A generalization of the functional equation of bisymmetry

by

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1. The function $F(x,y)$ is called *bisymmetric* if

$$F[F(x,y),F(u,v)]=F[F(x,u),F(y,v)]\ ^{1)}.$$

We shall examine the functional equation

$$F[G(x,y),H(u,v)]=f[g(x,u),h(y,v)], \tag{1}$$

which is a generalization of the equation of bisymmetry, moreover, in a certain sense the widest generalization.

We shall prove in the section 2 that all strictly monotonic and differentiable solutions of (1) have the form

$$z=Z[X(x)+Y(y)]\ ^{2)}. \tag{2}$$

It might be worth remarking that here one functional equation (1) determines the general form of six functions figuring in it.

In the section 3 we examine some special cases of the equation (1).

Section 4 gives another condition necessary and sufficient for the possibility of writing a function in the form (2).

2. THEOREM I. *If the strictly monotonic and differentiable functions* $F(x,y),G(x,y),H(x,y),f(x,y),g(x,y),h(x,y)$ *satisfy the functional equation* (1) *then (and only then) there exist nine strictly monotonic and differentiable functions* $\chi(t),\Phi(t),\Psi(t),\varphi(t),\psi(t),X_1(t),X_2(t),Y_1(t),Y_2(t)$ *and a constant* k *such that*

$$f(x,y)=\chi[\varphi(x)+\psi(y)], \tag{3}$$

$$F(x,y)=\chi[\Phi(x)+\Psi(y)], \tag{4}$$

[1]) This equation was solved by J. Aczél, *On mean values*, Bulletin of the Amer. Math. Soc. 54,4 (1948), p. 392-400. He proved that the most general strictly monotonic and continuous solution of the functional equation of bisymmetry is $F(x,y)=\varphi^{-1}[p\varphi(x)+q\varphi(y)+r]$.

[2]) Functions of this form (and only those) can be represented by nomograms with three straight scales.

$$G(x,y)=\Phi^{-1}[X_1(x)+Y_1(y)], \tag{5}$$

$$H(x,y)=\Psi^{-1}[X_2(x)+Y_2(y)], \tag{6}$$

$$g(x,y)=\varphi^{-1}[X_1(x)+X_2(x)+k], \tag{7}$$

$$h(x,y)=\psi^{-1}[Y_1(x)+Y_2(y)-k], \tag{8}$$

where $\Phi^{-1}(t)$, $\Psi^{-1}(t)$, $\varphi^{-1}(t)$, $\psi^{-1}(t)$ *denote the inverse functions of* $\Phi(t)$, $\Psi(t)$, $\varphi(t)$, $\psi(t)$.

Proof. First let us keep in (1) $u=y=y_0$ constant and define the functions $\alpha(t)$, $\beta(t)$ by

$$\alpha[G(t,y_0)]=g(t,y_0), \qquad \beta[H(y_0,t)]=h(y_0,t).$$

Thus we have

$$F[G(x,y_0),H(y_0,v)]=f\{\alpha[G(x,y_0)],\beta[H(y_0,v)]\}.$$

If we introduce new variables for $G(x,y_0)$ and $H(y_0,v)$, which we shall denote by x and y respectively, we get

$$F(x,y)=f[\alpha(x),\beta(y)]. \tag{9}$$

Hereafter we derive (1) with regard to u and v respectively; then we have

$$F_2[G(x,y),H(u,v)]H_1(u,v)=f_1[g(x,u),h(y,v)]g_2(x,u) \tag{10}$$

and

$$F_2[G(x,y),H(u,v)]H_2(u,v)=f_2[g(x,u),h(y,v)]h_2(y,v) \tag{11}$$

respectively, where the indices 1 and 2 denote the partial differential quotient of the respective functions with regard to the first and the second variable respectively. Dividing (10) by (11) we obtain

$$\frac{f_1[g(x,u),h(y,v)]}{f_2[g(x,u),h(y,v)]}=\frac{H_1(u,v)}{H_2(u,v)}\cdot\frac{h_2(y,v)}{g_2(x,u)}.$$

If we keep $u=u_0$, $v=v_0$ constant, define the new functions $\varphi(t),\psi(t)$ by the equations

$$\varphi'[g(t,u_0)]=\frac{H_1(u_0,v_0)}{g_2(t,u_0)}, \qquad \psi'[h(t,v_0)]=\frac{H_2(u_0,v_0)}{h_2(t,v_0)},$$

and further denote $g(x,u_0)$ and $h(y,v_0)$ by x and y respectively, then we get

$$\frac{f_1(x,y)}{f_2(x,y)}=\frac{\varphi'(x)}{\psi'(y)} \tag{12}$$

or, which is the same,

$$\begin{vmatrix} f_1(x,y) & f_2(x,y) \\ \varphi'(x) & \psi'(y) \end{vmatrix}=0.$$

This shows the dependence of the functions $f(x,y)$ and $\varphi(x)+\psi(y)$, whence we get (3). Taking (9) into account, we get also (4).

In order to determine also the functions $G(x,y), H(x,y), g(x,y)$, $h(x,y)$ we substitute (3) and (4) into (1):

$$\chi\{\Phi[G(x,y)]+\Psi[H(u,v)]\}=\chi\{\varphi[g(x,u)]+\psi[h(y,v)]\}.$$

The strict monotony of $F(x,y)$ implies the strict monotony of $\chi(t)$; consequently

$$\Phi[G(x,y)]+\Psi[H(u,v)]=\varphi[g(x,u)]+\psi[h(y,v)].$$

By keeping $u=u_0$, $v=v_0$ constant and writing $X_1(t)=\varphi[g(t,u_0)]$, $Y_1(t)=\psi[h(t,v_0)]-\Psi[H(u_0,v_0)]$ we have (5).

Similarly we get (6) and

(13) $$g(x,y)=\varphi^{-1}[X_3(x)+Y_3(y)],$$

(14) $$h(x,y)=\psi^{-1}[X_4(x)+Y_4(y)].$$

However (3), (4), (5), (6), (13), (14) do not yet satisfy (1) because substituting them into (1) we have

$$X_1(x)+Y_1(y)+X_2(u)+Y_2(v)=X_3(x)+Y_3(u)+X_4(y)+Y_4(v).$$

This shows that

$$X_3(x)=X_1(x)+a, \qquad X_4(y)=Y_1(y)+c,$$
$$Y_3(u)=X_2(u)+b, \qquad Y_4(v)=Y_2(v)+d,$$

with $a+b+c+d=0$.

Thus by writing $k=a+b$ we get (7) and (8). (3), (4), (5), (6), (7), (8) really satisfy the equation (1). We also see that all functions figuring in the functional equation (1) have the form (2) and this completes the proof of Theorem I.

3. We shall discuss the following specializations of the functional equation (1):

(15) $$F(x,y)=g(x,y)=h(x,y), \qquad f(x,y)=G(x,y)=H(x,y);$$

(16) $$G(x,y)=H(x,y)=g(x,y)=h(x,y);$$

(17) $$G(x,y)=H(y,x)=g(x,y)=h(y,x).$$

1° First let us examine (15).

THEOREM II. *If the strictly monotonic and derivable functions $F(x,y)$, $f(x,y)$ satisfy the functional equation*

(18) $$F[f(x,y),f(u,v)]=f[F(x,u),F(y,v)]\ ^{3}),$$

[3]) See [1]) and L. Fuchs, *On mean systems*, Acta Math. Acad. Sci. Hungar. 1 (1950), p. 303-319, § 4.

then (and only then) they can be written in the form

(19) $$F(x,y)=\varphi^{-1}[P\varphi(x)+Q\varphi(y)+R],$$

(20) $$f(x,y)=\varphi^{-1}[p\varphi(x)+q\varphi(y)+r],$$

where $\varphi(x)$ is an arbitrary function and five of the six constants P,Q,R,p,q,r are arbitrary and the sixth can be obtained from the equation

(21) $$(P+Q-1)r=(p+q-1)R.$$

Proof. Since now in (1) $g(x,y)=h(x,y)$, therefore if we keep $u=v$ constant, then the differential equation (12) can be written as

$$\frac{f_1(x,y)}{f_2(x,y)}=\frac{a}{b}\cdot\frac{\varphi'(x)}{\varphi'(y)},$$

where

$$\frac{a}{b}=\frac{H_1(u,v)}{H_2(u,v)}.$$

Thus we obtain

(22) $$f(x,y)=\chi[a\varphi(x)+b\varphi(y)]$$

in the same way as we obtained (3).

In order to determine $F(x,y)$ we put (22) into (18); then

$$F\{\chi[a\varphi(x)+b\varphi(y)],\chi[a\varphi(u)+b\varphi(v)]\}=\chi\{a\varphi[F(x,u)]+b\varphi[F(y,v)]\}.$$

Substituting both sides of this equation into the function $\chi^{-1}(t)$ and writing again the new variables, we have

$$\chi^{-1}\{F[\chi(ax+by),\chi(au+bv)]\}$$
$$=a\varphi\{F[\varphi^{-1}(x),\varphi^{-1}(u)]\}+b\varphi\{F[\varphi^{-1}(y),\varphi^{-1}(v)]\}.$$

By denoting

(23) $$M(x,y)=\chi^{-1}\{F[\chi(x),\chi(y)]\},$$

(24) $$N(x,y)=\varphi\{F[\varphi^{-1}(x),\varphi^{-1}(y)]\},$$

we have

$$M(ax+by,au+bv)=aN(x,u)+bN(y,v).$$

Let

$$x=y=\frac{t}{a+b}, \qquad u=v=\frac{s}{a+b};$$

then

(25) $$M(t,s)=(a+b)N\left[\frac{t}{a+b},\frac{s}{a+b}\right];$$

consequently

$$N\left[\frac{ax+by}{a+b}, \frac{au+bv}{a+b}\right] = \frac{aN(x,u)+bN(y,v)}{a+b}. \tag{26}$$

This is equivalent to Jensen's equation and the general solution is

$$N(x,y)=Px+Qy+R,$$

where P,Q,R are arbitrary constants[4]); thus by (24) we have (19). On the other hand, taking (25) into account, we obtain from (23)

$$\begin{aligned} F(x,y) &= \chi\{M[\chi^{-1}(x),\chi^{-1}(y)]\} \\ &= \chi\left\{(a+b)\left[P\frac{\chi^{-1}(x)}{a+b}+Q\frac{\chi^{-1}(y)}{a+b}+R\right]\right\} \\ &= \chi[P\chi^{-1}(x)+Q\chi^{-1}(y)+(a+b)R]; \end{aligned}$$

hence

$$\varphi^{-1}[P\varphi(x)+Q\varphi(y)+R]=\chi[P\chi^{-1}(x)+Q\chi^{-1}(y)+(a+b)R],$$

or with new variables and substituting both sides of this equation into the function $\varphi(t)$

$$P\varphi[\chi(x)]+Q\varphi[\chi(y)]+R=\varphi\{\chi[Px+Qy+(a+b)R]\}.$$

With $T(t)=\varphi[\chi(t)]$ we have

$$PT(x)+QT(y)+R=T[Px+Qy+(a+b)R]. \tag{27}$$

The general solution of (27) is [4])

$$T(t)=At+B.$$

Thus

$$\varphi[\chi(t)]=At+B$$

or

$$\chi(t)=\varphi^{-1}[At+B].$$

So we arrive at (20) from (22) by denoting $AP=p$, $AQ=q$, $AR+B=r$. Finally, by substituting (19) and (20) into (18), we get (21).

2° The specialization (16) of (1) gives the functional equation

$$F[g(x,y),g(u,v)]=f[g(x,u),g(y,v)]. \tag{28}$$

THEOREM III. *All strictly monotonic and differentiable solutions of* (28) *are of the form*

$$F(x,y)=f(x,y)=\chi[P\varphi(x)+Q\varphi(y)], \tag{29}$$

[4]) J. Aczél, *Über eine Klasse von Funktionalgleichungen*, Commentarii Mathematici Helvetici 21, 3 (1948), p. 247-252.

$$g(x,y)=\varphi^{-1}[pX(x)+qX(y)], \tag{30}$$

where $X(t),\varphi(t)$ *and* $\chi(t)$ *are arbitrary functions and* P,Q,p,q *are constants such that*

$$pQ=Pq. \tag{31}$$

Proof. $F(x,y)=f(x,y)$ is evident if we set $u=y$ in (28); further (29) follows from (28), just as (22) from (18); finally (30), (31) are evident if we substitute (29) into (28)[5]).

3° We obtain the functional equation

$$F[g(x,y),g(v,u)]=f[g(x,u),g(v,y)] \tag{32}$$

from (1) by the specialization (17).

THEOREM IV. *The strictly monotonic and differentiable solutions of the functional equation* (32) *are*

$$F(x,y)=f(x,y)=\chi[\varphi(x)+\varphi(y)]; \tag{33}$$

$$g(x,y)=\varphi^{-1}[X(x)+Y(y)]. \tag{34}$$

Proof. $F(x,y)=f(x,y)$ is a consequence of (32) with $u=y$; further (32) with $x=v$ implies the symmetry of $f(x,y)$ $\big(f(x,y)=f(y,x)\big)$; the further proof goes along the same lines as the proofs of (22) and (5).

4. If we wish to decide whether a given function $z(x,y)$ belongs to the class of functions (2) or not, we look for an equation containing the least possible number of functions. Such equations can be obtained by specializing (1), e. g. by the specialization (17), i. e. (32). The question whether $z(x,y)$ has the form of (2) or not can be settled practically by testing the differential equation (12)[6]).

The functions (2) can be characterised also by the following simple condition (which also can be reduced to (12)):

(D) *The strictly monotonic and differentiable functions* $z(x,u),z(x,v),z(y,u),z(y,v)$ — *all considered as functions of* x,y,u,v — *are not independent.*

THEOREM V. *The condition* (D) *is necessary and sufficient for the existence of three strictly monotonic and differentiable functions* $Z(t),X(t),Y(t)$ *by which the function* $z(x,y)$ *might be represented in the form* (2).

[5]) It might be observed that solutions (29), (30), (31) can be obtained also without supposing the derivability by reducing the equation (28) to the functional equation of bisymmetry. For theorems III and IV cf. M. Hosszu, *A biszimmetria függvényegyenletéhez*, MTA Alkalmazott Matematikai Intézetének Közleményei 1 (1952), p. 335-342.

[6]) J. Aczél, *Zur Charakterisierung nomographisch einfach darstellbarer Funktionen durch Differential- und Funktionalgleichungen*, Acta Scientiarum Math. 12, Pars A (1950), p. 74-80, § 1.

Proof. The condition (D) is necessary because

$$Z^{-1}[z(x,u)]+Z^{-1}[z(y,v)]=Z^{-1}[z(x,v)]+Z^{-1}[z(y,u)].$$

On the other hand, if (D) holds then

$$\begin{vmatrix} z_1(x,u) & 0 & z_2(x,u) & 0 \\ z_1(x,v) & 0 & 0 & z_2(x,v) \\ 0 & z_1(y,u) & z_2(y,u) & 0 \\ 0 & z_1(y,v) & 0 & z_2(y,v) \end{vmatrix}=0,$$

and taking $u=y$ this becomes

$$z_1(x,y)\,z_2(x,v)\,z_2(y,y)\,z_1(y,v)=z_2(x,y)\,z_1(x,v)\,z_1(y,y)\,z_2(y,v).$$

If v is constant, we arrive at (12) with

$$X'(x)=\frac{z_1(x,v)}{z_2(x,v)},\qquad Y'(y)=\frac{z_1(y,v)}{z_2(y,v)}\cdot\frac{z_2(y,y)}{z_1(y,y)}.$$

Thus really

$$z(x,y)=Z[X(x)+Y(y)],\qquad \text{q. e. d.}$$

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Sur certains corollaires du théorème de Titchmarsh

par

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Désignons respectivement par $f(x)$ et $g(x)$, $0\leqslant x\leqslant 2\pi$, deux fonctions complexes, intégrables avec leurs carrés,

$$f(x)\sim\sum_{n=0}^{\infty}(a_n\cos nx+b_n\sin nx),$$

$$g(x)\sim\sum_{n=0}^{\infty}(c_n\cos nx+d_n\sin nx)\qquad (b_0=d_0=0),$$

où les séries

$$(1)\qquad \sum_{n=0}^{\infty}|a_n|^2,\quad \sum_{n=0}^{\infty}|b_n|^2,\quad \sum_{n=0}^{\infty}|c_n|^2,\quad \sum_{n=0}^{\infty}|d_n|^2$$

sont convergentes.

Les coefficients de Fourier A_n et B_n de la fonction

$$h(x)=\int_0^x f(y)g(x-y)\,dy$$

sont alors

$$A_0=\pi a_0c_0+\sum_{m=1}^{\infty}\frac{a_0d_m+c_0b_m}{m}-\frac{1}{2}\sum_{m=1}^{\infty}\frac{a_md_m+b_mc_m}{m},$$

$$(2)\qquad A_n=\frac{\pi}{2}(a_nc_n-b_nd_n)-\frac{c_nb_n+a_nd_n}{4n}+\sum_{n\neq m=0}^{\infty}\frac{m(a_md_m+c_mb_m)}{n^2-m^2}+\sum_{n\neq m=0}^{\infty}\frac{n(a_md_n+c_mb_n)+m(a_nd_m+c_nb_m)}{m^2-n^2};$$

$$B_n=\frac{\pi}{2}(a_nd_n+c_nb_n)+\frac{a_nc_n+3b_nd_n}{4n}-\frac{2a_0c_0}{n}+\sum_{n\neq m=1}^{\infty}\frac{n(a_mc_m-b_md_m)}{m^2-n^2}+\sum_{n\neq m=0}^{\infty}\frac{m(b_nd_m-b_md_n)-n(a_nc_m-a_mc_n)}{m^2-n^2}$$

pour $n>0$.