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A GENERALIZATION OF THE ROSS–THOMAS SLOPE THEORY

Dedicated to Professor Toshiki Mabuchi on his sixtieth birthday

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Abstract

We give a formula for the Donaldson–Futaki invariants of certain type of semi test configurations, which essentially generalizes the Ross–Thomas slope theory [28]. The positivity (resp. non-negativity) of those "a priori special" Donaldson–Futaki invariants implies K-stability (resp. K-semistability). As an application, we prove the K-(semi)stability of certain polarized varieties with semi-log-canonical singularities, which generalizes some results of [28].

1. Introduction

Considering some algebro-geometric objects such as algebraic varieties, vector bundles or representations, the moduli "space" $\mathcal{M}$ (in very loose sense) which parametrizes all of them is often unseparated (not Hausdorff). The geometric invariant theory (in short, GIT) [20] provides a Zariski open subset $\mathcal{M}^s$ of $\mathcal{M}$ which is a quasi-projective scheme. The objects parametrized by points in $\mathcal{M}^s$ above, are said to be "GIT-stable".

The objects which we study here are polarized varieties. The topic has recently drawn much attention as the relation with the existence problem of "canonical" Kähler metrics become clearer. Along that development, the *K-stability* is formulated as a newer kind of the GIT stability by Tian [30] and reformulated by Donaldson [4], which is conjecturally an algebro-geometric equivalent of the existence of a Kähler metric with constant scalar curvature (*cscK* metric, in short). In this paper, we provide some basic results towards a concrete solution for the general problem "When a polarized variety is GIT-stable?". This paper will provide the foundation for our subsequent papers (cf. [22], [23], [24]). Mainly, we treat K-stability here.

The K-stability is defined as the positivity of the *Donaldson–Futaki invariants* (also called as the *generalized Futaki invariants*). Roughly speaking, they are a kind of GIT weights associated to the *test configurations*, which can be regarded as the "geometrization" of one-parameter subgroups from the GIT viewpoint. From the viewpoint of differential geometry, the Donaldson–Futaki invariant generalizes the Futaki's obstruction [7] to the existence of Kähler–Einstein metric on a Fano manifold. More precisely, it

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generalizes a value of the Futaki characters [7] at a generator of $\mathbb{C}^*$-action on a Fano manifold, which should vanish if there is a Kähler–Einstein metric on it.

Recently, Ross introduced and systematically studied with Thomas ([27], [28]) the concept of *slope stability* as the polarized variety analogue of the original stability which was defined for vector bundles by Mumford and Takemoto. Let (X, L) be a polarized variety which we are interested in. Then, essentially they gave an explicit formula for the Donaldson–Futaki invariants of some special test configurations of (X, L). It is the blow up of $X \times \mathbb{A}^1$ along a closed subscheme (scheme-theoritically) supported in $X \times \{0\}$, which is coined as *the deformation to the normal cone* by Fulton. The *slope stability* is defined as the positivity of those invariants. Therefore, K-stability implies slope stability. However, the converse does not hold in the sense that the blow up of 2 points in the projective plane is slope stable but K-unstable (Panov and Ross [26]). Please consult [27], [28], [26] for their theory.

In this paper, we generalize their theory by treating more general test configurations and give an explicit formula 3.2 for the Donaldson–Futaki invariants.

The formula 3.2 is useful in two senses. Firstly, the positivity (resp. non-negativity) of the Donaldson–Futaki invariants of those "a priori special" test configurations implies K-stability (resp. K-semistability) as we will see in Corollary 3.11.

Secondly, those Donaldson–Futaki invariants are described in an analyzable form as a sum of two parts, the *canonical divisor part* reflecting the global "positivity" of the canonical divisor, and the *discrepancy term* reflecting the singularities. Please consult Theorem 3.2 for the details of our formula.

As simplest applications, we provide straightforward algebro-geometric proofs of K-semistability (resp. K-stability) of Calabi–Yau varieties (resp. curves) which admit some mild singularities.

Corollary 1.1 (= Theorem 4.1). (i) *A semi-log-canonical canonically polarized curve $(X, L = \omega_X)$ is K-stable.*
(ii) *A semi-log-canonical polarized variety (X, L) with numerically trivial K_X is K-semistable.*

Some generalisations of the above results are in the sequel to this paper [23], although it was at last published earlier than this paper. The notion of semi-log-canonical singularities, which form a class of mild singularities, were first introduced by Kollár and Shepherd-Barron [14] for the 2-dimensional case and extended by Alexeev [1] to the higher dimensional cases. It is defined in terms of the so-called *discrepancy*, which has been developed along the log minimal model program as a fundamental invariant of singularities. A variety is simply said to be semi-log-canonical if it has only semi-log-canonical singularities. For the details, consult the original paper [1] and the textbook [13, Sections 2.3 and 5.4] on the basics of discrepancy.

We should remark that the affirmative solution to the Calabi conjecture [32] and the recent works [5], [2], [29], [17], [18] on Yau's conjecture, the polarized variety analogue of Kobayashi–Hitchin correspondence, altogether give a differential geometric proof of Corollary 1.1 for smooth X. The statement (i) of Corollary 1.1 for smooth X and the slope stability version of (ii) for X with at worst canonical singularities are also proved by an algebro-geometric method in [28, Corollary 6.7 and Theorem 8.4].

We should also note that, after having written the first draft of this paper, the author noticed that a similar formula for the Donaldson–Futaki invariants had already been discovered by Professor X. Wang [31, Proposition 19]. The two results are different in two senses. Firstly, we extend the setting to "semi" test configurations (cf. Definition 2.2), which was essential in the proof of Corollary 1.1. Our formula can be regarded as a generalization of the Ross–Thomas slope theory at the same time. Secondly, the proofs are totally different; Wang's proof depends on the relation between GIT weights and *heights* [31, Theorem 8], while ours depends on an old lemma of [21].

We refer the reader to [22], [23] and [24] as for further applications of the formula 3.2.

This paper is organized as follows. In the next section, we will review the basic stability notions for polarized varieties. For the readers' convenience, we include Mabuchi's proof [16] of the equivalence of asymptotic Hilbert stability and asymptotic Chow stability in a simplified but essentially the same form. In Section 3, we will introduce the key formula 3.2 for Donaldson–Futaki invariants and show that K-stability (resp. K-semistability) follows from those positivity (resp. non-negativity) alone. In Section 4, we give the applications.

CONVENTION. We work over the complex number field $\mathbb{C}$ throughout. An algebraic scheme means separated scheme of finite type. A variety means a reduced algebraic scheme.

A projective scheme means a complete (algebraic) scheme which has some ample invertible sheaves. (X, L) always denotes a polarized scheme, a projective scheme X with a polarization L, which means an ample invertible sheaf. Furthermore, we always assume X to be reduced, equidimensional, and Gorenstein in codimension 1 for simplicity. We also assume that X satisfies Serre's condition S_2.

For a divisor e over a normal variety X (cf. [13]), $a(e; X)$ denotes the discrepancy of e under the assumption of $\mathbb{Q}$-Gorensteiness of X and $a(e; (X, D))$ denotes the discrepancy of e on a log pair (X, D) (i.e. a pair of a normal variety X and its Weil divisor D with $\mathbb{Q}$-Cartier $K_X + D$). As for the notation about discrepancy we follow [13, Section 2.3], which we refer to for the details.

2. The stability notions

In this section, we will review the basic of the stability notions for polarized varieties. There are a few of well known versions: K-stability, asymptotic Chow stability, asymptotic Hilbert stability and their semistable versions. Originally, Gieseker [8], [9] introduced the asymptotic Hilbert stability which was confirmed for canonically polarized curves and surfaces with only mild singularities. Asymptotic Chow stability was introduced in [21] and K-stability was introduced firstly by Tian in [30], and extended and reformulated by Donaldson [4]. The motivation for introducing the K-(semi, poly)stability is to seek the GIT-counterpart of the existence of special Kähler metric, as an analogy of the Kobayashi–Hitchin correspondence for vector bundles. Let us recall that "$*$-unstable" means that "not $*$-*semi*stable".

At first, we review the definition of asymptotic stabilities.

DEFINITION 2.1. A polarized scheme (X, L) is said to be *asymptotically Chow stable* (resp. *asymptotically Hilbert stable*, *asymptotically Chow semistable*, *asymptotically Hilbert semistable*), if for an arbitrary $m \gg 0$, $\phi_m(X) \subset \mathbb{P}(H^0(X, L^{\otimes m}))$ is Chow stable (resp. Hilbert stable, Chow semistable, Hilbert semistable), where ϕ_m is the closed immersion defined by the complete linear system $|L^{\otimes m}|$.

To define the K-stability, we review the concept of test configuration following Donaldson [4]. Our notation (and even expression) almost follows [28], so we refer to it for details.

DEFINITION 2.2. A *test configuration* (resp. *semi test configuration*) for a polarized scheme (X, L) is a polarized scheme $(\mathcal{X}, \mathcal{M})$ with:
(i) a $\mathbb{G}_m$ action on $(\mathcal{X}, \mathcal{M})$,
(ii) a proper flat morphism $\alpha\colon \mathcal{X} \to \mathbb{A}^1$
such that α is $\mathbb{G}_m$-equivariant for the usual action on $\mathbb{A}^1$:

$$\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$$
$$(t, x) \mapsto tx,$$

$\mathcal{M}$ is relatively ample (resp. relatively semi ample), and $(\mathcal{X}, \mathcal{M})|_{\alpha^{-1}(\mathbb{A}^1\setminus\{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, L^{\otimes r}) \times (\mathbb{A}^1 \setminus \{0\})$ for some positive integer r, called *exponent*, with the natural action of $\mathbb{G}_m$ on the latter and the trivial action on the former.

Proposition 2.3 ([28, Proposition 3.7]). *In the above situation, a one-parameter subgroup of $GL(H^0(X, L^{\otimes r}))$ is equivalent to the data of a test configuration $(\mathcal{X}, \mathcal{M})$ whose polarization $\mathcal{M}$ is very ample (over $\mathbb{A}^1$) with exponent r of (X, L) for $r \gg 0$.*

We will call the test configuration which corresponds to a one parameter subgroup, called the *DeConcini–Procesi family*. (Its curve case appears in [20, Chapter 4 §6].) Therefore, the test configuration can be regarded as *geometrization* of one-parameter subgroup. This is a quite essential point for our study, as in Ross and Thomas' slope theory [27], [28].

The *total weight* of an action of $\mathbb{G}_m$ on some finite-dimensional vector space is defined as the sum of all weights. Here the *weights* mean the exponents of eigenvalues which should be powers of t. We denote the total weight of the induced action on $(\alpha_*\mathcal{M}^{\otimes K})|_0$ as $w(Kr)$ and $\dim X$ as n. It is a polynomial of K of degree $n+1$. We write $P(k) := \dim H^0(X, L^{\otimes k})$. Let us focus on the action of $\mathbb{G}_m$ on $(\alpha_*\mathcal{M})|_0$ and "normalize" it as follows. Let us take its $rP(r)$-th power (i.e., the new action obtained by composing the morphism of $rP(r)$-th power $\mathbb{G}_m \to \mathbb{G}_m$) and after that take a product with the suitable power of t so that the determinant of the action on $(\alpha_*\mathcal{M})|_0$ will be 1. Then the corresponding normalized weight of the $\mathbb{G}_m$-action on $(\alpha_*\mathcal{M}^{\otimes K})|_0$ is $\tilde{w}_{r,Kr} := w(k)rP(r) - w(r)kP(k)$, where $k := Kr$. It is a polynomial of the form $\sum_{i=0}^{n+1} e_i(r)k^i$ of degree $n+1$ in k for $k \gg 0$, whose coefficients $e_i(r)$ are also polynomial of r of degree $n+1$ for $r \gg 0$. Write $e_i(r) = \sum_{j=0}^{n+1} e_{i,j}r^j$ for $r \gg 0$. Since the action is normalized, $e_{n+1,n+1} = 0$. The next coefficient $e_{n+1,n}$ is the so-called *Donaldson–Futaki invariant* of the test configuration $(\mathcal{X}, \mathcal{M})$, which we will denote as $\mathrm{DF}(\mathcal{X}, \mathcal{M})$. Let us recall that $(n+1)!\, e_{n+1}(r)r^{n+1}$ is the Chow weight of $X \subset \mathbb{P}(H^0(X, L^{\otimes r}))$ ([21, Lemma 2.11]). For an arbitrary *semi* test configuration $(\mathcal{X}, \mathcal{M})$ of exponent r (cf. [28]), we can also define the (normalized) Chow weight or the Donaldson–Futaki invariant as well by setting $w(Kr)$ as the total weight of the induced action on $H^0(\mathcal{X}, \mathcal{M}^{\otimes K})/tH^0(\mathcal{X}, \mathcal{M}^{\otimes K})$.

Roughly speaking, the K-stability is positivity of the Donaldson–Futaki invariants above but recentely it is pointed out by [15] that some pathological test configurations of the following type should be "taken away" from our concern.

DEFINITION 2.4 ([15], [25]). A test configuration $(\mathcal{X}, \mathcal{L})$ is said to be *almost trivial* if $\mathcal{X}$ is $\mathbb{G}_m$-equivariantly isomorphic to the product test configuration, away from a closed subscheme of codimension at least 2.

Now we can define K-stability of Donaldson's version as follows.

DEFINITION 2.5. A polarized scheme (X, L) is said to be *K-stable* (resp. *K-semistable*, *K-polystable*) if for all $r \gg 0$, for any non-almost-trivial test configuration for (X, L) with exponent r the leading coefficient $e_{n+1,n}$ of $e_{n+1}(r)$ (the Donaldson–Futaki invariant) is positive (resp. non-negative, positive if $\mathcal{X} \not\cong X \times \mathbb{A}^1$ and non-negative otherwise).

We should note that the original K-stability of [4] is what is called K-*poly*stability in [28]. We follow the convention of [28]. These are related as follows.

Asymptotically Chow stable $\Rightarrow$ asymptotically Hilbert stable $\Rightarrow$ asymptotically Hilbert semistable $\Rightarrow$ asymptotically Chow semistable $\Rightarrow$ K-semistable.

The implications above are easy to prove, so we omit the proofs (see [21], [28]). We finish this section by proving the equivalence of two asymptotic stability notions, following the paper [16] but in much simplified form, for readers' convenience. We should note that its semistability version is not proved anywhere in literatures, as far as the author knows.

Theorem 2.6 ([16, Main Theorem (b)]). *For a polarized scheme over an arbitrary algebraically closed field, asymptotic Hilbert stability and asymptotic Chow stability are equivalent.*

Proof. We prove this along the idea of [16]. The formulations are different, but the essential ideas are the same. We make full use of the framework of test configurations. This proof is valid over an arbitrary algebraically closed field with any characteristic.

Let us recall the basic criteria of asymptotic stabilities ([28, Theorem 3.9]). (X, L) is asymptotically Chow stable (resp. asymptotically Hilbert stable) if and only if for all $r \gg 0$, any nontrivial test configuration for (X, L) with exponent r has $e_{n+1}(r) > 0$ (resp. $\tilde{w}_{r,k} > 0$ for all $k \gg 0$). Therefore, asymptotic Chow stability implies asymptotic Hilbert stability. (Actually, Chow stability implies Hilbert stability as well). To prove the converse, we assume that $\tilde{w}_{r,k} > 0$ for all $k \gg r \gg 0$.

Since

$$\left(\frac{\tilde{w}_{r,kk'}}{kk'P(kk')}\right) - \left(\frac{\tilde{w}_{r,k}}{kP(k)}\right) = \left(\frac{rP(r)}{k^2k'P(kk')P(k)}\right) \times \tilde{w}_{k,kk'}$$

and $\tilde{w}_{k,kk'}$ is positive by our assumption, the inequality $\tilde{w}_{r,kk'}/(kk'P(kk')) > \tilde{w}_{r,k}/(kP(k))$ holds for all $k' \gg k \gg r \gg 0$. Therefore, we can take a monotonely-increasing sequence k_i $(i = 0, 1, \ldots)$ divisible by r, and $k_0 = r$ with $\tilde{w}_{r,k_i}/(k_iP(k_i))$ increasing. $\tilde{w}_{r,k_i}/(k_iP(k_i))$ converges since the denominator is a polynomial of k_i of degree $n + 1$ and the numerator is a polynomial of k_i of degree at most $n + 1$. In our case, the initial term is $\tilde{w}_{r,k_0}/(k_0P(k_0)) = 0$, so the sequence converges to a positive number, which should have the same sign as $e_{n+1}(r)$. This completes the proof. □

3. A formula for Donaldson–Futaki invariants

In this section, we prove the main formula for the Donaldson–Futaki invariants of certain type of semi test configurations, and establish some results on the structures of semi test configurations which assure the usefulness of the formula. As we noted in the introduction, a same type formula of Donaldson–Futaki invariants had already been proved independently for a test configuration (with a relatively *ample polarization*) by Professor X. Wang [31], earlier than us. The differences are essentially twofolds, as we explained in the introduction. Firstly, we define a class of ideals, which we use for our study on stability.

DEFINITION 3.1. Let (X, L) be an n-dimensional polarized variety. A coherent ideal $\mathcal{J}$ of $X \times \mathbb{A}^1$ is called a *flag ideal* if $\mathcal{J} = I_0 + I_1t + \cdots + I_{N-1}t^{N-1} + (t^N)$, where $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{N-1} \subseteq \mathcal{O}_X$ is the sequence of coherent ideals. (It is equivalent to that the ideal is $\mathbb{G}_m$-invariant under the natural action of $\mathbb{G}_m$ on $X \times \mathbb{A}^1$.)

Let us introduce some notation. We set $\mathcal{L} := p_1^*L$ on $X \times \mathbb{P}^1$ or $X \times \mathbb{A}^1$, and denote the i-th projection morphism from $X \times \mathbb{A}^1$ or $X \times \mathbb{P}^1$ by p_i. Let us write the blowing up morphism as $\Pi\colon \bar{\mathcal{B}}$ $(:= Bl_{\mathcal{J}}(X\times\mathbb{P}^1)) \to X\times\mathbb{P}^1$ and the natural exceptional Cartier divisor as E, i.e., $\mathcal{O}(-E) = \Pi^{-1}\mathcal{J}$. Let us assume $\mathcal{L}^{\otimes r}(-E)$ is (relatively) semi-ample (over $\mathbb{A}^1$) and consider the Donaldson–Futaki invariant of the blowing up (semi) test configuration $(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$, where $\mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1)$. Now, we can state our main formula.

Theorem 3.2. *Let (X, L) and $\mathcal{B}$, $\mathcal{J}$ be as above. And we assume that exponent $r = 1$. (It is just to make the formula easier. For general r, put $L^{\otimes r}$ and $\mathcal{L}^{\otimes r}$ to the place of L and $\mathcal{L}$.) Furthermore, we assume that $\mathcal{B}$ is Gorenstein in codimension 1. Then the corresponding Donaldson–Futaki invariant $\mathrm{DF}((Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}(-E)))$ is*

$$\frac{1}{2(n!)((n+1)!)}\Big\{-n(L^{n-1}\,.\,K_X)(\mathcal{L}(-E))^{n+1} + (n+1)(L^n)((\mathcal{L}(-E))^n\,.\,\Pi^*(p_1^*K_X)) + (n+1)(L^n)\big((\mathcal{L}(-E))^n\,.\,K_{\bar{\mathcal{B}}/X\times\mathbb{A}^1}\big)\Big\}.$$

In the above, all the intersection numbers are taken on X or $\bar{\mathcal{B}}$, which are complete schemes.

We call the sum of first two terms the *canonical divisor part* since they involve intersection numbers with the canonical divisor K_X or its pullback, and the last term the *discrepancy term* since it reflects discrepancies over X. This division into two parts plays an important role in our applications (cf. Section 4, [23], [24]).

Proof of Theorem 3.2. By definition, the Donaldson–Futaki invariant is the coefficient of $k^{n+1}r^n$ in $w(k)rP(r) - w(r)kP(k)$ under the same notation as in the previous section. Therefore, it is enough to calculate $w(k)$ modulo $O(k^{n-1})$.

Firstly, we interpret the weight $w(k)$ as a dimension of a certain vector space, through the following lemma [21, Lemma (2.14)] which was called "droll lemma" by Mumford.

Lemma 3.3 ([21, Lemma (2.14)]). *Let V be a vector space over k and assume that $\mathbb{G}_m$ acts on $V \otimes_k k[t]$, where V is a vector space over k, by acting V trivially and t by weight (-1). For a sequence of subspaces of V, $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{N-1} \subseteq V_N = \cdots = V$, let us set $\mathcal{V} := \sum V_i t^i$ which is a sub $k[t]$ module of $V \otimes_k k[t]$. Then, the total weight on $\mathcal{V}/t\mathcal{V}$ is equal to $-\dim(V \otimes_k k[t]/\mathcal{V})$.*

From this lemma, it follows that

$$w(k) = -\dim(H^0(X \times \mathbb{A}^1, \mathcal{L}^{\otimes k})/H^0(X \times \mathbb{A}^1, \mathcal{J}^k\mathcal{L}^{\otimes k})).$$

Lemma 3.4. $h^i(X \times \mathbb{A}^1, \mathcal{J}^k\mathcal{L}^{\otimes k}) = O(k^{n-1})$ *for* $i > 0$.

Proof. By our assumption, $\mathcal{L}(-E)$ is (relatively) semiample (over $\mathbb{A}^1$). Therefore, its global section (the direct image sheaf of the projection onto $\mathbb{A}^1$) and $\mathcal{L}^{\otimes k_0}(-k_0E)$ for large enough k_0 induces a morphism $f\colon \mathcal{B} \to \mathcal{C}$, which is isomorphic over $\mathbb{A} \setminus \{0\}$. Let $\mathcal{N}$ be the canonical ample invertible sheaf with $f^*\mathcal{N} = \mathcal{L}^{k_0}(-k_0E)$. Since $H^i(X \times \mathbb{A}^1, \mathcal{J}^{kk_0}\mathcal{L}^{\otimes kk_0}) = H^i(\mathcal{B}, \mathcal{L}^{\otimes kk_0}(-kk_0E)) = H^0(\mathcal{C}, (R^i f_*\mathcal{O}_{\mathcal{B}}) \otimes \mathcal{N}^{\otimes k})$ and we have the support of $R^i f_*\mathcal{O}_{\mathcal{B}}$ only on the image of f-exceptional set (i.e., the locus in $\mathcal{C}$ where f is not finite) whose dimension is less than or equal to $(n-1)$, the lemma holds. □

Using Lemma 3.4, we can see that for $k \gg 0$;

$$\begin{aligned} &-\dim(H^0(X \times \mathbb{A}^1, \mathcal{L}^{\otimes k})/H^0(X \times \mathbb{A}^1, \mathcal{J}^k\mathcal{L}^{\otimes k})) \\ &= -h^0(\mathcal{L}^{\otimes k}/\mathcal{J}^k\mathcal{L}^{\otimes k}) + O(k^{n-1}) \\ &= \chi(X \times \mathbb{P}^1, \mathcal{J}^k\mathcal{L}^{\otimes k}) - \chi(X \times \mathbb{P}^1, \mathcal{L}^{\otimes k}) + O(k^{n-1}). \end{aligned}$$

Finally, using the weak Riemann–Roch formula of the following type, we obtain the formula by simple calculation, which we omit here.

Lemma 3.5 (Weak Riemann–Roch formula). *For an n-dimensional polarized variety (X, L) which is Gorenstein in codimension* 1,

$$\chi(X, L^{\otimes k}) = \frac{(L^n)}{n!}k^n - \frac{(L^{n-1}.K_X)}{2((n-1)!)}k^{n-1} + O(k^{n-2}),$$

where $(L^{n-1}\,.\,K_X)$ is well-defined since X is Gorenstein in codimension 1.

Proof. We can prove it by induction on $\dim(X)$. If $\dim(X) = 0$, then the assertion is obvious, and for the induction, we cut X by a general member $H \in |L^{\otimes m}|$ for $m \gg 0$. We note that H is reduced and Gorenstein in codimension 1. By fixing H and seeing the long exact sequence of coherent cohomologies, associated to

$$0 \to L^{\otimes k}(-H) \to L^{\otimes k} \to L^{\otimes k}|_H \to 0,$$

we have

$$\chi(X, L^{\otimes k}) - \chi(X, L^{\otimes(k-m)}) = \chi(H, L^{\otimes k}|_H).$$

Then the assertion on X follows from that of H. □

REMARK 3.6. The formula 3.2 can also be deduced from the formula of Chow weight by Mumford [21, Theorem (2.9)], as we did (implicitly) in [22]. As Mumford obtained it by using the *droll lemma* (Lemma 3.3), these proofs are essentially the same.

From now on, we will argue to ensure the usefulness of our formula 3.2 (cf. Corollary 3.11). Let us continue fixing a polarized variety (X, L) and think of its semi test configurations. We prepare the following notion.

Definition 3.7. A semi test configuration $(\mathcal{X}, \mathcal{M})$ is *partially normal* if any prime divisor supported on the singular locus of $\mathcal{X}$ projects surjectively onto $\mathbb{A}^1$.

For example, a normal semi test configuration is partially normal of course. This notion is defined to extend the notion of the normality of semi test configurations to that of not necessarily normal X.

Proposition 3.8. *For an arbitrary test configuration* $(\mathcal{X}, \mathcal{M})$, *there exists a finite surjective birational morphism* $f: \mathcal{Y} \to \mathcal{X}$, *where* $(\mathcal{Y}, f^*\mathcal{M})$ *is a partially normal test configuration, with* $\mathrm{DF}(\mathcal{Y}, f^*\mathcal{M}) \le \mathrm{DF}(\mathcal{X}, \mathcal{M})$.

Proof. If X is normal, we can simply take the normalization of the test configuration. Even if X is not normal, and $\mathcal{X}$ is not partially-normal, we can still "partially normalize" $\mathcal{X}$ as follows.

Let us take the normalization $\nu\colon \mathcal{X}^\nu \to \mathcal{X}$ and take $p\nu\colon (\mathcal{Y} :=) \operatorname{Spec}_{\mathcal{O}_{\mathcal{X}}}(i_*\mathcal{O}_{X\times(\mathbb{A}\setminus\{0\})} \cap \mathcal{O}_{\mathcal{X}^\nu}) \to \mathcal{X}$, where $i: X \times (\mathbb{A}^1 \setminus \{0\}) \to X \times \mathbb{A}^1$ is the open immersion. Obviously, $p\nu$ is finite as a morphism. We call this $\mathcal{Y}$ as the *partial normalization* of the semi test configuration $\mathcal{X}$. Since $\mathcal{X}^\nu$ is equidimensional and it dominates $\mathcal{Y}$ by a birational finite morphism, it is obvious that $\mathcal{Y}$ is equidimensional as well. Furthermore, since $\mathcal{X}$ is reduced, $\mathcal{Y}$ is reduced as well. Therefore, $\mathcal{Y}$ is flat over $\mathbb{A}^1$ (cf. [10, Chapter III, Proposition 9.7]) and it forms a test configuration with the natural $\mathbb{G}_m$-equivariant polarization $(p\nu)^*\mathcal{M}$.

This partial normalization is partially-normal as a test configuration (Definition 3.7) due to the following lemma.

Lemma 3.9. *The morphism* $\mathcal{X}^\nu \to \mathcal{Y}$ *is an isomorphism over an open neighborhood of the generic points of the central fiber.*

Proof. Let us take an open affine subscheme U ($\cong \operatorname{Spec} R$) $\subset \mathcal{X}$ which includes all the generic points of the central fiber in $\mathcal{X}$. Then the preimage of U in $\mathcal{Y}$ is $\operatorname{Spec}(R[t^{-1}] \cap R^\nu)$. If we take small enough U, $R[t^{-1}]$ is normal so that $R^\nu \subset R[t^{-1}]$. This completes the proof. □

The normalization or the partial normalization $\mathcal{Y}$ of semi test configuration has the canonical $\mathbb{G}_m$-linearized polazation, the pullback of the linearized polarization of the original test configuration.

Then, $\mathrm{DF}(\mathcal{Y}, f^*\mathcal{M}) \le \mathrm{DF}(\mathcal{X}, \mathcal{M})$ by [28, Proposition 5.1], whose claim holds and the proof essentially works without the normality condition of X. □

Proposition 3.10. *For an arbitrary partially normal test configuration* $(\mathcal{X}, \mathcal{M})$, *there is a flag ideal* $\mathcal{J}$ *and* $r,s \in \mathbb{Z}_{>0}$ *such that its blow up* $(\mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ *is a semi test configuration, which is Gorenstein in codimension* 1, *dominating* $(\mathcal{X}, \mathcal{M}^{\otimes s})$

by a morphism $f\colon \mathcal{B} \to \mathcal{X}$ *such that* $\mathcal{L}^{\otimes r}(-E) = f^*\mathcal{M}$ *and* $\mathrm{DF}(\mathcal{B}, \mathcal{L}^{\otimes r}(-E)) = \mathrm{DF}(\mathcal{X}, \mathcal{M}^{\otimes s})$.

Proof. Firstly, we take a $\mathbb{G}_m$-equivariant resolution of the indeterminancy of a natural birational map $h\colon X \times \mathbb{A}^1 \dashrightarrow \mathcal{X}$ as follows. Since the indeterminancy locus Z of h has codimension at least 2 in $X \times \mathbb{A}^1$, if we write $j\colon ((X \times \mathbb{A}^1) \setminus Z) \hookrightarrow X \times \mathbb{A}^1$ the natural open immersion, then $j_* h^* \mathcal{M}^{\otimes s}$ for $s \in \mathbb{Z}_{>0}$ is canonically isomorphic to $\mathcal{L}^{\otimes r}$ for some $r \in \mathbb{Z}_{>0}$, by the Serre's S_2 property of $X \times \mathbb{A}^1$ which follows from the S_2 condition of X, which is assumed in Convention. If we take sufficiently large s, then $\mathcal{M}^{\otimes s}$ is (relatively) very ample over $\mathbb{A}^1$ and so h is defined by the relative linear system over $\mathbb{A}^1$. Take a basis of $H^0(\mathcal{X}, \mathcal{M}^{\otimes s})$ as a free $k[t]$-module, which consists of eigenvectors of the naturally associated $\mathbb{G}_m$-action. They induces sections of $h|^*_{((X\times\mathbb{A}^1)\setminus Z)}\mathcal{M}^{\otimes s}$ and so, they also define global sections of $\mathcal{L}^{\otimes r}$ because Z has codimension at least 2, as we noted. Therefore, there is a flag ideal $\mathcal{J}'$ where those global sections of $\mathcal{L}^{\otimes r}$ generate the subsheaf $\mathcal{J}'\mathcal{L}^{\otimes r} \subset \mathcal{L}^{\otimes r}$. We note that $\mathcal{O}/\mathcal{J}'$ is not necessarily supported in Z. If we blow up the flag ideal $\mathcal{J}'$, we obtain a resolution of indeterminancy of h. Let us write it as $\mathcal{B}' := Bl_{\mathcal{J}'}(X \times \mathbb{A}^1) \to \mathcal{X}$ and let E' be the exceptional Cartier divisor with $\mathcal{O}_{\mathcal{B}'}(-E') = \mathcal{J}'\mathcal{O}_{\mathcal{B}'}$.

Furthermore, we can take the partial normalization $\mathcal{B}$ of $\mathcal{B}'$ as before. We note that $\mathcal{B}$ is Gorenstein in codimension 1. To prove it, it is sufficient to prove that for an arbitrary prime divisor e, a general point of e has an open neighborhood which is Gorenstein. If e is supported on the central fiber, it follows from Lemma 3.9 which implies that the generic points of e is regular. If it is not the case, the Gorenstein property of the generic point of e follows from the assumption that X is Gorenstein in codimension 1. Let us write the projection $\mathcal{B} \to X \times \mathbb{A}^1$ as Π. Let us put $\mathcal{J} := \Pi_*(p\nu)^*\mathcal{O}_{\mathcal{B}'}(-mE')$ for sufficiently large $m \in \mathbb{Z}_{>0}$. Then, it is a flag (coherent) ideal since $X \times \mathbb{A}^1$ satisfies Serre's S_2 condition. And its blow up is $\mathcal{B}$ itself since $p\nu^*\mathcal{O}_{\mathcal{B}'}(-E')$ over $\mathbb{A}^1$ is relatively ample. The relative ampleness follows from e.g., the relative Nakai–Moishezon criterion (cf. [13, Theorem 1.42]). Furthermore, if we write f the morphism from $\mathcal{B}$ to $\mathcal{X}$, $f^*\mathcal{M}^{\otimes s} = \mathcal{L}^{\otimes r}(-E)$ where $E = (p\nu)^*E'$.

We want to prove $\mathrm{DF}(\mathcal{B}, \mathcal{L}^{\otimes r}(-E)) = \mathrm{DF}(\mathcal{X}, \mathcal{M}^{\otimes s})$. For that, we note that there exists a closed subset Z' of the central fiber of $\mathcal{X}$ with $\mathrm{codim}_{\mathcal{X}}(Z') \geq 2$ such that f is isomorphism outside Z', since $\mathcal{X}$ is assumed to be partially normal. Therefore the equality $\mathrm{DF}(\mathcal{B}, \mathcal{L}^{\otimes r}(-E)) = \mathrm{DF}(\mathcal{X}, \mathcal{M}^{\otimes s})$ follows from the proof of [28, Proposition 5.1], in particular the equation on each weights $w(-)$ written at the 3 line above from the end of the proof. We note again that the proof of [28, Proposition 5.1] works essentially without the assumption of normality of X. □

A remark is that if $(\mathcal{X}, \mathcal{L})$ is almost trivial test configuration, the flag ideal $\mathcal{J}$ is of the form (t^M) with some $M \in \mathbb{Z}_{>0}$. ([25, Proposition 3.5]). Hence, Propositions 3.8 and 3.10 imply the following corollary. The "only if" part simply follows the fact that for an arbitrary semi test configuration $(\mathcal{Y}, \mathcal{N})$, by taking $(Proj \bigoplus_{a\geq 0} H^0(\mathcal{Y}, \mathcal{N}^{\otimes a}), \mathcal{O}(r))$

with sufficiently divisible positive integer r, we can associate a test configuration with the same Donaldson–Futaki invariant as $(\mathcal{Y}, \mathcal{N}^{\otimes r})$.

Corollary 3.11. (i) *A polarized variety (X, L) is K-semistable if and only if for all semi test configurations of the type in* Theorem 3.2 (*i.e.*, $(\mathcal{B} = Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ *with $\mathcal{B}$ Gorenstein in codimension* 1), *the Donaldson–Futaki invariant is non-negative.*
(ii) *A polarized variety (X, L) is K-stable if and only if for all semi test configurations of the type in* Theorem 3.2 (*i.e.*, $(\mathcal{B} = Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ *with $\mathcal{B}$ Gorenstein in codimension* 1 *and $\mathcal{J}$ is not of the form (t^M) with some $M \in \mathbb{Z}_{\geq 0}$*), *the Donaldson–Futaki invariant is positive.*

Corollary 3.11 (i) provides further corollary as follows, since the Donaldson–Futaki invariants of the type in Theorem 3.2 is continuous with respect to a variation of $\mathbb{G}_m$-linearized polarizations, if we extend the framework to $\mathbb{Q}$-*line bundles*.

We can extend the definition of the Donaldson–Futaki invariants naturally to those of the case where $\mathcal{M}$ is semiample $\mathbb{Q}$-line bundle. It is because the Donaldson–Futaki invariant behaves in homogeneous way, if we take the powers of the ($\mathbb{G}_m$-linearized) line bundle. From its natural extension, it is obvious that our formula 3.2 works even under this $\mathbb{Q}$-polarized setting.

Let us fix a flag ideal $\mathcal{J}$ and consider the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{B}, \mathcal{L}(-cE))$ with $c \in \mathbb{Q}_{>0}$ and $\mathcal{L}(-cE)$ semiample. We introduce the following Seshadri constants.

- $\mathrm{Sesh}(\mathcal{J}; (X \times \mathbb{A}^1, L \times \mathbb{A}^1)) := \sup\{c \in \mathbb{Q}_{>0} \mid \mathcal{L}(-cE) \text{ is ample}\}$,
- $\mathrm{Sesh}(I_i; (X, L)) := \sup\{c \in \mathbb{Q}_{>0} \mid (\pi_i)^* L(-ce_i) \text{ is ample}\}$, where $\pi_i\colon B_i := Bl_{I_i}(X) \to X$ is the blow up of X along the coherent ideal I_i and $\mathcal{O}_{B_i}(-e_i) = (\pi_i)^{-1} I_i$.

Recall that $\mathrm{Sesh}(\mathcal{J}; (X \times \mathbb{A}^1, L \times \mathbb{A}^1)) = \min_i\{\mathrm{Sesh}(I_i; (X, L))\}$ ([28, Corollary 5.8]). Therefore, $\mathrm{Sesh}(\mathcal{J}; (X \times \mathbb{A}^1, L \times \mathbb{A}^1))$ depends only on the numerical class of L (and $\mathcal{J}$).

The parameter c runs over all rational numbers in the interval $(0, \mathrm{Sesh}(\mathcal{J}; (X \times \mathbb{A}^1, L \times \mathbb{A}^1))$ or possibly $c = \mathrm{Sesh}(\mathcal{J}; (X \times \mathbb{A}^1, L \times \mathbb{A}^1))$. We point out that the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{B}, \mathcal{L}(-cE))$ depend only on the numerical class of L, $\mathcal{J}$ and the parameter c. Moreover, the invariant is continuous with respect to c. Therefore, we have

Corollary 3.12. *K-semistability of (X, L) only depends on X and the numerical equivalent class of L.*

4. Some K-(semi)stabilities

In this section, we give the first direct applications of the formula 3.2. That is a concise and algebro-geometric proof of some K-(semi)stabilities.

Theorem 4.1. (i) *A semi-log-canonical polarized curve (X, L), where $L = \omega_X$ (i.e., canonically polarized curve) is K-stable.*

(ii) *A semi-log-canonical polarized variety* (X, L) *with numerically trivial canonical divisor* K_X *is K-semistable.*

REMARK 4.2. Let us recall that a polarized manifold which admits a constant scalar curvature Kähler metric is K-polystable, due to the works of [5], [2], [29], [17] and [18].

Therefore, the classical result of the existence of constant curvature metric on an arbitrary compact Riemann surface gives another way of proof of (i) for smooth X, as well as the famous result by Yau on the existence of Ricci-flat Kähler metric on an arbitrary polarized Calabi–Yau manifold gives another proof of (ii) for smooth X.

Proof of Theorem 4.1. Due to Corollary 3.11, it is sufficient to prove the positivity or non-negativity of the test configurations of the form $(\mathcal{B} = Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ with $\mathcal{B}$ Gorenstein in codimension 1, for which we have a formula of Donaldson–Futaki invariants in Theorem 3.2.

Let us assume that X is semi-log-canonical, and denotes its normalization as $\nu\colon X^\nu \to X$ with its conductor $\mathrm{cond}(\nu)$. Then $(X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1 + X^\nu \times \{0\})$ is log-canonical, which can be shown by seeing the discrepancy of the exceptional divisors of the log resolution of $X^\nu \times \mathbb{A}^1$ of the form $\tilde{X} \times \mathbb{A}^1 \to X^\nu \times \mathbb{A}^1$, where $\tilde{X} \to X^\nu$ is a log resolution of $(X^\nu, \mathrm{cond}(\nu))$, which exists by [11] and [12]. This upshot is an easy case of the inversion of adjunction of log-canonicity. Now, we want to prove that for an arbitrary (not necessarily closed) point $\eta \in X^\nu \times \{0\}$ with $\dim\{\bar{\eta}\} \leq n-1$, $\mathrm{mindiscrep}(\eta; (X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1) \geq 0$, where "mindiscrep" means the associated minimal discrepancy. We take an exceptional prime divisor E above $X^\nu \times \mathbb{A}^1$ with $\mathrm{center}_{X^\nu \times \mathbb{A}^1}(E) = \{\bar{\eta}\}$. Then;

$$\begin{aligned} &a(E; (X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1)) \\ &= a(E; (X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1 + X^\nu \times \{0\})) + v_E(t) \\ &\geq \mathrm{mindiscrep}(\eta; (X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1 + X^\nu \times \{0\})) + 1, \end{aligned}$$

where, $v_E(-)$ denotes the corresponding discrete valuation for prime divisor E. Here, $a(-)$ denotes the corresponding discrepancy (cf. [13, Section 2.3] or Convention of this paper). Since $(X^\nu \times \mathbb{A}^1, \mathrm{cond}(\nu) \times \mathbb{A}^1 + X^\nu \times \{0\})$ is log-canonical as we proved, the last line is nonnegative.

Therefore, we proved that the relative canonical divisor $K_{\mathcal{B}/X \times \mathbb{A}^1}$ is effective so that the discrepancy term is nonnegative, if X is semi-log-canonical.

The canonical divisor part vanishes in this case, since the canonical divisor is assumed to be numerically trivial and the canonical divisor parts consist of the intersection numbers with the canonical divisor K_X or its pullback. This completes the proof of (ii).

For the case (i), the signature of the canonical divisor part is that of $((\mathcal{L}^{\otimes r} - E) \,.\, (\mathcal{L}^{\otimes r} + E)) = -(E^2)$. We note that dividing the flag ideal $\mathcal{J}$ by power of t does not change the associated Donaldson–Futaki invariants. Therefore, we can assume that $\mathcal{O}/\mathcal{J}$ is supported in a *proper* closed subset of $X \times \{0\}$, not whole of $X \times \{0\}$, without loss of generality, by dividing by suitable power of t. Consider the normalization $\mu\colon \mathcal{B}^{\mu} \to \mathcal{B}$. We note that there is some connected component S of $\mathcal{B}^{\mu}$, which is a blow up of 0-dimensional closed subscheme in some connected component of $X^{\nu} \times \mathbb{A}^1$, by the assumption above. Then, we have $(-\mu^* E|_S^2) > 0$ and $(-\mu^* E|_{\mathcal{B}^{\mu}\setminus S}^2) \geq 0$. Therefore, we complete the proof of (i) as well. □

We end with reviewing that for *asymptotic stability* of these polarized varieties, following is obtained so far by [21], [9] and [3], in comparison with Theorem 4.1.

Theorem 4.3. (i) ([21], [9]) *A semi-log-canonical polarized curve* (X, L), *where* $L = \omega_X$ (*i.e., canonically polarized curve*) *is asymptotically stable.*
(ii) (*the combination of* [32] *and* [3]) *A smooth polarized manifold* (X, L) *with numerically trivial canonical divisor* K_X *is asymptotically stable.*

The proof of (i) is purely algebro-geometric and done by calculation of weights, although the proof of (ii) is only done by differential geometric methods, which depends on the existence of Ricci-flat Kähler metrics.

We also note that we can *not* admit semi-log-canonical singularities for Theorem 4.3 (ii), and the naturally conceivable extension of (i) to higher dimensional semi-log-canonical canonically polarized varieties does *not* hold, as we will show explicit counterexamples in [23].

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