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A Generalization of the Erdős–Szekeres Theorem to Disjoint Convex Sets*

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Abstract. Let $\mathcal{F}$ denote a family of pairwise disjoint convex sets in the plane. $\mathcal{F}$ is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others. For any fixed $k \geq 3$, we estimate $P_k(n)$, the maximum size of a family $\mathcal{F}$ with the property that any k members of $\mathcal{F}$ are in convex position, but no n are. In particular, for $k = 3$, we improve the triply exponential upper bound of T. Bisztriczky and G. Fejes Tóth by showing that $P_3(n) < 16^n$.

1. Introduction

In their classical paper [ES1], Erdős and Szekeres proved that any set of more than $\binom{2n-4}{n-2}$ points in general position in the plane contains n points which are in convex position, i.e., they form the vertex set of a convex n-gon. Bisztriczky and Fejes Tóth [BF1], [F] extended this result to families of convex sets.

Throughout this paper, by a *family* $\mathcal{F} = \{A_1, \ldots, A_t\}$ we always mean a family of pairwise disjoint compact convex sets in the plane in *general position*, i.e., no three of them have a common supporting line. $\mathcal{F}$ is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others, i.e., if $\operatorname{bd}\operatorname{conv}(\bigcup \mathcal{F})$,

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the boundary of the convex hull of the union of all members of $\mathcal{F}$, contains a piece of the boundary of each A_i. Evidently, any two members of $\mathcal{F}$ are in convex position.

Bisztriczky and Fejes Tóth proved that there exists a function $P(n)$ such that if $|\mathcal{F}| > P(n)$ and any *three* members of $\mathcal{F}$ are in convex position, then $\mathcal{F}$ has n members in convex position. Improving their initial result, in [BF2], they showed that this statement is true with a function $P(n)$, triply exponential in n. They also remarked that "it seems that none of the" previous proofs of the Erdős–Szekeres theorem "can be modified so as to obtain a proof of our theorem." One of the aims of the present paper is to show that the idea of the original proof of Erdős and Szekeres can be applied to deduce the Bisztriczky–Fejes Tóth theorem with a much better function $P(n) < 16^n$.

Theorem 1. *Let $\mathcal{F}$ be a family of n pairwise disjoint compact convex sets in the plane, any three of which are in convex position. If*

$$|\mathcal{F}| > \binom{2n-4}{n-2}^2,$$

then $\mathcal{F}$ has n members in convex position.

If any k members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P_k. If no n members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P^n. *Property* P_k^n means that both P_k and P^n are satisfied. Using these notions, Theorem 1 states that if a family $\mathcal{F}$ satisfies property P_3^n, then $|\mathcal{F}| \leq \binom{2n-4}{n-2}^2$.

Bisztriczky and Fejes Tóth [BF2] raised the following more general question. What is the maximum size $P_k(n)$ of a family $\mathcal{F}$ satisfying property P_k^n? They gave an exponential upper bound on $P_4(n)$, and quadratic upper bounds on $P_k(n)$ for any fixed $k \geq 5$, as n tends to infinity. Some of these estimates can be improved as follows:

Theorem 2. $2\lfloor \frac{n+1}{4} \rfloor^2 \leq P_4(n) < n^3$.

Theorem 3. $P_{11}(n) \leq cn \log n$.

Obviously, $P_l(n) \leq P_k(n)$ holds for every $l \geq k$.

2. Proof of Theorem 1

The combinatorial seed of the original proof of the Erdős–Szekeres theorem was isolated and generalized by Chvátal and Komlós. A complete graph, whose edges are arbitrarily oriented, is called a *tournament*. An acyclic tournament is said to be *transitive*.

Lemma 2.1 [CK]. *Let T be a transitive tournament with more than $\binom{2n-4}{n-2}$ vertices, and let f be any real-valued function defined on its edge set.*

Then there is an oriented path $v_1 v_2 \cdots v_n$ with n vertices such that the sequence $f(\overrightarrow{v_1 v_2}), f(\overrightarrow{v_2 v_3}), \ldots, f(\overrightarrow{v_{n-1} v_n})$ is either monotone increasing or strictly decreasing.

We use this statement to establish the following result, whose part (ii) was proved in [BF2].

Lemma 2.2. *Let $\mathcal{F}$ be a family of compact convex sets in the plane, satisfying property P_3^n, and at least one of the two following conditions*:

(i) *any two members of $\mathcal{F}$ can be separated by a vertical line*; *and*
(ii) *there is a line intersecting all members of $\mathcal{F}$.*

Then $\mathcal{F}$ has at most $t = \binom{2n-4}{n-2}$ members.

Proof. In case (ii), we can assume without loss of generality that the common transversal of the elements of $\mathcal{F}$ is horizontal.

Let $A_1, A_2, \ldots, A_t$ be the members of $\mathcal{F}$ listed from left to right (with respect to their projections onto the x-axis in case (i), and with respect to their intersections with the common transversal in case (ii)). For any $1 \leq i < j \leq t$, there are four uniquely determined points $p_1, q_1 \in \operatorname{bd} A_i$; $p_2, q_2 \in \operatorname{bd} A_j$ such that the segments $p_1 p_2, q_1 q_2$ belong to the boundary of $\operatorname{conv}(A_i \cup A_j)$, and along this boundary the counterclockwise order of these points is p_2, p_1, q_1, q_2. Let $f(i, j)$ and $g(i, j)$ denote the counterclockwise angles from the direction of the positive x-axis to $\overrightarrow{p_2 p_1}$ and $\overrightarrow{q_2 q_1}$, respectively (see Fig. 1).

Since $\mathcal{F}$ satisfies property P_3, for any $i < j < k$ with $f(i, j) \leq f(j, k)$, we have $g(i, j) < g(j, k)$.

Define a transitive tournament with vertices $v_1, v_2, \ldots, v_t$, such that every edge is oriented toward its endpoint of larger index. For any $i < j$, assign to the edge $\overrightarrow{v_i v_j}$ the value $f(i, j)$. By Lemma 2.1, if $t > \binom{2n-4}{n-2}$, then there is a directed path $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ such that either

$$f(i_1, i_2) \leq f(i_2, i_3) \leq \cdots \leq f(i_{n-1}, i_n)$$

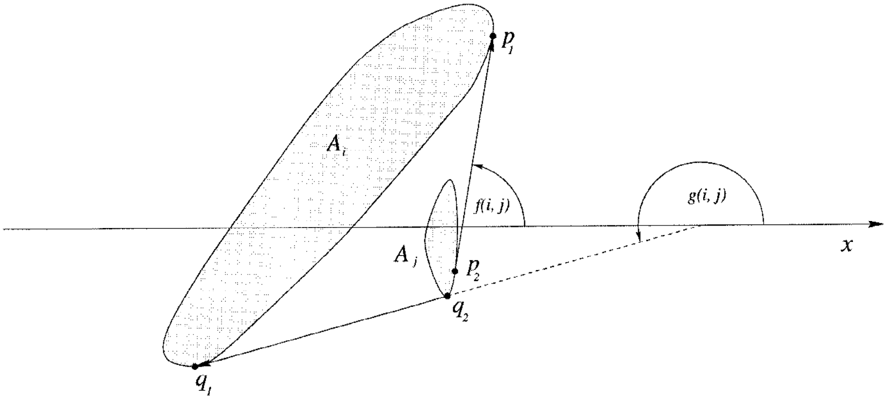

Fig. 1

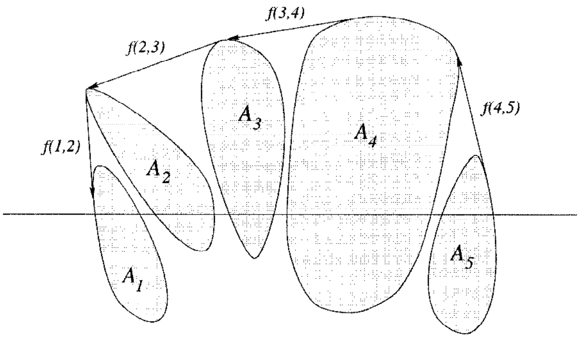

Fig. 2. $f(1,2) > f(2,3) > f(3,4) > f(4,5)$.

or

$$f(i_1, i_2) > f(i_2, i_3) > \cdots > f(i_{n-1}, i_n).$$

In both cases, it is easy to verify that $(A_{i_1}, A_{i_2}, \ldots, A_{i_n})$ are in convex position (see Fig. 2). □

Now we are ready to prove Theorem 1. Let $\mathcal{F}$ be a family of more than $\binom{2n-4}{n-2}^2$ convex sets in the plane satisfying property P_3. Projecting these sets onto the x-axis, we obtain a system of intervals $\mathcal{I}$. A well-known result by Gallai (see [B, p. 373]) implies that $\mathcal{I}$ has more than $\binom{2n-4}{n-2}$ elements that are either pairwise disjoint or all of them have a point in common. In the first case, the corresponding elements of $\mathcal{F}$ can be separated by vertical lines, in the second case all of them can be intersected by one line. In either case, we can apply Lemma 2.1 to finish the proof. □

3. Proof of Theorem 2

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_t\}$ be a family of pairwise disjoint convex sets in general position in the plane. Denote the convex hull of $\bigcup \mathcal{F} = \bigcup_{i=1}^{t} A_i$ by $\operatorname{conv} \mathcal{F}$. The boundary of $\operatorname{conv} \mathcal{F}$, $\operatorname{bd} \operatorname{conv} \mathcal{F}$, consists of finitely many boundary pieces of the A_i's, called *vertex-arcs*, connected by straight-line segments, called *edge-arcs*. (This terminology reflects the picture in the special case when every set A_i is a single point.)

The elements $A_i \in \mathcal{F}$ contributing at least one vertex-arc to the boundary of $\operatorname{conv} \mathcal{F}$ will be called *vertices of* $\operatorname{conv} \mathcal{F}$ or, simply, *vertices of* $\mathcal{F}$. If A is not a vertex, then it is said to be an *internal member of* $\mathcal{F}$.

Lemma 3.1 [BF2]. *Let $k \geq 4$ and let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane satisfying property P_k. If $\mathcal{F}$ has m vertices, then there are $\lfloor (m-3)/(k-3) \rfloor$ lines such that any internal member of $\mathcal{F}$ is intersected by at least one of them.*

Lemma 3.2. *Let $\mathcal{F}$ be a family of disjoint convex sets satisfying property P_4^n, and* *assume that there is a line ℓ intersecting all members of $\mathcal{F}$. Then $\mathcal{F}$ has at most $(n-2)^2+1$ members.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the members of $\mathcal{F}$ listed in the order of their intersections with ℓ. For any A_i, A_j, $1 \le i < j \le t$, define $f(i, j)$ and $g(i, j)$ exactly as in the proof of Lemma 2.2.

If $f(i_1, i_2) > f(i_2, i_3) > \cdots > f(i_{k-1}, i_k)$ for some $i_1 < i_2 < \cdots < i_k$, then $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, are said to form an *upper chain* of length k. They form a *lower chain* of length k if $g(i_1, i_2) < g(i_2, i_3) < \cdots < g(i_{k-1}, i_k)$. It is easy to see that, in both cases $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are in convex position.

For any $2 \le i < j \le t$, let u_i (resp., l_i) be the length of the *longest* upper (resp., lower) chain that ends with A_i. Clearly, $u_i, l_i \ge 2$.

Claim. *If $i \ne k$, then $(u_i, l_i) \ne (u_k, l_k)$.*

Indeed, if $u_i = u_k = u$, $l_i = l_k = l$ for some $i < k$, then neither the longest upper chain $A_{i_1}, \ldots, A_{i_u} = A_i$ nor the longest lower chain $A_{j_1}, \ldots, A_{j_l} = A_i$ ending with A_i could be extended by A_k to a longer (upper, resp., lower) chain. Therefore, $f(i_{u-1}, i) < f(i, k)$ and $g(j_{l-1}, i) > g(i, k)$, which would imply $\mathrm{conv}(A_{i_{u-1}} \cup A_{j_{l-1}} \cup A_k) \supset A_i$, contradicting property P_4. (See Fig. 3.)

It follows from the claim and from the fact that $u_i, l_i \ge 2$ for every $2 \le i \le t$ that, if $t > (n-2)^2 + 1$, then there is an i such that either $u_i \ge n$ or $l_i \ge n$. So, there is an upper (resp., lower) chain of length n, and its elements are in convex position. $\square$

Proof of Theorem 2. First we prove the upper bound. Let $\mathcal{F}$ be a family satisfying property P_4^n and suppose for contradiction that $|\mathcal{F}| \ge (n-4)((n-2)^2+1)+n$.

By Lemma 3.1, we can select at most $n-4$ lines such that every internal member of $\mathcal{F}$ intersects at least one of them. Since $\mathcal{F}$ has at least $(n-4)((n-2)^2+1)+1$ internal

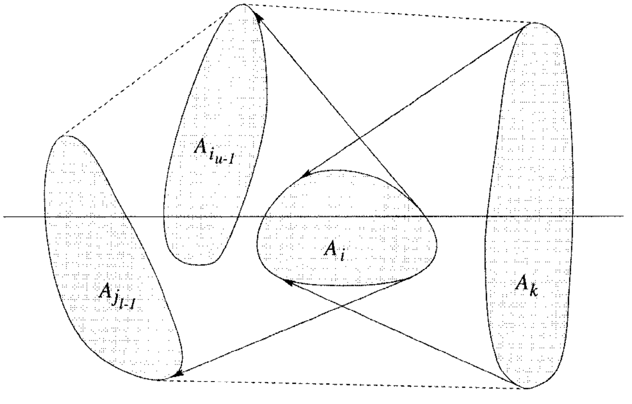

Fig. 3. $f(i_{u-1}, i) < f(i, k)$, $g(j_{l-1}, i) > g(i, k)$.

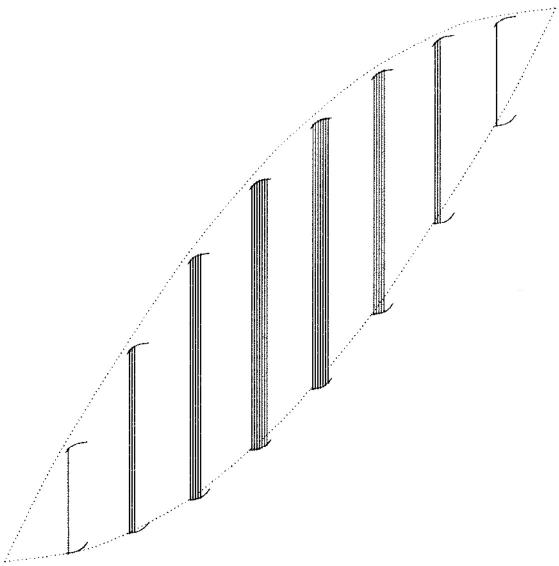

Fig. 4

members, one of the lines intersects at least $(n-2)^2+2$ members of $\mathcal{F}$. By Lemma 3.2, $\mathcal{F}$ has n members in convex position, contradicting property P^n.

The lower bound is shown by the following construction. Suppose for simplicity that $n = 4k+3$ for some k, and let $\mathcal{F}$ denote the family of vertical segments

$$S_{ij} = \{(x, y) \mid x = x_{ij},\ y_{ij} \le y \le y'_{ij}\},$$

$1 \le i \le 2k+2,\ 1 \le j \le 2\min(i, 2k-i+3) - 1$, where

$$x_{ij} = i + \varepsilon j, \qquad y_{ij} = (2k-i+2)^2 + (\varepsilon j)^2, \qquad y'_{ij} = (2k+3)^2 - i^2 - (\varepsilon(k-j))^2,$$

and ε is an extremely small positive number (see Fig. 4). Clearly, $|\mathcal{F}| = 2(k+1)^2 > n^2/8$.

For any $S = S_{i,j} \in \mathcal{F}$, let $i(S) = i$, $j(S) = j$.

Let $\mathcal{F}'$ be a subfamily of $\mathcal{F}$, $S_{ij} \in \mathcal{F}'$. Observe that if (x_{ij}, y'_{ij}) is not a vertex of $\operatorname{conv}\mathcal{F}'$, then there are $S_1, S_2 \in \mathcal{F}'$ such that $i(S_1) > i$, $i(S_2) = i$, and $j(S_2) < j$. Similarly, if (x_{ij}, y_{ij}) is not a vertex of $\operatorname{conv}\mathcal{F}'$, then there are $S_3, S_4 \in \mathcal{F}'$ such that $i(S_3) < i$, $i(S_4) = i$, and $j(S_4) > j$. Therefore, if S_{ij} is not a vertex of $\mathcal{F}'$, then $\mathcal{F}'$ has at least four other members. This shows that $\mathcal{F}'$ satisfies property P_4.

It remains to show that $\mathcal{F}$ satisfies property P^n. To see this, consider a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \ge n > 4k+2$. It is easy to see that there are $S_1, S_2, S_3, S_4, S_5 \in \mathcal{F}'$ such that $i(S_1) < i(S_2) = i(S_3) = i(S_4) < i(S_5)$ and $j(S_2) < j(S_3) < j(S_4)$. Then, by the above observation, S_3 is not a vertex of $\mathcal{F}'$, so the members of $\mathcal{F}'$ are not in convex position. This completes the proof of Theorem 2. $\square$

4. Proof of Theorem 3

Lemma 4.1. *Let $\mathcal{F}$ be a family of disjoint convex sets in the plane, satisfying property P_5 and at least one of the two following conditions*:

(i) *any two members of $\mathcal{F}$ can be separated by a vertical line*; *and*
(ii) *there is a line intersecting all members of $\mathcal{F}$*.

Then $\mathcal{F}$ is in convex position.

Proof. Case (ii) was settled by Bisztriczky and Fejes Tóth [BF2]. So we have to prove the assertion only in case (i).

Let $A_1, A_2, \ldots, A_t$ denote the members of $\mathcal{F}$ listed from left to the right. Clearly, A_1 and A_t are vertices of $\mathcal{F}$, so we can choose two points, $x \in A_1$, $y \in A_t$, that belong to the boundary of $\operatorname{conv}\mathcal{F}$. Let $a(xy)$ (and $a(yx)$) denote the counterclockwise oriented arcs from x to y (from y to x, respectively).

Suppose that A_j is not a vertex of $\operatorname{conv}\mathcal{F}$ for some $1 < j < t$. Let

$$\alpha = \max\{i \mid i < j, A_i \text{ meets } a(xy)\},$$

$$\beta = \min\{i \mid i > j, A_i \text{ meets } a(xy)\},$$

$$\gamma = \max\{i \mid i < j, A_i \text{ meets } a(yx)\},$$

$$\delta = \min\{i \mid i > j, A_i \text{ meets } a(yx)\}.$$

(Since A_1 and A_t meet both $a(xy)$ and $a(yx)$, these numbers are well defined.) Notice that $\operatorname{conv}(A_\alpha \cup A_\beta \cup A_\gamma \cup A_\delta) \supset A_j$, contradicting property P_5. □

Lemma 4.2. *Let $\mathcal{F}$ be a family of disjoint convex sets in the plane, satisfying property P_{11}^n. Suppose that there are m vertical lines such that every member of $\mathcal{F}$ intersects at least one of them*.

Then we can choose at most $\lfloor m/2 \rfloor$ vertical lines so that every internal member of $\mathcal{F}$ intersects at least one of them.

Proof. Suppose that every member of $\mathcal{F}$ intersects at least one of the vertical lines $\ell_1, \ell_2, \ldots, \ell_m$, ordered from left to right. For any i, let $\mathcal{F}_i$, $\mathcal{F}_{<i}$, and $\mathcal{F}_{>i}$ denote the families of all members of $\mathcal{F}$ *intersecting* ℓ_i, lying in the open half-plane to the *left* of ℓ_i, and in the open half-plane to the *right* of ℓ_i, respectively.

It is sufficient to show that every internal member of $\mathcal{F}$ intersects at least two distinct lines ℓ_i, and then it follows that $\ell_2, \ell_4, \ldots, \ell_{2\lfloor m/2 \rfloor}$ meet the requirements of the lemma.

Suppose, for contradiction, that there is an internal member $A \in \mathcal{F}$ which intersects only one line ℓ_i, and assume that $1 < i < m$. (The cases when $i = 1$ or m are similar, but somewhat simpler.)

Let X and Y be two vertex-arcs on the boundary of $\operatorname{conv}\mathcal{F}$ such that there is a point $x \in X$ in the closed half-plane to the left of ℓ_1, and there is a point $y \in Y$ in the closed half-plane to the right of ℓ_m. Let $a(xy)$ and $a(yx)$ denote the counterclockwise oriented arcs of the boundary of conv $\mathcal{F}$ from x to y, and from y to x, respectively.

Let V_1 (and V_4) denote the last (resp., first) vertex-arc along $a(xy)$, which belongs to a member of $\mathcal{F}_{<i}$ (of $\mathcal{F}_{>i}$, respectively). If there is no such vertex-arc, let $V_1 = X$ (resp. $V_4 = Y$). Clearly, if there is any vertex-arc on $a(x, y)$ between V_1 and V_4, it must belong to an element of $\mathcal{F}_i$. Let V_2 (resp., V_3) denote the vertex-arc succeeding V_1 (resp., preceding V_4) along $a(x, y)$. Similarly, define the vertex-arcs U_1, U_2, U_3, U_4 along the oriented arc $a(yx)$.

Let $A_1, A_2, \ldots, A_s$ denote the members of $\mathcal{F}_i$ listed from top to bottom, in order of their intersections with ℓ_i. (A appears in this list, i.e., $A = A_r$ for some $1 \leq r \leq s$.) By Lemma 5.1(ii), $\mathcal{F}_i$ is in *convex position*. Let $x' \in \operatorname{bd} A_1$ and $y' \in \operatorname{bd} A_s$ be two boundary points of conv $\mathcal{F}_i$. Let $a(x'y')$ (and $a(y'x')$) denote the oriented arcs connecting x' to y' (resp., y' to x') along bd conv $\mathcal{F}_i$. Assume without loss of generality that A has a boundary point on $a(y'x')$. We distinguish two cases.

If A *has* a boundary point on $a(x'y')$, then let us define $\mathcal{G}$ as the collection of those members (vertices) of $\mathcal{F}$ which correspond to the vertex-arcs $V_1, \ldots, V_4, U_1, \ldots, U_4$.

If A *does not have* a boundary point on $a(x'y')$, then let

$$\alpha = \max\{i \mid i < r,\ A_i \text{ has a point on } a(x'y')\},$$
$$\beta = \min\{i \mid i > r,\ A_i \text{ has a point on } a(x'y')\}. \qquad \square$$

Since both $x' \in A_1$ and $y' \in A_s$ belong to $a(x'y')$, α and β are well defined. Now let $\mathcal{G}$ consist of A_α, A_β, and the members of $\mathcal{F}$, corresponding to $V_1, \ldots, V_4, U_1, \ldots, U_4$.

In both cases, $\mathcal{G}$ has at most 10 members. It is easy to check that none of the edge-arcs of conv $\mathcal{G}$ can be met by A. Since $A \cap \ell_i \subseteq \operatorname{conv} \mathcal{G}$, we obtain that A must be contained in the convex hull of $\mathcal{G}$, contradicting property P_{11} (see Fig. 5).

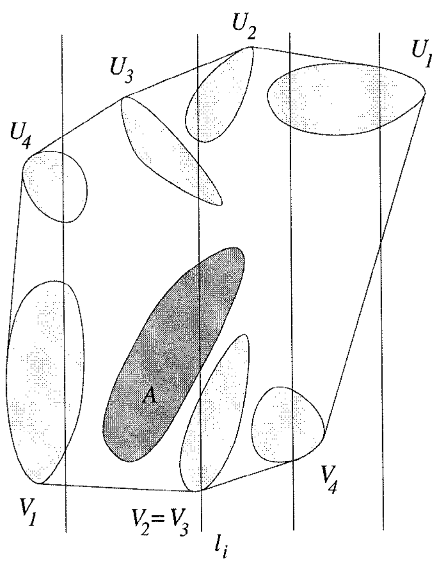

Fig. 5

Now we can prove Theorem 3. Let $\mathcal{F}$ be a family of disjoint convex sets in the plane satisfying property P_{11}^n. In view of Lemma 4.1(i), no n members of $\mathcal{F}$ can be separated from each other by vertical lines. Thus, according to a well-known result of T. Gallai (cited before), we can find $n-1$ vertical lines such that every member of $\mathcal{F}$ intersects at least one of them.

Let $\mathcal{F}_1$ denote the family of all internal members of $\mathcal{F}$. Clearly, $|\mathcal{F}_1| > |\mathcal{F}| - n$. By Lemma 4.2, all members of $\mathcal{F}_1$ can be pierced by $\lfloor (n-1)/2 \rfloor < n/2$ vertical lines. Similarly, the family $\mathcal{F}_2$ of all internal members of $\mathcal{F}_1$ has more than $|\mathcal{F}| - 2n$ members, and all of them can be intersected by fewer than $n/4$ vertical lines. Applying Lemma 4.2 repeatedly, after at most $\lfloor \log_2 n \rfloor$ steps, we end up with a subfamily of $\mathcal{F}$, which has more than $|\mathcal{F}| - n\log_2 n$ members, and they all intersect the same line. By Lemma 4.1(ii), this implies that

$$|\mathcal{F}| - n\log_2 n < n,$$

concluding the proof of Theorem 3. □

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