# Part II. Further Developments of the Reductive Perturbation Method

## A

## Generalization of the Reductive Perturbation Method to Multi-Wave Systems

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The reductive perturbation method applies with some generalizations to nonlinear and dispersive multi-wave systems. Such systems evolve under the effect of the self-interaction of each wave and of the mutual-interactions between them. The systems can be described, in the lowest order, as assemblies of the "quasi-simple" waves or the nonlinearly self-modulated waves, both of which include the effects due to the self-interactions and a part of the mutual-interactions. The rest of the mutual-interactions gives rise to higher order corrections in the wave forms.

## § 1. Introduction

The reductive perturbation method can apply to one-dimensional, unidirectional and nonlinear wave motions in a dispersive or dissipative system, to yield a single non-linear equation with simple structure as the approximate governing equation.[1] The object so far treated, however, has been limited to the self-interaction of the single wave. When more than two waves coexist, the mutual-interactions between them give rise to additional effects both on the wave characteristics and on the wave profiles. A generalization of the reductive perturbation method to such multi-wave systems has been attempted by the present authors.[2),3)]

First, we consider the nonlinear wave propagation in a weakly dispersive or dissipative system, which is governed by the system of equations considered in §4 of Part I. It is assumed that, if the dispersive (or dissipative) effect is disregarded, there exist $n$ simple waves corresponding to the $n$ possible families of characteristics. Each simple wave is distorted under the effect of the dispersion (or dissipation). Such a wave (often called the "quasi-simple" wave[4)])

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is governed by a single nonlinear equation with a simple structure, which reduces to the Korteweg-de Vries (KdV) equation, the modified KdV equation, the Burgers equation and so on according to the properties of the dispersions or dissipations and the degree of the nonlinearities. Our concern is with the systems in which there exist $n$ mutually interacting "quasi-simple" waves. In the many-"quasi-simple" wave system, the profiles and the speed of each wave are both affected by the mutual-interactions. In this case, a naive perturbation approach meets with failure, bringing about secular terms in the higher order solutions. The expansion procedure to be used here is a generalization of the reductive perturbation method in which the same type of $\varepsilon$ dependence of the wave-amplitude is kept but the stretched coordinates include the effect of the variations of wave velocity. The condition that higher order terms of the expansion be bounded, that is to say the non-secularity condition, leads to equations for the correct approximating wave motions. The results given by this expansion procedure show that the $n$ "quasi-simple" waves, whose orbits are modified due to the mutual-interactions, can be superposed in the lowest order approximation and the change in the wave profiles are due to the higher order corrections. The generalized reductive perturbation method, together with an example, is shown in §2.

Next, consider a wide class of nonlinear and strongly dispersive wave systems, which are governed by the equation given in §6 of Part I but consist of several waves interacting with each other. The many-wave problem becomes in general complicated due to the resonance coupling between them. The discussion presented here is then restricted to a simple system consisting of only two interacting waves. When the difference in the wave-numbers and the frequencies of the two waves are of the order of $\varepsilon$, the effect of mutual-interaction can be included in the self-modulation phenomena (see §6 of Part I).[5] On the other hand, if they are of the order of unity, the mutual-interaction affects the orbits and the frequencies of the self-modulated waves. In §3, the reductive perturbation method is generalized so as to include such effects beforehand. The result is that, quite similar to the weakly dispersive systems considered in §2, the wave systems are approximated in the lowest order as the superposition of two nonlinearly self-modulated waves which are governed by their respective nonlinear Schrödinger equation. As an example, the system, which is governed by the Klein-Gordon equation with cubic interaction, is also considered in §3.

## § 2. Weakly dispersive systems[2]

Let us consider the following equation:

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(H_{\alpha}^{\beta}\frac{\partial}{\partial t}+K_{\alpha}^{\beta}\frac{\partial}{\partial x}\right)U=0, \tag{1}$$

where $U$ is a column vector with $n$ components $u_1, u_2, \cdots, u_n$ $(n \geq 2)$, $A$, $H_\alpha^\beta$'s and $K_\alpha^\beta$'s are $n \times n$ matrices, the elements of which are functions of $U$, being assumed sufficiently smooth, and $p \geq 2$. Here, we shall investigate the possibility that $n$ "quasi-simple" waves are superposed to describe the wave motions.

The function $U$ is now expanded for a smallness parameter $\varepsilon$;

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots. \tag{2}$$

The eigenvalues of $A_0(=A(U_0))$, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are assumed to be real and distinct. Following the Taniuti-Wei example,[1] we introduce the stretched variables

$$\xi_j = \varepsilon^a(x - \lambda_j t - \varepsilon^{1-a}\varphi_j(x, t)), \tag{3 a}$$

$$\tau = \varepsilon^{a+1} t, \tag{3 b}$$

where $a = (p-1)^{-1} \leq 1$. We consider that $U$ is a function of $x$ and $t$ through the variables $\xi_j (j = 1, 2, \ldots, n)$ and $\tau$. In Eq. (3a), $\varphi_j(x, t)$ is introduced in anticipation that the velocities of waves vary in space and time due to the mutual-interactions. The factor $\varepsilon^{1-a}$ comes from the following consideration: The variation in the wave velocity due to the two-wave interaction is expected to be proportional to the product of the wave-amplitude and the interaction time. The former is of the order $\varepsilon$. The latter is considered to be the time during which the two waves pass through each other, and then estimated by dividing the width of wave $(\sim O(\varepsilon^{-a}))$ with their relative velocity $(\sim O(1))$, i.e., being of the order $\varepsilon^{-a}$. Therefore, the variation in the wave velocity is of the order $\varepsilon \times \varepsilon^{-a} = \varepsilon^{1-a}$.

Substituting Eqs.(2), (3a) and (3b) into Eq.(1) and equating the successive power of $\varepsilon$ to zero, then we get a sequence of equations.

In the lowest order, we have

$$\sum_{l=1}^{n} (A_0 - \lambda_l) \frac{\partial}{\partial \xi_l} U_1 = 0. \tag{4}$$

Let $R_l$ and $L_l$ be the right and left eigenvectors of $A_0$ for the eigenvalue $\lambda_l$, respectively;

$$A_0 R_l = \lambda_l R_l, \qquad L_l A_0 = \lambda_l L_l. \tag{5}$$

Expanding $U_1$ with the set $\{R_j\}$,

$$U_1 = \sum_{j=1}^{n} f_j(\xi_1, \cdots, \xi_n, \tau) R_j, \tag{6}$$

and using the orthogonality of eigenvectors

$$(L_j, R_k) = \delta_{jk}, \tag{7}$$

we get the equation

$$\sum_{l=1}^{n}(\lambda_j-\lambda_l)\frac{\partial f_j}{\partial \xi_l}=0. \qquad (j=1,\cdots,n) \tag{8}$$

The general solution of Eq. (8) is written as

$$f_j=f_j(\eta_1^{(j)},\eta_2^{(j)},\cdots,\eta_n^{(j)},\tau), \tag{9 a}$$

$$\eta_i^{(j)}=\xi_i-n_i^{(j)}\sum_{l=1}^{n}n_l^{(j)}\xi_l, \tag{9 b}$$

$$n_i^{(j)}=(\lambda_j-\lambda_i)/\Lambda_j, \qquad \Lambda_j=[\sum_i(\lambda_j-\lambda_i)^2]^{1/2}, \tag{9 c}$$

where $\eta_i^{(j)}$ is the perpendicular component of the vector $\xi=\{\xi_1,\ldots,\xi_n\}$ to the $n$-dimensional unit vector $n^{(j)}$. Since we are interested in "quasi-simple" wave systems, we restrict ourselves to the case that $f_j$ is a function of only one variable $\eta_j^{(j)}(=\xi_j)$. We then have

$$f_j=f_j(\xi_j,\tau). \tag{10}$$

We must note here that for the two-"quasi-simple" wave problem Eq.(8) has only the solution (10).

In the next order, we have

$$\begin{aligned}-\sum_j(\lambda_j-\lambda_l)\frac{\partial}{\partial\xi_j}g_l+\sum_{j\neq l}\sum_m(L_l,(R_m\nabla_U)A_0R_j)f_m\frac{\partial f_j}{\partial\xi_j}\\+\sum_{j\neq l}(L_l,\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(K_{\alpha0}^{\beta}-\lambda_jH_{\alpha0}^{\beta})R_j)\frac{\partial^p}{\partial\xi_j^p}f_j\\+\frac{\partial f_l}{\partial\tau}+(L_l,(R_l\nabla_U)A_0R_l)f_l\frac{\partial}{\partial\xi_l}f_l\\+(L_l,\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(K_{\alpha0}^{\beta}-\lambda_lH_{\alpha0}^{\beta})R_l)\frac{\partial^p}{\partial\xi_l^p}f_l\\+\sum_{j\neq l}\left\{(\lambda_j-\lambda_l)\frac{\partial\varphi_l}{\partial\xi_j}+(L_l,(R_j\nabla_U)A_0R_l)f_j\right\}\frac{\partial f_l}{\partial\xi_l}=0,\end{aligned} \tag{11}$$

where the $g_l$'s are the expansion coefficients of $U_2$ with $\{R_l\}$, $U_2=\sum_{l=1}^{n}g_l(\xi_1,\ldots,\xi_n,\tau)R_l$. Now suppose that the variables $\varphi_l$'s satisfy

$$\sum_{j\neq l}(\lambda_j-\lambda_l)\frac{\partial\varphi_l}{\partial\xi_j}=-\sum_{j\neq l}(L_l,(R_j\nabla_U)A_0R_l)f_j, \tag{12}$$

i.e.,

$$\begin{aligned}\varphi_l=\sum_{j\neq l}(\lambda_l-\lambda_j)^{-1}(L_l,(R_j\nabla_U)A_0R_l)\int^{\xi_j}f_j(\xi)d\xi\\+\theta_l(\eta_1^{(l)},\cdots,\eta_n^{(l)},\tau),\end{aligned} \tag{13}$$

where $\theta_l$ is determined by the boundary conditions for $\varphi_l$.

We can solve Eq. (11) to obtain[2)]

$$g_l = -\sum_{j\neq m,}\sum_{j\neq l, m\neq l} \Lambda_l^{-1}(L_l, (R_m\nabla_U)A_0R_j)$$

$$\int^{s_l} ds' f_m(\eta_m^{(l)} + n_m^{(l)}s') \frac{\partial}{\partial\eta_j^{(l)}} f_j(\eta_j^{(l)} + n_j^{(l)}s')$$

$$-\sum_{j\neq l} S_{l,j}(\xi_j, \xi_l) - \Lambda_l^{-1}T_l(\xi_l)s_l + h_l(\eta_1^{(l)}, \cdots, \eta_n^{(l)}), \tag{14}$$

$$S_{l,j} = (\lambda_l - \lambda_j)^{-1}\Big\{(L_l, (R_j\nabla_U)A_0R_j)f_j^2/2 + (L_l, (R_l\nabla_U)A_0R_j)f_lf_j$$

$$+ (L_l, \sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(K_{\alpha 0}^{\beta} - \lambda_j H_{\alpha 0}^{\beta})R_j)\frac{\partial^{p-1}}{\partial\xi_j^{p-1}}f_j\Big\}, \tag{15}$$

$$T_l = \frac{\partial f_l}{\partial\tau} + a_l f_l \frac{\partial f_l}{\partial\xi_l} + \beta_l \frac{\partial^p}{\partial\xi_l^p} f_l, \tag{16 a}$$

$$a_l = (L_l, (R_l\nabla_U)A_0R_l), \tag{16 b}$$

$$\beta_l = (L_l, \sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(K_{\alpha 0}^{\beta} - \lambda_l H_{\alpha 0}^{\beta})R_l), \tag{16 c}$$

where $s_l = \sum_{j=1}^{n} n_j^{(l)}\xi_j$ and $h_l$ is an arbitrary function to be determined in the next step. By imposing the boundedness of $g_l$ in Eq. (14), i.e., the non-secularity condition for $g_l$, the term proportional to $s_l$ in Eq. (14) must vanish, i.e.,

$$T_l = 0. \tag{17}$$

Equations (13) and (17) with Eqs. (3a), (3b), (16a), (16b) and (16c) govern the $n$-"quasi-simple" wave systems, that apply not only to the study of special problems such as collisions of solitary waves but also to the study of more general problems, for example, the time development of nonlinear wave motions. The result implies that the $n$-"quasi-simple" waves can be superposed to describe the nonlinear systems, playing an essential role as well as the $n$ families of characteristics in the usual hyperbolic system. Each "quasi-simple" wave satisfies the simple nonlinear equation (17), which becomes, for a special value of the parameter $p$, the Burgers equation ($p=2$) and the KdV equation ($p=3$). The interactions between these "quasi-simple" waves are included in the variables $\varphi_l$'s.

As an example we deal with the interactions between two ion acoustic solitons travelling opposite to another in a collisionless plasma. Let the electron number density and the ion fluid velocity be $n$ and $u$, which are normalized in terms of the mean number density $n_0$ and the sound velocity of ion acoustic wave $(T_e/m_i)^{1/2}$, respectively, where $T_e$ is the constant electron temperature and $m_i$ the ion mass. For a collisionless plasma of cold ions and warm electrons, the following system of equations applies:[6)]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial n}{\partial x}\right) = 0, \tag{18 a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{n}\frac{\partial n}{\partial x}=0, \tag{18 b}$$

where $x$ and $t$ are also normalized by the Debye length $(T_e/4\pi n_0e^2)^{1/2}$ and the inverse of the ion plasma frequency $(m_i/4\pi n_0e^2)^{1/2}$. Writing Eqs. (18a) and (18b) in the form of Eq.(1), expanding $n$ and $u$ in terms of a smallness parameter $\varepsilon$ as

$$n=1+\varepsilon n_1+\varepsilon^2 n_2+\cdots \quad \text{and} \quad u=\varepsilon u_1+\varepsilon^2 u_2+\cdots,$$

and using the systematic expansion method developed above, we obtain

$$\frac{\partial f_i}{\partial \tau}+f_i\frac{\partial f_i}{\partial \xi_i}+\frac{1}{2}\frac{\partial^3 f_i}{\partial \xi_i^3}=0, \tag{19}$$

where $i=1, 2$ and

$$f_1=(n_1+u_1)/2, \qquad f_2=(n_1-u_1)/2,$$

$$\xi_1=\varepsilon^{1/2}\left\{x-t+(\varepsilon^{1/2}/2)\int^{\xi_2} f_2(\xi)d\xi\right\}, \tag{20 a}$$

$$\xi_2=\varepsilon^{1/2}\left\{x+t+(\varepsilon^{1/2}/2)\int^{\xi_1} f_1(\xi)d\xi\right\}, \tag{20 b}$$

$$\tau=\varepsilon^{3/2}t. \tag{20 c}$$

For the two-soliton problem, we put $f_i=f_i(\xi_i-c_i\tau)$ and integrate Eq. (19) under the boundary conditions that $f_i=(\partial f_i/\partial\xi_i)=(\partial^2 f_i/\partial\xi_i^2)=(\partial^3 f_i/\partial\xi_i^3)=0$ at $\xi_i=\pm\infty$, to obtain

$$f_1=A\ \mathrm{sech}^2\{(A/6)^{1/2}(\xi_1-A\tau/3)\}, \tag{21 a}$$

$$f_2=B\ \mathrm{sech}^2\{(B/6)^{1/2}(\xi_2-B\tau/3)\}, \tag{21 b}$$

$$\xi_1=\varepsilon^{1/2}\{x-t+(3\varepsilon B/2)^{1/2}\tanh[(B/6)^{1/2}(\xi_2-B\tau/3)]-x_A\}, \tag{22 a}$$

$$\xi_2=\varepsilon^{1/2}\{x+t+(3\varepsilon A/2)^{1/2}\tanh[(A/6)^{1/2}(\xi_1-A\tau/3)]-x_B\}, \tag{22 b}$$

where $x_A$ and $x_B$ are initial phases of the two solitons. The phase shift of each soliton in the whole process of collision can be estimated as

$$\begin{aligned}\delta_A&=[x-t]_{\xi_1=0,\,\xi_2=\infty}-[x-t]_{\xi_1=0,\,\xi_2=-\infty}\\&=-(6\varepsilon B)^{1/2},\end{aligned} \tag{23 a}$$

$$\begin{aligned}\delta_B&=[x+t]_{\xi_1=-\infty,\,\xi_2=0}-[x+t]_{\xi_1=\infty,\,\xi_2=0}\\&=(6\varepsilon A)^{1/2}.\end{aligned} \tag{23 b}$$

Tatsumi and Tokunaga presented another example by making use of the generalized reductive perturbation method, that is, the interaction of weak nonlinear disturbances in a compressible fluid including shocks, expansion waves and contact surfaces.[7] According to them, the nonlinear waves belonging to different families of characteristics behave almost independently of each other, while those belonging to the same family are governed either by the Burgers equation or by the equation of heat conduction. They applied the result to one-dimensional shock turbulence in a compressible fluid and found that the law of energy decay of shock turbulence is identical to that of the Burgers turbulence.

## § 3. Strongly dispersive systems[3)]

Here we consider the system of interacting two waves with the frequencies and the wave-numbers $(\omega_1, k_1)$ and $(\omega_2, k_2)$, respectively, which is governed by the equation

$$\frac{\partial U}{\partial t}+A(U)\frac{\partial U}{\partial x}+B(U)=0, \tag{24}$$

where $U$ is a column vector with $n$ components, $u_1, u_2, \ldots, u_n$, $A$ an $n\times n$ matrix and $B$ a column vector. The frequency and the wave-number of each wave, $\omega_i$ and $k_i$ $(i=1, 2)$, satisfies the dispersion relation

$$\det(\omega_i I-k_i A_0+i\nabla B_0)=0, \tag{25}$$

where $I$ is the unit matrix, $A_0=A(U^{(0)})$, $(\nabla B_0)_{ij}=(\partial B_i/\partial u_j)_{U=U^{(0)}}$ and $U^{(0)}$ is a constant solution of Eq. (24), satisfying

$$B(U^{(0)})=0. \tag{26}$$

The present system consists of the fundamental two modes with $(\omega_1, k_1)$ and $(\omega_2, k_2)$ and their higher harmonics caused by the nonlinear interaction, and undergoes a slight modulation of their amplitudes and frequencies under the nonlinear effects. The form of $U$ is then anticipated as

$$U=U^{(0)}+\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\sum_{l,n=-\infty}^{\infty}U_{l,n}^{(\alpha)}(\xi_1, \xi_2, \tau)Z_{l,n}, \tag{27}$$

where $Z_{l,n}$ is the exponentially oscillating factor,

$$\begin{aligned}Z_{l,n}=\exp[&il\{k_1x-\omega_1t+\sum_{\gamma=1}^{\infty}\varepsilon^{\gamma}\Omega_1^{(\gamma)}(\xi_1, \xi_2, \tau)\}\\&+in\{k_2x-\omega_2t+\sum_{\gamma=1}^{\infty}\varepsilon^{\gamma}\Omega_2^{(\gamma)}(\xi_1, \xi_2, \tau)\}],\end{aligned} \tag{28}$$

and $\xi_1$, $\xi_2$ and $\tau$ are the stretched variables introduced through

$$\left.\begin{aligned}&\xi_1=\varepsilon\{x-\lambda_1t-\sum_{\gamma=0}^{\infty}\varepsilon^{\gamma}\psi_1^{(\gamma)}(\xi_1, \xi_2, \tau)-\gamma_1\},\\&\xi_2=\varepsilon\{x-\lambda_2t-\sum_{\gamma=0}^{\infty}\varepsilon^{\gamma}\psi_2^{(\gamma)}(\xi_1, \xi_2, \tau)-\gamma_2\},\\&\tau=\varepsilon^2t.\end{aligned}\right\} \tag{29}$$

In the above expressions, $\varepsilon$ is a smallness parameter, $\lambda_1$ and $\lambda_2$ are the group velocities, that is,

$$\lambda_i=(\partial\omega/\partial k)_{k=k_i}, \qquad i=1, 2, \tag{30}$$

$\gamma_1$ and $\gamma_2$ are arbitrary constants and $\Omega_i^{(T)}$, $\psi_i^{(T)}$ are introduced to take into account the frequency shifts and the orbit modifications due to the nonlinear interaction. Here the relative velocity $|\lambda_1-\lambda_2|$ is assumed to be of the order of unity.

The sequence of equations to be solved is obtained by inserting Eqs.(27)~(30) into Eq.(24), corresponding to the successive powers of $\varepsilon$ of the same harmonics. In order to ensure the reality of $U$, we assume that

$$U_{l,n}^{(\alpha)*}=U_{-l,-n}^{(\alpha)}, \tag{31 a}$$

$$\Omega_1^{(T)*}=\Omega_1^{(T)} \quad \text{and} \quad \Omega_2^{(T)*}=\Omega_2^{(T)}, \tag{31 b}$$

$$\psi_1^{(T)*}=\psi_1^{(T)} \quad \text{and} \quad \psi_2^{(T)*}=\psi_2^{(T)}. \tag{31 c}$$

In the lowest order, we have

$$W_{l,n}U_{l,n}^{(1)}=0, \tag{32}$$

$$W_{l,n}=-i(l\omega_1+n\omega_2)I+i(lk_1+nk_2)A_0+\nabla B_0. \tag{33}$$

Now, suppose that

$$\det \quad W_{l,n}=0 \qquad \text{for} \quad |l|+|n|=1, \tag{34 a}$$

$$\neq 0 \qquad \text{otherwise.} \tag{34 b}$$

Equation (34a) corresponds to the dispersion relation (25). Although Eq. (34b) is not always valid for arbitrary $l$ and $n$, we here assume that it hold so far as a few order of perturbation expansion, at most $|l|+|n|\leq 4$, is considered. Equations (34) then yield

$$U_{1,0}^{(1)}=\varphi_1(\xi_1, \xi_2, \tau)R_1, \tag{35 a}$$

$$U_{0,1}^{(1)}=\varphi_2(\xi_1, \xi_2, \tau)R_2, \tag{35 b}$$

$$U_{l,n}^{(1)}=0 \quad \text{for} \quad |l|+|n|\neq 1, \tag{35 c}$$

where $R_1$ and $R_2$ are the right eigenvectors of $W_{1,0}$ and $W_{0,1}$, respectively;

$$W_{1,0}R_1=0 \quad \text{and} \quad W_{0,1}R_2=0, \tag{36}$$

and $\varphi_1$, $\varphi_2$ are scalar functions to be determined later.

Following the discussion in §6 of Part I, we can proceed to the next order to get

$$W_{l,n}U^{(2)}_{l,n}-(\lambda_1 I-A_0)\frac{\partial U^{(1)}_{l,n}}{\partial\xi_1}-(\lambda_2 I-A_0)\frac{\partial U^{(1)}_{l,n}}{\partial\xi_2}$$
$$+i\sum_{l',n'}(l'k_1+n'k_2)(\nabla A_0\cdot U^{(1)}_{l-l',\,n-n'})U^{(1)}_{l',n'}$$
$$+\frac{1}{2}\sum_{l',n'}\nabla\nabla B_0 : U^{(1)}_{l-l',\,n-n'}U^{(1)}_{l',n'}=0. \tag{37}$$

This corresponds to Eq.(I.6·17). ((I.6·17) denotes Eq.(6·17) in Part I. In what follows, this notation will be used.)

Multiplying Eq.(37) by the left eigenvector $L_1$ corresponding to $R_1$,

$$L_1 W_{1,0}=0, \tag{36'}$$

and using Eq. (I. 6·14), we have

$$(\lambda_1-\lambda_2)\frac{\partial\varphi_1}{\partial\xi_2}=0,\quad \text{i.e.,}\quad \varphi_1=\varphi_1(\xi_1,\tau), \tag{38}$$

corresponding to Eq. (I. 6·17), we readily obtain

$$U^{(2)}_{1,0}=\varphi^{(2)}_1(\xi_1,\xi_2,\tau)R_1-i\frac{\partial\varphi_1}{\partial\xi_1}\frac{dR_1}{dk_1}. \tag{39}$$

Equation (37) has in general the non-trivial solutions for $|l|+|n|\leq 2$;

$$\left.\begin{aligned}&U^{(2)}_{2,0}=U^{(2)*}_{-2,0}=S_1\varphi_1^2, \qquad U^{(2)}_{0,2}=U^{(2)*}_{0,-2}=S_2\varphi_2^2,\\ &U^{(2)}_{1,1}=U^{(2)*}_{-1,-1}=T\varphi_1\varphi_2, \quad U^{(2)}_{1,-1}=U^{(2)*}_{-1,1}=V\varphi_1\varphi_2^*,\\ &U^{(2)}_{0,0}=X_1|\varphi_1|^2+X_2|\varphi_2|^2,\end{aligned}\right\} \tag{40}$$

where

$$\left.\begin{aligned}&S_1=-W^{-1}_{2,0}\left\{ik_1(\nabla A_0\cdot R_1)R_1+\frac{1}{2}\nabla\nabla B_0 : R_1R_1\right\},\\ &S_2=-W^{-1}_{0,2}\left\{ik_2(\nabla A_0\cdot R_2)R_2+\frac{1}{2}\nabla\nabla B_0 : R_2R_2\right\},\\ &T=-W^{-1}_{1,1}\{ik_1(\nabla A_0\cdot R_2)R_1+ik_2(\nabla A_0\cdot R_1)R_2+\nabla\nabla B_0 : R_1R_2\},\\ &V=-W^{-1}_{1,-1}\{ik_1(\nabla A_0\cdot R_2^*)R_1+ik_2(\nabla A_0\cdot R_1)R_2^*+\nabla\nabla B_0 : R_1R_2^*\},\\ &X_i=-W^{-1}_{0,0}\{ik_i(\nabla A_0\cdot R_i^*)R_i-ik_i(\nabla A_0\cdot R_i)R_i^*+\nabla\nabla B_0 : R_iR_i^*\}.\end{aligned}\right\} \tag{41}$$

The solutions $U^{(2)}_{2,0}$ and $U^{(2)}_{0,2}$ are the same as those obtained by putting $k=k_1$ and $k=k_2$, respectively, in Eq. (I. 6·18b), and $U^{(2)}_{0,0}$ agrees with the sum of $U^{(2)}_0(k=k_1)$ and $U^{(2)}_0(k=k_2)$ given by Eq. (I. 6·18a). The solution $U^{(2)}_{1,1}$, $U^{(2)}_{1,-1}$ and their complex conjugates are due to the mutual-interactions.

In the third order of $\varepsilon$, the expressions becomes lengthier. However, we are now interested in how $\varphi_1$, $\Omega^{(1)}_1$and $\psi^{(1)}_1$ may be determined. Only the third order terms with $l=1$ and $n=0$ are written out;

$$W_{1,0}U^{(3)}_{1,0}-(\lambda_1 I-A_0)\frac{\partial U^{(2)}_{1,0}}{\partial\xi_1}-(\lambda_2 I-A_0)\frac{\partial U^{(2)}_{1,0}}{\partial\xi_2}$$
$$+\frac{\partial U^{(1)}_{1,0}}{\partial\tau}+\left[(\lambda_1 I-A_0)\frac{\partial\psi^{(0)}_1}{\partial\xi_1}+(\lambda_2 I-A_0)\frac{\partial\psi^{(0)}_1}{\partial\xi_2}\right]\frac{\partial U^{(1)}_{1,0}}{\partial\xi_1}$$
$$-i\left[(\lambda_1 I-A_0)\frac{\partial\Omega^{(1)}_1}{\partial\xi_1}+(\lambda_2 I-A_0)\frac{\partial\Omega^{(1)}_1}{\partial\xi_2}\right]U^{(1)}_{1,0}$$
$$+i\sum_{l'n'}(l'k_1+n'k_2)[(\nabla A_0\cdot U^{(2)}_{1-l',-n'})U^{(1)}_{l',n'}+(\nabla A_0\cdot U^{(1)}_{1-l',-n'})U^{(2)}_{l',n'}]$$
$$+\frac{i}{2}\sum_{l'n'l''n''}(l''k_1+n''k_2)[(\nabla\nabla A_0: U^{(1)}_{1-l'-l'',-n'-n''}U^{(1)}_{l',n'})U^{(1)}_{l'',n''}]$$
$$+\sum_{l',n'}\nabla\nabla B_0 : U^{(2)}_{1-l',-n'}U^{(1)}_{l',n'}$$
$$+\frac{1}{6}\sum_{l'n'l''n''}\nabla\nabla\nabla B^0 : U^{(1)}_{1-l'-l'',-n'-n''}U^{(1)}_{l',n'}U^{(1)}_{l'',n''}=0. \qquad (42)$$

Multiplying Eq. (42) by $L_1$ from the left and using Eqs. (35), (36′), (39)∼(41) and the identical relations given in §6 of Part I,

$$\left(L_1, (\lambda_1 I-A_0)\frac{dR_1}{dk_1}\right)=-\frac{1}{2}\frac{d^2\omega_1}{dk_1^2}(L_1, R_1),$$

we obtain

$$(\lambda_1-\lambda_2)\frac{\partial\varphi^{(2)}_1}{\partial\xi_2}-\mathrm{Im}(\beta_1)|\varphi_2|^2\varphi_1$$
$$+\left\{\frac{\partial\varphi_1}{\partial\tau}-\frac{i}{2}\frac{d^2\omega_1}{dk_1^2}\frac{\partial^2\varphi_1}{\partial\xi_1^2}+i\alpha_1|\varphi_1|^2\varphi_1\right\}$$
$$+i\left\{(\lambda_1-\lambda_2)\frac{\partial\Omega^{(1)}_1}{\partial\xi_2}+\mathrm{Re}(\beta_1)|\varphi_2|^2\right\}\varphi_1$$
$$+(\lambda_2-\lambda_1)\frac{\partial\psi^{(0)}_1}{\partial\xi_2}\frac{\partial\varphi_1}{\partial\xi_1}=0, \qquad (43)$$

where $\mathrm{Re}(\beta_1)$ and $\mathrm{Im}(\beta_1)$ are the real and the imaginary part of $\beta_1$, respectively, and $\alpha_1$, $\beta_1$ are given by

$$\alpha_1=(L_1, R_1)^{-1}L_1\Big[k_1\Big\{(\nabla A_0\cdot X_1)R_1-(\nabla A_0\cdot S_1)R_1^*+2(\nabla A_0\cdot R_1^*)S_1$$
$$+(\nabla\nabla A_0 : R_1R_1^*)R_1-\frac{1}{2}(\nabla\nabla A_0 : R_1R_1)R_1^*\Big\}$$
$$-i\Big\{\nabla\nabla B_0 : (S_1R_1^*+X_1R_1)+\frac{1}{2}\nabla\nabla\nabla B_0 : R_1R_1R_1^*\Big\}\Big], \qquad (44\text{ a})$$

$$\beta_1=(L_1, R_1)^{-1}L_1[k_1\{(\nabla A_0\cdot X_2)R_1+(\nabla A_0\cdot R_2^*)T+(\nabla A_0\cdot R_2)V$$
$$+(\nabla\nabla A_0 : R_2^*R_2)R_1\}$$
$$+k_2\{(\nabla A_0\cdot V)R_2-(\nabla A_0\cdot R_2)V+(\nabla A_0\cdot R_2^*)T$$

$$-(\nabla A_0 \cdot T)R_2^* + (\nabla\nabla A_0 : R_1 R_2^*)R_2 - (\nabla\nabla A_0 : R_1 R_2)R_2^*\}$$
$$-i\{\nabla\nabla B_0 : (X_2 R_1 + T R_2^* + V R_2)$$
$$+\nabla\nabla\nabla B_0 \vdots R_2 R_2^* R_1\}]. \tag{44 b}$$

We now suppose that

$$(\lambda_1 - \lambda_2)\frac{\partial \Omega_1^{(1)}}{\partial \xi_2} + \mathrm{Re}(\beta_1)|\varphi_2|^2 = 0, \tag{45}$$

$$\frac{\partial \psi_1^{(0)}}{\partial \xi_2} = 0. \tag{46}$$

Substituting Eqs. (45) and (46) into Eq. (43) and imposing the non-secularity requirement on $\varphi_1{}^{(2)}$, we obtain the nonlinear Schrödinger equation

$$i\frac{\partial \varphi_1}{\partial \tau} + \frac{1}{2}\frac{d^2\omega_1}{dk_1^2}\frac{\partial^2 \varphi_1}{\partial \xi_1^2} - \alpha_1|\varphi_1|^2\varphi_1 = 0, \tag{47}$$

in which $\boldsymbol{\alpha}_1$ agrees with $Q$ in Eq. (I.6·25) if $k=k_1$ is substituted. From Eqs. (43)~(47), we finally obtain the equation for $\varphi_1^{(2)}$,

$$(\lambda_1 - \lambda_2)\frac{\partial \varphi_1^{(2)}}{\partial \xi_2} = \mathrm{Im}(\beta_1)|\varphi_2(\xi_2, \tau)|^2 \varphi_1(\xi_1, \tau). \tag{48}$$

Equations (45)~(47) are readily integrated to yield

$$\Omega_1^{(1)} = \frac{1}{\lambda_2 - \lambda_1}\mathrm{Re}(\beta_1)\int^{\xi_2}|\varphi_2(\xi)|^2 d\xi + \Theta_1(\xi_1, \tau), \tag{49}$$

$$\varphi_1^{(2)} = \frac{1}{\lambda_1 - \lambda_2}\mathrm{Im}(\beta_1)\int^{\xi_2}|\varphi_2(\xi)|^2 d\xi + \Phi_1(\xi_1, \tau), \tag{50}$$

$$\psi_1^{(0)} = \Psi_1(\xi, \tau), \tag{51}$$

where $\Theta_1$, $\Phi_1$ and $\Psi_1$ are due to the self-interaction, not to the mutual-interaction, and are determined in the next order equation. Among them, $\Theta_1$ and $\Phi_1$ can be absorbed into the first order solution $\varphi_1(\xi_1, \tau)$, and hence, without loss of generality, may be put to zero. The quantity $\Psi_1$, however, cannot be taken zero, because it denotes a modification of the orbit of the wave packet and is affected by the spatial and the temporal variations of $\varphi_1(\xi_1, \tau)$. Since the present perturbation method is based on the assumption that the orbit of the wave packet has a practical meaning, then $\varphi_1(\xi_1, \tau)$ is expected to be written as the product of a function of $\xi_1$ and a oscillating function of $\tau$ with modulus *1*; i.e., taking into account Eq. (47) we can write

$$\varphi_1(\xi_1, \tau) = f_1(\xi_1)e^{-i\Delta\nu_1\tau}. \tag{52}$$

In this case, without performing the higher order calculations, $\Psi_1$ is estimated as follows: The $\tau$-dependent exponential factor in the expression (52) changes the frequency of the carrier wave by $\varepsilon^2 \Delta\nu_1$ and then the group velocity by $\varepsilon^2 d(\Delta\nu_1)/dk_1$. While, the orbit modification $\Psi_1$ changes the velocity of the wave packet by $\varepsilon^2 d\Psi_1/d\tau$ ($\Psi_1$ is assumed to be independent of $\xi_1$). Thus, we obtain

$$\Psi_1 = \frac{d}{dk_1}(\Delta\nu_1)\tau. \tag{53}$$

For the case with $l=0$ and $n=1$, the results are obtained by replacing the subscript 1 with 2, and vice versa, in Eqs. (44)~(53).

The next order correction on the orbit of wave packet, $\psi_1^{(1)}$, is determined by the fourth order equation. However, this can be also done without actually performing complicated calculations: Let $\Delta k_1$ and $\Delta\omega_1$ be the variations of the wave-number and the frequency due to the mutual-interaction, respectively. These are obtained from $\Omega_1^{(1)}$, accurate to the second order of $\varepsilon$,

$$\Delta k_1 = \frac{\partial}{\partial x}(\varepsilon\Omega_1^{(1)}) = \varepsilon^2 \frac{\partial \Omega_1^{(1)}}{\partial \xi_2}, \tag{54 a}$$

$$\Delta\omega_1 = -\frac{\partial}{\partial t}(\varepsilon\Omega_1^{(1)}) = \varepsilon^2 \lambda_2 \frac{\partial \Omega_1^{(1)}}{\partial \xi_2} = \lambda_2 \Delta k_1. \tag{54 b}$$

The group velocity is then modified as

$$\frac{d(\omega_1+\Delta\omega_1)}{d(k_1+\Delta k_1)} \simeq \frac{d\omega_1}{dk_1} + \frac{d\Delta\omega_1}{dk_1} - \frac{d\Delta k_1}{dk_1}\frac{d\omega_1}{dk_1}$$
$$= \lambda_1 - (\lambda_1 - \lambda_2)\frac{d}{dk_1}(\Delta k_1). \tag{55}$$

Alternatively, the group velocity is obtained from the time derivative of the orbit of the wave packet, i.e., $[dx/dt]_{\xi_1=\mathrm{const}}$. Using Eq. (29), we get

$$\left[\frac{dx}{dt}\right]_{\xi_1=\mathrm{const}} \simeq \lambda_1 + \left[\frac{d}{dt}\varepsilon\psi_1^{(1)}(\xi_1, \xi_2, \tau)\right]_{\xi_1=\mathrm{const}}$$
$$\simeq \lambda_1 + \varepsilon \frac{\partial \psi_1^{(1)}}{\partial \xi_2}\left[\frac{d\xi_2}{dt}\right]_{\xi_1=\mathrm{const}}$$
$$\simeq \lambda_1 + \varepsilon^2(\lambda_1 - \lambda_2)\frac{\partial \psi_1^{(1)}}{\partial \xi_2}. \tag{56}$$

Substituting Eqs. (49) and (54a) into Eq. (55) and taking into account that $\varphi_2(\xi_2, \tau)$ is independent of $k_1$, we have

$$\frac{\partial \psi_1^{(1)}}{\partial \xi_2} = \frac{d}{dk_1}\left(\frac{\mathrm{Re}(\beta_1)}{\lambda_1 - \lambda_2}\right)|\varphi_2(\xi_2)|^2$$
$$= -\frac{d}{dk_1}\left\{\log\left(\frac{\mathrm{Re}(\beta_1)}{\lambda_1 - \lambda_2}\right)\right\}\frac{\partial \Omega_1^{(1)}}{\partial \xi_2}. \tag{57}$$

It will be shown later that the fourth order calculation for an example leads to the same results as that estimated from the above intuitive picture.

As an example, we deal with the Klein-Gordon equation with a cubic interaction term,[8)]

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+m^2 y-\beta y^3=0, \tag{58}$$

where $m$ and $\beta$ are real constants. Introducing $\chi$ and $\phi$ by the equations

$$\chi-\frac{\partial y}{\partial x}=0, \tag{59 a}$$

$$\phi-\frac{\partial y}{\partial t}=0, \tag{59 b}$$

and differentiating Eq.(59a) with respect to $t$, we can bring Eq.(58) into the matrix form (24):

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+B=0. \tag{60}$$

Here, $U$, $A$ and $B$ take the forms

$$U=\begin{pmatrix}\phi\\ \chi\\ y\end{pmatrix}, \tag{61 a}$$

$$A=\begin{pmatrix}0&-1&0\\ -1&0&0\\ 0&0&0\end{pmatrix}, \tag{61 b}$$

$$B=\begin{pmatrix}m^2 y-\beta y^3\\ 0\\ -\phi\end{pmatrix}. \tag{61 c}$$

In what follows, Eq.(59a) will be regarded as a subsidiary condition which perpetuates if it is valid initially. We now assume the expansion (27) about the constant solution

$$U_0=0.$$

Then the matrices $W_{l,n}$'s defined by Eq.(33) are expressed by

$$W_{l,n}=\begin{pmatrix}-i(l\omega_1+n\omega_2)&-i(lk_1+nk_2)&m^2\\ -i(lk_1+nk_2)&-i(l\omega_1+n\omega_2)&0\\ -1&0&-i(l\omega_1+n\omega_2)\end{pmatrix}. \tag{62}$$

The dispersion relations, $\det W_{\pm1,0}=0$ and $\det W_{0,\pm1}=0$, become

$$\omega_1^2=k_1^2+m^2, \qquad \omega_2^2=k_2^2+m^2, \tag{63}$$

which imply

$$\det W_{l,n}\neq 0 \tag{64}$$

unless $|l|+|n|$ is zero or unity. Hence $U_{l,n}^{(1)}$ except $U_{0,0}^{(1)}$ are given by Eq. (32), i.e.,

$$U_{l,n}^{(1)}=0 \quad \text{for} \quad |l|+|n|\geq 2, \tag{65 a}$$

$$U_{1,0}^{(1)}=\varphi_1 R_1, \qquad U_{0,1}^{(1)}=\varphi_2 R_2, \tag{65 b}$$

where $R_1$ and $R_2$ are the column vectors introduced in Eqs.(36) and take the form

$$R_1=\begin{pmatrix}\omega_1\\ -k_1\\ i\end{pmatrix}, \qquad R_2=\begin{pmatrix}\omega_2\\ -k_2\\ i\end{pmatrix}. \tag{66 a}$$

The corresponding row vectors $L_1$ and $L_2$ may be given by

$$L_1=(\omega_1, -k_1, -im^2), \qquad L_2=(\omega_2, -k_2, -im^2). \tag{66 b}$$

Since $\det W_{0,0}$ vanishes, a different method is required to account for $U_{0,0}^{(1)}$ and $U_{0,0}^{(2)}$. Consider the component $l=n=0$ of Eq.(60). For the first order in $\varepsilon$, it yields

$$\phi_{0,0}^{(1)}=\psi_{0,0}^{(1)}=0,$$

hence, the subsidiary condition (59a) yields

$$\chi_{0,0}^{(1)}=0,$$

and, consequently, we have

$$U_{0,0}^{(1)}=0. \tag{67}$$

Putting $l=n=0$ in Eq.(37), taking into account $\nabla A_0=\nabla\nabla A_0=\nabla\nabla B_0=0$ and using Eq.(67), similarly we obtain

$$U_{0,0}^{(2)}=0. \tag{68}$$

Since $\nabla A_0=\nabla\nabla B_0=0$, we find from Eqs. (40) and (41),

$$U_{l,n}^{(2)}=0 \quad \text{for} \quad |l|+|n|\geq 2. \tag{69}$$

Introducing Eqs. (66) and (68) into Eqs. (44)

$$\alpha_1=-(3/2)\beta/\omega_1, \tag{70 a}$$

$$\beta_1=-3\beta/\omega_1. \tag{70 b}$$

Hence, we have from Eqs. (47)∼(51)

$$i\frac{\partial\varphi_1}{\partial\tau}+\frac{1}{2}\frac{m^2}{\omega_1^3}\frac{\partial^2\varphi_1}{\partial\xi_1^2}+\frac{3\beta}{2\omega_1}|\varphi_1|^2\varphi_1=0, \tag{71 a}$$

$$i\frac{\partial\varphi_2}{\partial\tau}+\frac{1}{2}\frac{m^2}{\omega_2^3}\frac{\partial^2\varphi_2}{\partial\xi_2^2}+\frac{3\beta}{2\omega_2}|\varphi_2|^2\varphi_2=0, \tag{71 b}$$

$$\Omega_1^{(1)}=\frac{3\beta/\omega_1}{\lambda_1-\lambda_2}\int^{\xi_2}|\varphi_2(\xi)|^2 d\xi, \tag{72 a}$$

$$\Omega_2^{(1)}=\frac{3\beta/\omega_2}{\lambda_2-\lambda_1}\int^{\xi_1}|\varphi_1(\xi)|^2 d\xi, \tag{72 b}$$

$$\varphi_1^{(2)}=\varphi_2^{(2)}=0, \tag{73}$$

$$\psi_1^{(0)}=\Psi_1(\xi_1,\tau), \qquad \psi_2^{(0)}=\Psi_2(\xi_2,\tau), \tag{74}$$

where $\lambda_1=k_1/\omega_1$ and $\lambda_2=k_2/\omega_2$. Substituting Eqs.(70) and (72) into Eq.(57), we obtain

$$\frac{\partial\psi_1^{(1)}}{\partial\xi_2}=\frac{3\beta}{\omega_1^2}\frac{1-\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}|\varphi_2(\xi_2)|^2, \tag{75 a}$$

$$\frac{\partial\psi_2^{(1)}}{\partial\xi_1}=\frac{3\beta}{\omega_2^2}\frac{1-\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}|\varphi_1(\xi_1)|^2. \tag{75 b}$$

We now show that the above results (75) coincide with those obtained by the fourth order calculation. Computation of $U_{1,0}^{(3)}$ is straightforward; from Eq. (42)

$$U_{1,0}^{(3)}=\varphi_1^{(3)}R_1+\left\{i\frac{\partial\varphi_1}{\partial\tau}-\frac{3\beta}{\omega_1}|\varphi_2|^2\varphi_1\right\}\frac{\omega_1^3}{m^2}\frac{d^2R_1}{dk_1^2}$$
$$+\left\{\frac{3\beta}{\omega_1}\frac{1}{\lambda_1-\lambda_2}|\varphi_2|^2\varphi_1+i\frac{\partial\Psi_1}{\partial\xi_1}\frac{\partial\varphi_1}{\partial\xi_1}\right\}\frac{dR_1}{dk_1}. \tag{76}$$

Introducing Eq.(76) into the fourth order equation with $l=1$ and $n=0$, we obtain after tedious calculation

$$(\lambda_1-\lambda_2)\frac{\partial}{\partial\xi_2}\left\{\varphi_1^{(3)}+\frac{3\beta m^2}{2\omega_1^2\omega_2^2(\lambda_1-\lambda_2)^2}\varphi_1|\varphi_2|^2\right\}$$
$$-i\left\{\frac{k_1}{\omega_1^2}\frac{\partial^2\varphi_1}{\partial\xi_1\partial\tau}-i\frac{\partial\varphi_1}{\partial\xi_1}\frac{\partial\Psi_1}{\partial\tau}-\frac{m^2}{2\omega_1^3}\frac{\partial\varphi_1}{\partial\xi_1}\frac{\partial^2\Psi_1}{\partial\xi_1^2}-\frac{m^2}{\omega_1^3}\frac{\partial^2\varphi_1}{\partial\xi_1^2}\frac{\partial\Psi_1}{\partial\xi_1}\right\}$$
$$-(\lambda_1-\lambda_2)\frac{\partial\varphi_1}{\partial\xi_1}\left\{\frac{\partial\psi_1^{(1)}}{\partial\xi_2}-\frac{1-\lambda_1\lambda_2}{\omega_1(\lambda_1-\lambda_2)}\frac{\partial\Omega_1^{(1)}}{\partial\xi_2}\right\}$$
$$+i(\lambda_1-\lambda_2)\varphi_1\left\{\frac{\partial\Omega_1^{(2)}}{\partial\xi_2}-\frac{\partial\psi_2^{(0)}}{\partial\xi_2}\frac{\partial\Omega_1^{(1)}}{\partial\xi_2}+\frac{1}{\lambda_1-\lambda_2}\frac{\partial\Omega_1^{(1)}}{\partial\tau}\right\}=0. \tag{77}$$

Imposing that the third and fourth brackets vanish individually, we obtain the equations to determine $\psi_1^{(1)}$ and $\Omega_1^{(2)}$;

$$\frac{\partial\psi_1^{(1)}}{\partial\xi_2}=\frac{1-\lambda_1\lambda_2}{\omega_1(\lambda_1-\lambda_2)}\frac{\partial\Omega_1^{(1)}}{\partial\xi_2}, \tag{78}$$

$$\frac{\partial\Omega_1^{(2)}}{\partial\xi_2}=\frac{\partial\psi_2^{(0)}}{\partial\xi_2}\frac{\partial\Omega_1^{(1)}}{\partial\xi_2}-\frac{1}{\lambda_1-\lambda_2}\frac{\partial\Omega_1^{(1)}}{\partial\tau}. \tag{79}$$

From the boundedness of $\varphi_1^{(3)}$, we have the equation for $\Psi_1$,

$$i\frac{\partial\varphi_1}{\partial\xi_1}\frac{\partial\Psi_1}{\partial\tau}+\frac{m^2}{2\omega_1^3}\left\{\frac{\partial\varphi_1}{\partial\xi_1}\frac{\partial^2\Psi_1}{\partial\xi_1^2}+2\frac{\partial^2\varphi_1}{\partial\xi_1^2}\frac{\partial\Psi_1}{\partial\xi_1}\right\}$$
$$=\frac{k_1}{\omega_1^3}\frac{\partial^2}{\partial\xi_1\partial\tau}\varphi_1. \tag{80}$$

Finally we have

$$\varphi_1^{(3)}=-\frac{3\beta m^2}{2\omega_1^2\omega_2^2(\lambda_1-\lambda_2)^2}\varphi_1|\varphi_2|^2. \tag{81}$$

Consequently, it is found that the wave profile is corrected in the third order of $\varepsilon$.

We put the subscript 1 in place of 2, and vice versa, in Eqs.(78)~(81), to obtain the equations for $\psi_2^{(1)}$, $\Omega_2^{(1)}$, $\Psi_2$ and $\varphi_2^{(3)}$.

Substituting Eq.(72a) into Eq.(78), we can readily obtain the same result as Eq.(75a). Equation (52) is substituted into Eq.(80) to yield

$$\frac{d\Psi_1}{d\tau}=-\frac{k_1}{\omega_1^2}\Delta\nu_1, \tag{82}$$

in which $\Psi_1$ is assumed to be independent of $\xi_1$. It follows from Eqs. (52) and (71a) that $\Delta\nu_1$ is connected with $\omega_1$ and the amplitude of $f_1$ as $\Delta\nu_1\propto$ (amplitude)$^2/\omega_1$. Thus we get

$$\frac{d\Psi_1}{dt}=\frac{d}{dk_1}(\Delta\nu_1). \tag{83}$$

These ensure the intuitive derivation of Eq. (53) and (57).

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