A Generalization of the Isoperimetric Inequality on the 2-Sphere

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§1. In [1] T. F. Banchoff and W. F. Pohl generalize the classical isoperimetric inequality for simple closed plane curves to an inequality which applies to compact oriented immersed submanifolds of E^n of class C^2. There is a classical isoperimetric inequality for simple closed curves in the 2-sphere of unit radius, S. In this paper the techniques of Banchoff and Pohl are used to generalize the isoperimetric inequality on S.

Let C be a finite collection of circles each with a fixed orientation. Also fix an orientation on S. Let $f : C \to S$ be a C^2 immersion.

Definition 1. The winding number of f with respect to $(p, q) \in S \times S$, denoted $w_f(p, q)$, is the algebraic number of intersections of $f(C)$ with either geodesic arc pq (with some orientation). This number is determined up to sign; also, $w_f(p, q) = w_f(q, p)$.

Remark. Imbed S in E^3 and let $\ell(p, q)$ be the line through p and q, for distinct points p and q in S. Then $w_f(p, q) = \lambda(\ell(p, q), f)$, the linking number of $\ell(p, q)$ with f.

If C is one circle and f an imbedding (or equivalently $f(C)$ is a simple closed curve), then $f(C)$ separates S into two regions of area A and $\bar{A}$. Let the length of $f(C)$ be L. The classical isoperimetric inequality states

$$L^2 \geqq A\bar{A},$$

with equality holding only for geodesic circles. We want to generalize the idea of the product $A\bar{A}$ of the complementary regions into which the simple closed curve $f(C)$ separates S.

Definition 2. For a C^2 immersion $f : C \to S$, set

$$\mathfrak{A}\bar{\mathfrak{A}}(f) = \frac{1}{2}\int_{S\times S} w_f^2 \, dS_1 \wedge dS_2 \, ,$$

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where dS_1, dS_2 denote the elements of area on the first and second factor of $S \times S$ respectively.

It is easy to see when C is one circle and f is an imbedding that $\mathfrak{A}\bar{\mathfrak{A}}(f) = A\tilde{A}$.

Theorem. *Let $f : C \to S$ be a C^2 immersion. Let L equal the length of $f(C)$, then*

$$L^2 \geqq \mathfrak{A}\bar{\mathfrak{A}}(f).$$

Equality holds if and only if $f(C)$ is one or several coincident geodesic circles all traversed in the same direction each a number of times.

The remainder of this paper is devoted to the proof of this theorem. The presentation of the proof follows very closely that of the proof of Theorem 1 in [1].

§2. First, we show that $\mathfrak{A}\bar{\mathfrak{A}}(f)$ can be given in terms of an integral over the space of geodesics of S.

A compact oriented 0-dimensional manifold is a finite set of points, to each of which is assigned a multiplicity ± 1. The sum of the multiplicities must be 0. Let $M = \{x_1, x_2, \cdots, x_q\}$ be a 0-dimensional manifold in the circle G of unit radius. Let i_j be the multiplicity of x_j, $1 \leqq j \leqq q$. Pick an orientation for G. Any ordered pair of distinct points $(p, q) \in G \times G$ determines a pair of arcs G^+, G^-. The arc G^+ is the one that is traversed in proceeding from p to q in a direction consistent with the orientation of G. The winding number of M about (p, q) is defined by

$$w_M(p, q) = \pm \sum_{x_j \in G^+} i_j = \mp \sum_{x_k \in G^-} i_k .$$

Let $r : G \times G \to R$ be the metric on G.

Lemma 1.

$$\int_{G\times G} w_M{}^2 \sin r \, ds_1 \wedge ds_2 = -2 \sum_{j,k=1}^{q} i_j i_k [r(x_j, x_k) + \sin r(x_j, x_k)],$$

where ds_1, ds_2 denote the elements of arc length on the first and second factor of $G \times G$, respectively.

Proof.

$$\int_{G\times G} w_M{}^2 \sin r \, ds_1 \wedge ds_2 = -\int_{G\times G} \sum_{x_j \in G^+} i_j \sum_{x_k \in G^-} i_k \sin r \, ds_1 \, ds_2$$

$$= -\sum_{j,k=1}^{q} i_j i_k \int_{G\times G} F_{jk} \sin r \, ds_1 \wedge ds_2 ,$$

where $F_{jk}(p, q) = 1$ if $x_j \in G^+$ and $x_k \in G^-$; otherwise $F_{jk}(p, q) = 0$. Now $(x_j, x_k) \in G^+ \times G^-$ if and only if $(p, q) \in G^-{}_{jk} \times G^+{}_{jk}$, where $G^+{}_{jk}$, $G^-{}_{jk}$ are the arcs of G determined by (x_j, x_k). Thus

$$\int_{G\times G} F_{jk} \sin r \, ds_1 \wedge ds_2 = \int_{G^-_{jk}\times G^+_{jk}} \sin r \, ds_1 \wedge ds_2 .$$

As observed by Santalo [3, p. 709], the last integral equals

$$2[r(x_j, x_k) + \sin r(x_j, x_k)]. \qquad \square$$

Let $\mathcal{G}$ be the set of (unoriented) geodesics in S. One can show that $S \times S$ can be identified with $[0, 2\pi]^2 \times \mathcal{G}$ where the identification is one-to-one *a.e.* Under this identification $dS_1 \wedge dS_2 = \sin r \, ds_1 \wedge ds_2 \wedge d\mathcal{G}$, where ds_1, ds_2 are the elements of arc length of the first and second factor of $[0, 2\pi]^2$, and $d\mathcal{G}$ is the invariant measure on the set of geodesics; see Santalo [3, p. 708]. Here $r(s_1, s_2) = |s_1 - s_2|$ if $|s_1 - s_2| \leqq \pi$ and $r(s_1, s_2) = 2\pi - |s_1 - s_2|$ if $|s_1 - s_2| > \pi$; equivalently, $r : S \times S \to R$ is the metric on S.

Lemma 2.

$$\mathfrak{A}\bar{\mathfrak{A}}(f) = -\int_{\mathcal{G}} \sum_{x_j, x_k} i_j i_k [r(x_j, x_k) + \sin r(x_j, x_k)] \, d\mathcal{G},$$

where x_1, x_2, $\cdots$ are the points of intersection of $f(C)$ with a moving geodesic G, and i_j is the intersection number of G (with some orientation) and $f(C)$ at x_j.

Proof.

$$\mathfrak{A}\bar{\mathfrak{A}}(f) = \frac{1}{2}\int_{S\times S} w_f^2 \, dS_1 \wedge dS_2 = \frac{1}{2}\int_{\mathcal{G}} \left[\int_{G\times G} w_f^2 \sin r \, ds_1 \wedge ds_2\right] d\mathcal{G}.$$

Using Lemma 1, we can evaluate the integral in the brackets. $\square$

§3. We now show that $\mathfrak{A}\bar{\mathfrak{A}}(f)$ is given by an integral over $C \times C$. If $p \in S$, let $-p$ be the antipode of p. Let $B \subset C \times C$ denote the set of all (x, y) such that $f(x) \neq f(y)$ and $f(x) \neq -f(y)$, and note that $C \times C \backslash B$ is a set of measure zero. Let $g(x, y)$ denote the geodesic joining $f(x)$ to $f(y)$ oriented from $f(x)$ to $f(y)$ along the shorter geodesic arc, so that $g : B \to \mathcal{G}$, and let $e_1(x, y)$ denote the unit vector oriented along $g(x, y)$ at $f(x)$. Let $X(x, y) = f(x)$, and $\omega_1 = dX \cdot e_1$.

As in [2, pp. 1324–1326], we can show that

$$(1) \qquad d\omega_1 = -(\sin r)^{-1} \sin \sigma_1 \sin \sigma_2 \, ds_1 \wedge ds_2 ,$$

where σ_i, $i = 1, 2$, are certain angles to be described and ds_1, ds_2 are elements of arc length on the first and second factor of $C \times C$, respectively. Let $a_2(x, y)$ be a unit vector at $f(x)$ oriented along $f(C)$ and $b_2(x, y)$ be a unit vector at $f(y)$ oriented along $f(C)$. Then σ_1 is the signed angle that a_2 makes with e_1, and σ_2 is the signed angle that b_2 makes with $\bar{e}_1$, the unit vector oriented along $g(x, y)$ at $f(y)$. (The reader is reminded that each tangent space of S is oriented so that there is a sense of positive rotation at each point of S.)

Lemma 3.

$$\mathfrak{a}\bar{\mathfrak{a}}(f) = \int_{C\times C} (r + \sin r)\, d\omega_1 .$$

Proof. One can easily show that $d\omega_1 = -g^*(d\mathcal{G})$. If the intersection numbers of $f(C)$ at x and y with $G = g(x, y)$ are i_x and i_y, respectively, then the sign of $i_x i_y$ is the same as the sign of $\sin \sigma_1 \sin \sigma_2$. We see from (1), by inspection, that the sign of $\sin \sigma_1 \sin \sigma_2$ is the sign of the Jacobian of g. Hence by Lemma 1 of [2], and Lemma 2,

$$\mathfrak{a}\bar{\mathfrak{a}}(f) = -\int_{\mathcal{G}} \sum_{x_i, x_k} i_i i_k [r(x_i, x_k) + \sin r(x_i, x_k)]\, d\mathcal{G}$$

$$= \int_{C\times C} (r + \sin r)\, d\omega_1 . \qquad \square$$

§4. In this section we prove the theorem. First we obtain an integral formula by integrating over a certain "secant space". For the definition of secant space see [1] or [2].

Definition 3. We say $f : C \to S$ is spherically generic if f is a self-transversal immersion and f intersects its antipodal image, $-f$, transversally.

Let $\pi : S \to P$ be the canonical projection of S onto the projective plane P. Clearly f is spherically generic if πf is a self-transversal immersion.

Let $S(C)$ denote the secant space of C. Let $A \subset S(C)$ denote the set of points (x, y) such that $f(x) = f(y)$ or $f(x) = -f(y)$. Assume that f is spherically generic; this implies that A is finite. Let $S(C)_*$ denote the directed dilitation of $S(C)$ along A; *i.e.*, $S(C)_* = S(C)\backslash A \cup N(A)$, where $N(A)$ is the set of unit vectors in $C \times C$ tangent to points of A. We are using the metric induced on C by f and the product metric on $C \times C$. Thus the boundary of $S(C)_*$ is $T(C) \cup N(A)$, where $T(C)$ is the unit tangent bundle of C.

Consider the smooth map $G_f : S(C)\backslash A \to \mathcal{G}$ which associates to each $(x, y) \in S(C)\backslash A$ the geodesic directed from $f(x)$ to $f(y)$ along the shorter geodesic arc and to the unit vector t_x tangent to C at x the geodesic determined by $f_*(t_x)$. The map G_f extends smoothly to $S(C)_*$. To see this suppose S is standardly situated in E^3 and identify the geodesic on S through p in the direction $v(\|v\| = 1)$ with the unit vector $p \times v$. Also regard f as an immersion into E^3. Let $(x, y) \in A$ and say that $p = f(x) = -f(y)$. Let s_1, s_2 denote the arc length along f, so taken that $s_1(x) = s_2(y) = 0$. Then G_f on the ϵ-neighborhood $0 < (s_1)^2 + (s_2)^2 < \epsilon^2$ of (x, y) assigns to each (s_1, s_2) a vector proportional to

$$(p + f'(x)s_1 + O(s_1^2)) \times (-p + f'(y)s_2 + O(s_2^2))$$
$$= p \times (f'(x)s_1 + f'(y)s_2) + \text{second order terms},$$

where f' denotes the derivative of f with respect to arc length. Since f is spherically generic $f'(x)s_1 + f'(y)s_2$ is never 0 unless $s_1 = s_2 = 0$. One sees that the direction of G_f is dominated by first order terms so that G_f extends smoothly to the set of unit vectors tangent to $C \times C$ at (x, y). Denote the extension by $G_f : S(C)_* \to \mathcal{G}$.

Let $\pi : S(C)_* \to C \times C$ be the canonical projection map, and $\pi_i : C \times C \to C$ be the projection onto the i-th factor, $i = 1, 2$. For $z \in S(C)_*$, let $X(z) = f\pi_1\pi(z)$. Let $e_1(z)$ denote the unit vector at $X(z)$ in S directed along $G_f(z)$ and $\omega_1 = dX \cdot e_1$, which is a differential 1-form of $S(C)_*$. Let $r(z)$ be the spherical distance from $f\pi_1\pi(z)$ to $f\pi_2\pi(z)$. Consider the differential form $r\omega_1$. The orientation of C induces an orientation on $C \times C$ and hence on $S(C)_*$. By applying Stokes' Theorem, we find that

$$\int_{T(C)\cup N(A)} r\omega_1 = \int_{S(C)^*} dr \wedge \omega_1 + \int_{S(C)^*} r\, d\omega_1 .$$

The left-hand term is zero since $r = 0$ on $T(C)$ and $\omega_1 = 0$ on $N(A)$. In addition, $S(C)_*$ and $C \times C$ differ by sets of measure zero. Hence we may write

$$(2) \qquad -\int_{C\times C} dr \wedge \omega_1 = \int_{C\times C} r\, d\omega_1 .$$

A local analysis similar to that on page 188 of [1] shows that

$$(3) \qquad dr \wedge \omega_1 = -\cos \sigma_1 \cos \sigma_2\, ds_1 \wedge ds_2 ,$$

where σ_1, σ_2, ds_1, ds_2 are the same quantities that appear in (1).

If f is not spherically generic, we can find a sequence of spherically generic immersions $f_k : C \to S$, $k = 1, 2, \cdots$, which converge uniformly to f and whose first derivatives also converge uniformly to f. Since (2), *i.e.*,

$$\int_{C\times C} \cos \sigma_1 \cos \sigma_2\, ds_1 \wedge ds_2 = -\int_{C\times C} r(\sin r)^{-1} \sin \sigma_1 \sin \sigma_2\, ds_1 \wedge ds_2$$

holds for all f_k, it holds for f.

Lemma 4. *Let $f : C \to S$ be a C^2 immersion. Then*

$$\mathfrak{A}\bar{\mathfrak{A}}(f) = \int_{C\times C} (\cos \sigma_1 \cos \sigma_2 - \sin \sigma_1 \sin \sigma_2)\, ds_1 \wedge ds_2 .$$

Proof. The proof follows immediately from Lemma 3, (1), (2), and (3). □

Proof of Theorem. By Lemma 4, we have

$$L^2 - \mathfrak{A}\bar{\mathfrak{A}}(f) = \int_{C\times C} (1 - \cos \sigma_1 \cos \sigma_2 + \sin \sigma_1 \sin \sigma_2)\, ds_1 \wedge ds_2$$

$$= \int_{C\times C} 2 \sin^2 \tfrac{1}{2}(\sigma_1 + \sigma_2)\, ds_1 \wedge ds_2 > 0.$$

Equality holds if and only if $\sigma_1 = -\sigma_2$. This holds for one or more geodesic circles gone around in the same direction any number of times. The converse of this last statement follows in the same manner as the corresponding result on page 190 of [1]. □

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