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A generalization of the Fourier pseudospectral method

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ABSTRACT

The Fourier pseudospectral (PS) method is generalized to the case of derivatives of nonnatural order (fractional derivatives) and irrational powers of the differential operators. The generalization is straightforward because the calculation of the spatial derivatives with the fast Fourier transform is performed in the wavenumber domain, where the operator is an irrational power of the wavenumber. Modeling constant-Q propagation with this approach is highly efficient because it does not require memory variables or additional spatial derivatives. The classical acoustic wave equation is modified by including those with a space fractional Laplacian, which describes wave propagation with attenuation and velocity dispersion. In particular, the example considers three versions of the uniform-density wave equation, based on fractional powers of the Laplacian and fractional spatial derivatives.

INTRODUCTION

The concept of fractional time derivative has been used to simulate constant-Q wave propagation in the time domain using the classical power-law stress-strain relation. An example is Kjartansson's constant-Q model (Kjartansson, 1979). Because of its simplicity, this model is used in many seismic applications, mainly in its frequency-domain form. The wave equation becomes parabolic because the phase velocity has no upper bound. The case of compressional-wave (P-wave) propagation in heterogeneous media has been solved by Carcione et al. (2002), whereas the case of P and shear (S) waves has been developed and solved numerically by Carcione (2009). Caputo and Carcione (2010) use distributed-order fractional derivatives, i.e., the derivatives are integrated with respect to the order of differentiation. The equations are solved with the Grünwald-Letnikov approximation for the time discretization and the classical Fourier method to compute the spatial derivatives. One of the drawbacks of this approach is that the implementation of the fractional time derivative requires storing the wavefield from $t = 0$ to present time. Another way of modeling attenuation without using memory variables is to use the Kelvin-Voigt rheology, which requires additional spatial derivatives (Carcione et al., 2004).

Chen and Holm (2004) propose a linear integro-differential equation wave model for the anomalous attenuation by using the space-fractional Laplacian, i.e., the fractional derivatives are taken with respect to the space variables. Spatial fractional derivatives are also used to describe anomalous diffusion processes (Hanyga, 2001; Gorenflo et al., 2002). The properties of the Fourier transform when it acts on fractional derivatives are well established and a rigorous treatment is available in the literature (e.g., Dattoli et al., 1998). On this basis, I generalize the Fourier method and describe attenuation by using spatial fractional derivatives.

THE GENERALIZATION OF THE FOURIER METHOD

The Fourier pseudospectral (PS) method is a collocation technique in which a continuous function $u(x)$ is approximated by a truncated series

$$u_N(x_j) = \sum_{r=0}^{N-1} \tilde{u}_r \exp(\mathrm{i}k_r x_j) = \sum_{r=0}^{N-1} \tilde{u}_r \exp(2\pi \mathrm{i} rj/N), \quad (1)$$

where $\tilde{u}_r$ are spectral coefficients; N is the number of grid points;

$$x_j = jdx \quad \text{and} \quad k_r = \frac{2\pi r}{Ndx}, \quad r = 0,\ldots,N-1 \quad (2)$$

are the collocation points and wavenumbers, respectively; dx is the grid spacing; and $\mathrm{i} = \sqrt{-1}$. The spectral (expansion) coefficients are chosen such that the approximate solution u_N coincides with the solution $u(x)$ at the collocation points. The Fourier PS method is ap-

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propriate for problems with periodic boundary conditions — for example, a wave which exits the grid on one side and reenters it on the opposite side.

The sequence $u(x_j)$ is the inverse discrete Fourier transform of

$$\tilde{u}_r = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)\exp(-2\pi \mathrm{i} rj/N),\ r=0,\ldots,N-1. \quad (3)$$

The computation of the fractional derivative of order β by the Fourier method conveniently reduces to a set of multiplications of the different coefficients $\tilde{u}_r$, with factors $(\mathrm{i}k_r)^\beta$, because

$$\partial_x^\beta u_N(x_j) = \sum_{r=0}^{N-1} (\mathrm{i}k_r)^\beta \tilde{u}_r \exp(\mathrm{i}k_r x_j). \quad (4)$$

The spectral coefficients $\tilde{u}_r$ are computed by the fast Fourier transform (FFT). The steps of the calculation of the first-order fractional partial derivative are as follows:

$$u(x_j) \rightarrow \mathrm{FFT} \rightarrow \tilde{u}_r \rightarrow (\mathrm{i}k_r)^\beta \tilde{u}_r \rightarrow \mathrm{FFT}^{-1} \rightarrow \partial_x^\beta u(x_j). \quad (5)$$

This involves the definition of fractional derivative according to Ozaktas et al. (2001) and fractional Fourier transform as in Dattoli et al. (1998). The classical Fourier method is obtained for $\beta = 1$. See details in Carcione (2007).

EXAMPLES: ANELASTIC WAVE EQUATIONS

I consider a generalization of the lossless acoustic equation of motion (e.g., Carcione, 2007, Chapter 9). It is an example of a constant-Q wave equation whose dispersion equation is similar to that of Kjartansson (1979) and Carcione et al. (2002) and is widely used in seismology (see the associated complex velocity in equation 9 below). The uniform-density pressure formulation is

$$\omega_0^{2-2\beta} c^{2\beta} (\partial_x^2 + \partial_z^2)^\beta p + s = \partial_t^2 p, \quad (6)$$

where x and z are Cartesian coordinates, $p(x,z)$ is the pressure, $c(x,z)$ is the velocity of the compressional wave, $s(x,z,t)$ is the body force, and ω_0 is a scaling frequency. The range of β is $1 \le \beta \le 2$, with β slightly greater than 1 describing seismic wave propagation and $\beta = 2$ describing infinite attenuation ($Q=0$, see equation 12). One possible choice of the differential operator is

$$\rho(\partial_x^\beta \rho^{-1} \partial_x^\beta + \partial_z^\beta \rho^{-1} \partial_z^\beta), \quad (7)$$

if one desires to solve the more general variable-density wave equation, where $\rho(x,z)$ is the mass density. Another choice is

$$-(-\partial_x^2 - \partial_z^2)^\beta \quad (8)$$

(Chen and Holm, 2004). If $\beta = 1$, I have the classical wave equation in all the cases.

Velocity and attenuation

Let us analyze the propagation characteristics of the medium defined by equation 6. Assume constant properties and the kernel $p = \exp(\mathrm{i}\omega t - \mathrm{i}k_x x - \mathrm{i}k_z z)$, where ω is the angular frequency and k_x and k_z are complex wavenumber components. We obtain the complex velocity

$$v = \frac{\omega}{k} = \left(\frac{\mathrm{i}\omega}{\omega_0}\right)^\gamma c, \quad \gamma = 1 - \frac{1}{\beta}, \quad (9)$$

where $k = \sqrt{k_x^2 + k_z^2}$. The phase velocity, attenuation factor, and quality factor are given by (e.g., Carcione, 2007)

$$v_p = \left[\mathrm{Re}\left(\frac{1}{v}\right)\right]^{-1} = \left[\cos\left(\frac{\pi\gamma}{2}\right)\right]^{-1} \left(\frac{\omega}{\omega_0}\right)^\gamma c, \quad (10)$$

$$\alpha = -\mathrm{Im}\left(\frac{\omega}{v}\right) = \frac{\omega}{c}\left(\frac{\omega_0}{\omega}\right)^\gamma \sin\left(\frac{\pi\gamma}{2}\right) \quad (11)$$

and

$$Q = \frac{\mathrm{Re}(v^2)}{\mathrm{Im}(v^2)} = \cot(\pi\gamma). \quad (12)$$

This constant-Q model is similar to that proposed in Carcione et al. (2002) and Carcione (2009), with the difference here being that the fractional derivative is taken with respect to the spatial variables.

It can be shown that the velocity associated with the Laplacian 8 is real — $v = (\omega/\omega_0)^\gamma c$. Therefore, Q is infinite and there is no attenuation. Chen and Holm (2004) use this operator in the context of a different wave equation where the propagation is lossy.

Computation of the Laplacian

The steps of the calculation of the Laplacian with a fractional exponent $(\partial_x^2 + \partial_z^2)^\beta$ in equation 6 are as follows:

$$p(x,z) \rightarrow \mathrm{FFT2} \rightarrow \tilde{p} \rightarrow (-1)^\beta (k_x^2 + k_z^2)^\beta \tilde{p}$$
$$\rightarrow \mathrm{FFT2}^{-1} \rightarrow (\partial_x^2 + \partial_z^2)^\beta p(x,z), \quad (13)$$

where FFT2 denotes the 2D Fourier transform. On the other hand, the calculation of the Laplacian in equations 7 and 8 requires the multiplication of $\tilde{p}$ by

$$(-1)^\beta (k_x^{2\beta} + k_z^{2\beta}) \quad (14)$$

(if the density is constant), and

$$-(k_x^2 + k_z^2)^\beta, \quad (15)$$

respectively. In equations 13 and 14, the wavenumber-domain kernel is complex, whereas in the last case it is real.

Simulations

I consider equation 6 and the properties $c = 2$ km/s and $\omega_0 = 2\pi$/s. Figure 1 shows the phase velocity and attenuation factor versus frequency. The dispersion is significant, with a velocity of approximately 2.4 km/s at 18 Hz. The simulations are based on an 88 $\times$ 88 mesh, with square cells of 20-m size and a second-order finite-difference solver for the time stepping, with a time step of 1 ms. A Ricker point source with 18 Hz central frequency is used. Figure 2 shows two snapshots at 300-ms propagation time, corresponding to $Q = 200$ (Figure 2a) and $Q = 5$ (Figure 2b). As expected, the wavefront of the quasielastic case travels with a lower velocity (close to 2 km/s) compared to the lossy case (Figure 2b). The opposite situa-

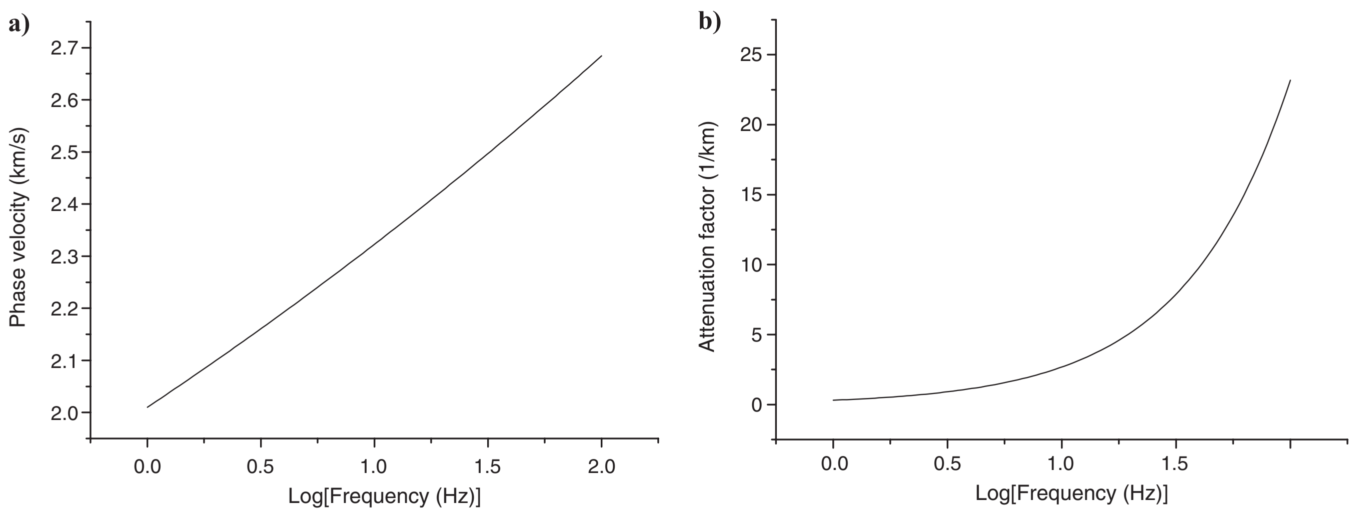

Figure 1. (a) Phase velocity and (b) attenuation factor corresponding to $Q = 5$.

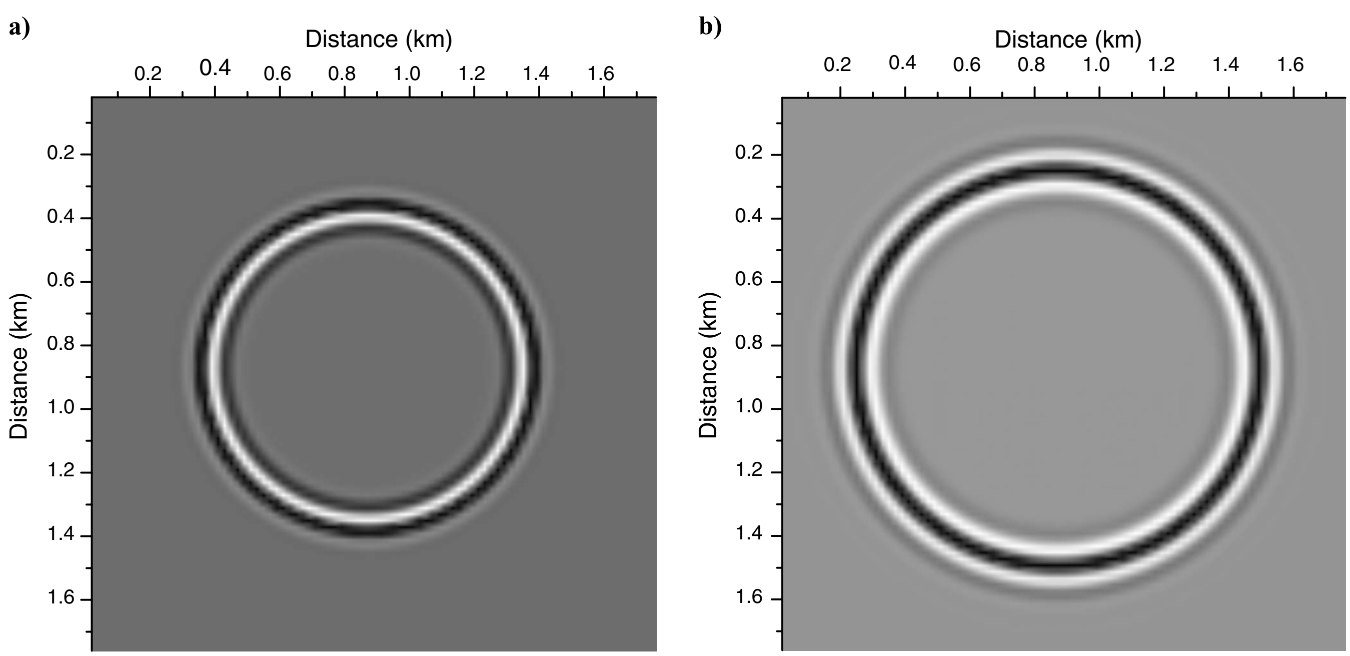

Figure 2. Snapshots computed at 300 ms for (a) $Q = 200$ and (b) $Q = 5$.

tion can be obtained by taking ω_0 much larger than the source dominant frequency. A comparison of time histories at 300 m from the source is displayed in Figure 3, where the solid and dashed lines correspond to $Q = 5$ and $Q = 200$, respectively. Simulations comparing the time histories computed with equations 6–8, using the kernels 13–15, are shown in Figure 4, where $Q = 5$. The comparison reflects the fact that the first two rheologies model velocity dispersion and attenuation, whereas the last one (dotted line) describes lossless propagation.

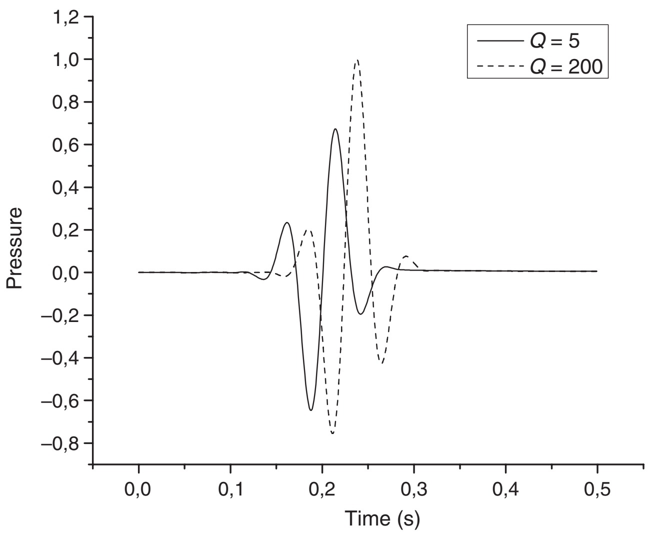

Figure 3. Pressure seismograms at 300 m from the source location for two values of the quality factor. The wave equation involves a fractional power of the Laplacian.

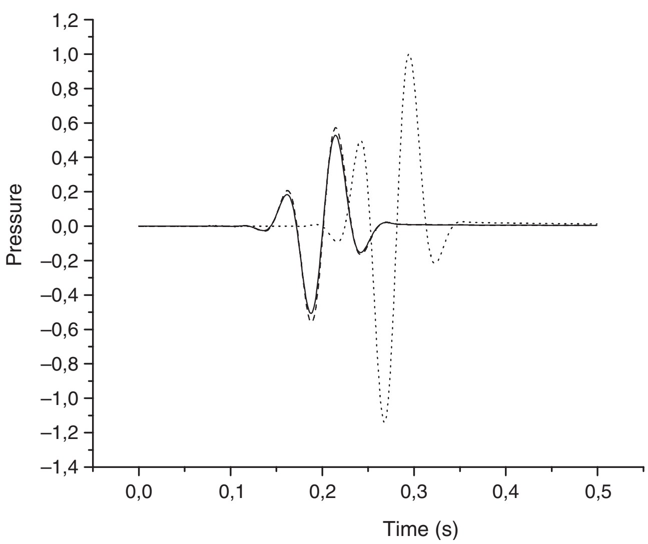

Figure 4. Comparison of pressure seismograms at 300 m from the source location for $Q = 5$. The solid and dotted lines correspond to a fractional power of the Laplacian (equations 6 and 8), whereas the dashed line corresponds to fractional spatial derivatives (equation 7).

CONCLUSIONS

I have simulated wave attenuation by generalizing the Fourier method. Unlike the temporal fractional derivatives, which require the use of significant memory storage, the present method can be implemented with a straightforward generalization of the classical Fourier pseudospectral method to the case of a fractional power of the Laplacian and a fractional order of differentiation. The method can also be applied to the solution of the elastic wave equation (P and S waves) and to the diffusion equation, such as, for instance, the fluid-flow equation if one requires modeling permeability with memory effects (anomalous diffusion).

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