The corresponding (unique) representation π of $\mathfrak{B}(M)$ in $\mathfrak{H}$ such that $\pi(f(h)) = f(\overline{A(h)})$, $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$ and $\forall h = h^* \in \mathscr{D}(M)$, satisfies:*

$$\begin{cases} \pi(\mathfrak{B}(\mathscr{O}))'' = R(\mathscr{O}), \forall \mathscr{O} \in \mathscr{F}_b \\ \pi(\alpha_a(x)) = U(a)\pi(x)\, U(a)^{-1} \qquad \forall a \in M . \end{cases}$$

$\forall x \in \mathfrak{B}(M)$ where $R(\mathscr{O})$ is the von Neumann algebra generated by the spectral resolutions of the family $\overline{A(h)}$ with $h=h^* \in \mathscr{D}(M)$ such that $\operatorname{supp}(h) \subset \mathscr{O}$.

Proof. a) The proof of a) is the same that the proof given by Borchers and Zimmermann in the case of an analytic vacuum [4] except that Nelson's theorem has to be replaced by Nussbaum's Theorem A. Here we shall directly use Nussbaum's Theorem B combined with Reeh-Schlieder theorem [18]. Let $h_1 = h_1^* \in \mathscr{D}$ and $h_2 = h_2^* \in \mathscr{D}$ be such that $x_1 \in \operatorname{Supp}(h_1)$ and $x_2 \in \operatorname{Supp}(h_2)$ imply $(x_1 - x_2, x_1 - x_2) < 0$ and let $\mathscr{O}$ be a no empty open subset of M which is space-like separated with respect to $\operatorname{Supp}(h_1) \cup \operatorname{Supp}(h_2)$. Then the linear hull $D(\mathscr{O}) \subset D$ of the vectors $A(h'_1) \dots A(h'_n)\Omega$, $n=1,2,3,\dots$ where the h'_k are functions of $\mathscr{D}(M)$ with

[5] As usual if S is self-adjoint ($S=S^+$), $f(S)$ is defined by $f(S) = \int f(t)\, dE(t)$ where $E(t)$ is the spectral resolution of S.

supports in $\mathcal{O}$, is dense in $\mathfrak{H}$ (by Reeh-Schlieder theorem). We have for $\alpha=1$ or $\alpha=2$:

$$\|A(h_\alpha)^m A(h'_1)\dots A(h'_n)\Omega\|^2 = (A(h'_1)\dots A(h'_n)\Omega \mid A(h_\alpha)^{2m} A(h'_1)\dots A(h'_n)\Omega)$$
$$= (A(h'^*_n)\dots A(h'^*_1)A(h'_1)\dots A(h'_n)\Omega \mid A(h_\alpha)^{2m}\Omega) \leqq K\,.\;\; \|A(h_\alpha)^{2m}\Omega\|\,,$$

where we used $h^*_\alpha = h_\alpha$, assumptions a) and f) and Schwartz inequality ($K = \|A(h'^*_n)\dots A(h'^*_1)A(h'^*_1)\dots A(h'_n)\Omega\|$). In the same way we have more generally:

$$\|A(h_\alpha)^m \Phi\|^2 \leqq K(\Phi).\;\; \|A(h_\alpha)^{2m}\Omega\|, \text{ for } \alpha=1 \text{ or } \alpha=2, \quad \forall \Phi \in D(\mathcal{O})\,.$$

Therefore, $\forall \Phi \in D(\mathcal{O})$, we have for $\alpha=1$ or $\alpha=2$:

$$\sum_n \|A(h_\alpha)^n \Phi\|^{-1/n} \geqq C^{te} \times \sum_n \|A(h_\alpha)^{2n}\Omega\|^{-1/2n} = \infty\,,$$

since the sequence $u_n = \|A(h_\alpha)^n \Omega\|^{1/n}$ is an increasing positive sequence (and therefore $\Sigma u_n^{-1} = \infty \Leftrightarrow \Sigma u_{2n}^{-1} = \infty$). It follows that $D(\mathcal{O})$ is a dense set (in $\mathfrak{H}$) of quasi-analytic vectors for both $A(h_1)$ and $A(h_2)$ and, since $D(\mathcal{O}) \subset D$ and since $\operatorname{supp}(h_1)$ and $\operatorname{supp}(h_2)$ are space-like separated, $D(\mathcal{O}) \subset \operatorname{dom}(A(h_1)^n A(h_2)^m)$, $D(\mathcal{O}) \subset \operatorname{dom}(A(h_2)^m A(h_1)^n)$ and $(A(h_1)^n A(h_2)^m - A(h_2)^m A(h_1)^n) D(\mathcal{O}) = 0$, $\forall n, m \geqq 1$; so, applying Nussbaum's Theorem B, we immediatly obtain the statement a) of the theorem.

b) Let ω_1 be an arbitrary solution of the $m(\Gamma_{\mathscr{D}'})$-problem for $\mathfrak{W}$, let $(\pi_{\omega_1}, \mathfrak{H}_{\omega_1}, \Omega_{\omega_1})$ be the cyclic *-representation π_{ω_1} of $\mathfrak{B}(T(\mathscr{D}), \Gamma_{\mathscr{D}'})$ in the Hilbert space $\mathfrak{H}_{\omega_1}$ with cyclic vector Ω_{ω_1} associated with ω_1 by G.N.S. construction and let Ψ_{ω_1} be defined as in the Proposition 3. Then $A(x)\Omega \mapsto \Psi_{\omega_1}(x)$ defines an isomorphism of the separated prehilbertian space $A(T(\mathscr{D}))\Omega$ on $\Psi_{\omega_1}(T(\mathscr{D}))$ (by Proposition 3). Therefore, we may identify $\mathfrak{H}$ with the closure of $\Psi_{\omega_1}(T(\mathscr{D}))$ in $\mathfrak{H}_{\omega_1}$ in such a way that we have:

$$\text{(31)} \qquad \begin{cases} A(x)\Omega = \Psi_{\omega_1}(x), & \forall x \in T(\mathscr{D})\,, \\ \text{so,} \quad \Omega = \Omega_{\omega_1}\,. \end{cases}$$

With these identifications, we have:

$$\text{(32)} \qquad \pi_{\omega_1}(x) A(y)\Omega = \Psi_{\omega_1}(xy)\,, \quad \forall x \in \mathfrak{B}(T(\mathscr{D}), \Gamma_{\mathscr{D}'}),\ \forall y \in T(\mathscr{D})\,.$$

Let $\mathcal{O}$ be a bounded open subset of $M(\mathcal{O} \in \mathscr{F}_b)$ and let $\mathcal{O}'$ be any no empty open subset of M which is space-like separated with respect to $\mathcal{O}$. If $h = h^* \in \mathscr{D}$ has his support in $\mathcal{O}$, then any $\Phi \in D(\mathcal{O}')$ is an analytic vector for $A(h)$. This means that the positive linear form on $\mathbb{C}[X]$ defined by $P(X) \mapsto (\Phi \mid P(A(h))\Phi)$ satisfies the condition of Carleman's theorem (see Section 5), so the classical (Hamburger's) moment problem for this positive linear form on $\mathbb{C}[X]$ is determined. Remembering that if μ is a bounded measure on $\mathbb{R}$ which is solution of a determined Hamburger's problem, then the polynomials are dense in $L^2(d\mu)$; it follows that $\{\pi_{\omega_1}(f(h))\Phi \mid f \in \mathscr{C}_{(0)}(\mathbb{R})\}$ is a dense subspace of the closure of $\{A(P(h))\Phi \mid P(X) \in \mathbb{C}[X]\}$ in $\mathfrak{H}$. This implies that $\pi_{\omega_1}(f(h)) \restriction \mathfrak{H}$ is unique and that $\pi_{\omega_1}(f(h))\mathfrak{H} \subset \mathfrak{H}$, $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$ [since $D(\mathcal{O}')$ is dense in $\mathfrak{H}$ and $\pi_{\omega_1}(f(h))$ is bounded]. Furthermore, $A(h)$ is essentially self adjoint on $D(\mathcal{O}')$ so we must have: $\pi_{\omega_1}(f(h)) = f(\overline{A(h)})$.

It follows that $\pi_{\omega_1}(x) \upharpoonright \mathfrak{H}$ is unique for any x in $\hat{\mathfrak{B}}(M)$, ($\hat{\mathfrak{B}}(M)$ is generated by the $f(h)$), and is a bounded operator in $\mathscr{L}(\mathfrak{H})$. This implies that the restriction ω of ω_1 to $\hat{\mathfrak{B}}(M)$ is unique and we have:

$$\omega(f_1(h_1)\dots f_n(h_n)) = (\Omega \,|\, \pi_{\omega_1}(f_1(h_1)) \dots \pi_{\omega_1}(f_n(h_n))\Omega)$$
$$= (\Omega \,|\, f_1(\overline{A(h_1)}) \dots f_n(\overline{A(h_n)})\Omega), \qquad \forall h_k = h_k^* \in \mathscr{D}, \ \forall f_k \in \mathscr{C}_{(0)}(\mathbb{R}).$$

c) The last part of the theorem follows immediately from above and from the fact that if h is an arbitrary real function of $\mathscr{D}(M)$, $(h = h^*)$ then the spectral resolutions of $\overline{A(h)}$ and $\overline{A(h_a)}$, $E(\lambda)$ and $E_a(\lambda)$, must be connected by:

$$E_a(\lambda) = U(a)\, E(\lambda)\, U(a)^{-1}, \qquad (\forall a \in M). \quad \square$$

Remark 9. a) The choice of $\mathscr{D}$ as space of test functions and $\Gamma_{\mathscr{D}'}$ as directed set of C^*-semi-norms on $T(\mathscr{D})$ is not essential. Other choices work as well. However the Proposition 5 must be slightly modified.

b) On most *-algebras (with units) generated by fields operators there are no C^*-semi-norms at all. Nevertheless, the Theorem 6 shows that the non-commutative moment problem, lifted to the tensor algebra, gives the local rings (at least, for the free field)[6].

10. Conclusion

In this paper we give the formulation of a generalization of the classical moment problem on *-algebras (the m-problem). In a forthcoming paper we shall describe (and apply) some important properties of this construction. In particular we shall deal with the connexion between the m-problem and self-adjointeness properties of operators in Hilbert space.

There exist other generalizations of the classical moment problem on *-algebras. For instance, the problem of integral decomposition of states on *-algebras is a very natural generalization of the classical moment problem. A recent work by Borchers and Yngvason deals with this problem [19]. In this paper, it is the measure itself which is replaced by something else (namely a positive linear from on a suitable C^*-algebra). At this point, it is worth noticing that Segal gave a non-commutative generalization of integration theory in reference [20]. However, the positive linear forms ω on the C^*-algebra $\mathfrak{B}(\mathfrak{A})$ which are solutions of m-problems are generally not central forms (or traces) on $\mathfrak{B}(\mathfrak{A})$; so $(\mathfrak{B}(\mathfrak{A}), \omega)$ is not an integration algebra in the sense of reference [21] when ω is a solution of some m-problem on $\mathfrak{A}$. There exist other (distinct) non-commutative problems of moments; for instance, in Ref. [22], a "quantum problem of moments" is introduced in order to study the representations of the *-algebra generated by Heisenberg canonical commutation relations (= the envelopping algebra of the Heisenberg Lie algebra).

[6] This remark has been suggested by a comment made by R. T. Powers (private communication).

The whole construction given in this paper is based on the properties of C^*-semi-norms. It is worth noticing that, in Ref. [23], Borchers had shown that the properties of the continuous C^*-semi-norms on the algebra of test functions for quantum fields may be used to prove various usefull results.

Let us end this paper with some remarks on the "localizable systems" which represent the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ (this situation may be easily generalized). According to the analysis of Haag and Kastler [24], the corresponding "algebraic systems" (= class of physically equivalent systems) are in one to one correspondance with primitive ideals of $\hat{\mathfrak{B}}(M)$. On the other hand, if $P(\hat{\mathfrak{B}}(M))$ denotes the set of pure states on $\hat{\mathfrak{B}}(M)$ equipped with the weak topology and if $\mathrm{Prim}(\hat{\mathfrak{B}}(M))$ denotes the set of primitive ideals of $\hat{\mathfrak{B}}(M)$ equipped with Jacobson topology, then the canonical mapping of $P(\hat{\mathfrak{B}}(M))$ on $\mathrm{Prim}(\hat{\mathfrak{B}}(M))$ is continuous and open [8]. It follows that the Jacobson topology has something to do with the notion of approximation of an algebraic system by another algebraic system. Notice also that if $\mathfrak{I} \in \mathrm{Prim}(\hat{\mathfrak{B}}(M))$, the corresponding algebraic system is translation invariant if and only if $\mathfrak{I}$ is translation invariant; it is a local system if and only if $\mathfrak{I}$ contains $xy - yx$ whenever $x \in \mathfrak{B}(\mathcal{O})$ and $y \in \mathfrak{B}(\mathcal{O}')$ where $\mathcal{O}$ and $\mathcal{O}'$ are space-like separated bounded open regions.

Acknowledgements. It is a pleasure to thank H. Epstein for his interest in this work and for helpfull suggestions. The author is also indebted to H. J. Borchers and J. Yngvason for stimulating discussions.

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Communicated by H. Araki

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