BOEING SCIENTIFIC RESEARCH LABORATORIES DOCUMENT D1-82-0572

BOEING

SCIENTIFIC RESEARCH LABORATORIES

A Generalized Bivariate Exponential Distribution

also available from the Author

Albert W. Marshall

Ingram Olkin

MATHEMATICS RESEARCH OCTOBER 1966

D1-82-0572

A GENERALIZED BIVARIATE EXPONENTIAL DISTRIBUTION

by

Albert W. Marshall

and

Ingram Olkin
Stanford University

Mathematical Note No. 486

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

October 1966

ABSTRACT

In a previous paper ("A Multivariate Exponential Distribution," Boeing Scientific Research Laboratories Document D1-82-0505) the authors have derived a multivariate exponential distribution from points of view designed to indicate the applicability of the distribution. Two of these derivations are based on "shock models" and one is based on the requirement that residual life is independent of age.

The practical importance of the univariate exponential distribution is partially due to the fact that it governs waiting times in a Poisson process. In this paper, the distribution of joint waiting times in a bivariate Poisson process is investigated. There are several ways to define "joint waiting time." Some of these lead to the bivariate exponential distribution previously obtained by the authors, but others lead to a generalization of it. This generalized bivariate exponential distribution is also derived from shock models. The moment generating function and other properties of the distribution are investigated.

1. Introduction.

It is common in applications to encounter random vectors with elements that are clearly dependent. In such cases, the marginal distributions are often known, but these alone do not determine the joint distribution; to the contrary, it is well known that the joint distribution can take many different forms. However, it may be that the joint distribution can be obtained through a study of the mechanism causing dependence. In the case of exponential marginals, we have in a previous paper (Marshall and Olkin, 1966) presented such derivations of the bivariate distribution given by

$$P\{X > x,\ Y > y\} \equiv \overline{F}(x,y) = e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x,y)},\ \text{for } x,y \geq 0. \tag{1.1}$$

If X and Y are life lengths of devices subjected to shocks, this distribution arises when the occurrence of shocks is governed by one or more Poisson processes. In addition, (1.1) is the unique bivariate distribution (with exponential marginals) which satisfies the requirement that residual life given survival to a common age t has a distribution independent of t.

The viewpoint of this paper is suggested by the fact that the univariate exponential distribution governs waiting times in a Poisson process. More precisely, if $\{Z(t),\ t \geq 0\}$ is a Poisson process with parameter λ and s is a fixed point, then the waiting time X from s to the next event (jump) of the process has an exponential distribution with parameter λ. Likewise, if s is replaced by a random variable S, the occurrence time of the i^{th} event, then the waiting time X from S to the $i+1^{st}$ event is again exponentially distributed with parameter λ. More generally, S can be any stopping time, i.e., any non-negative random variable for which one can check whether $S < s$ by observing $X(t)$ only for $t < s$. To avoid ambiguities and problems of separability, we assume throughout this paper that sample functions of Poisson processes are right continuous with probability one.

Following Dwass and Teicher (1957), we define (in §2.1) a bivariate Poisson process $\{Z(t) = (Z_1(t), Z_2(t)),\ t \geq 0\}$ where $Z_1(t)$ and $Z_2(t)$ are correlated Poisson processes. There are several ways to define a "joint waiting time" for this process:

(*i*) We may choose a fixed time s, and consider the waiting time X to the next event in the Z_1 process together with the time Y to the next event in the Z_2 process.

(*ii*) In place of a fixed time s as in (*i*), we may gin waiting at a random time S that is a stopping time.

(*iii*) We may consider the waiting time X from a fixed point s to the next event in the Z_1 process together with the waiting time Y from the fixed point $s+\delta$ to the next event in the Z_2 process. Alternatively, it may be that replaced by a stopping time.

(*iv*) F ... zation is obtained if in (*iii*), both s and $s+\delta$ are stopping times.

It is not difficult to show that in cases (*i*) and (*ii*), the joint distribution of X and Y is given by (1.1). On the other hand, (*iii*) leads to the generalization

$$(1.2) \quad P\{X > x,\ Y > y\} \equiv \overline{F}(x,y;\delta) = \begin{cases} e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max[x,y+\min(x,\delta)]}, & \delta \geq 0 \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max[x,y+\min(x,-\delta)]}, & \delta \leq 0,\ x,y \geq 0. \end{cases}$$

The parameter δ might be called a "shift" parameter, though it is not simply a location parameter. Of course when $\delta = 0$, (1.2) reduces to (1.1).

Case (*iv*) above results in a still more general family of distributions. These can be obtained from (1.2) by first conditioning on δ, so that the result is a mixture of such distributions:

$$(1.3) \qquad P\{X > x,\ Y > y\} = \int_{-\infty}^{\infty} \overline{F}(x,y;\delta)\,dG(\delta).$$

Because of this representation, we confine ourselves in this paper almost exclusively to a discussion of the distribution given by (1.2).

In addition to governing the waiting times in the bivariate Poisson process, (1.2) can be derived from shock models (§2.3). These derivations shed light on the applicability of (1.2), particularly in reliability problems.

For convenience, we say that X and Y are bivariate exponential, $\mathrm{BVE}(\lambda_1,\lambda_2,\lambda_{12};\delta)$, if (1.2) holds, and refer to the distribution of (1.2) as the $\mathrm{BVE}(\lambda_1,\lambda_2,\lambda_{12};\delta)$ distribution. When δ is the only significant parameter, it is convenient to abbreviate $\mathrm{BVE}(\lambda_1,\lambda_2,\lambda_{12};\delta)$ by writing $\mathrm{BVE}(\delta)$. We refer to $\overline{F}(x,y;\delta) \equiv P\{X > x,\ y > y\}$ as the *survival probability*; this has a much simpler form than the distribution function $F(x,y;\delta) \equiv P\{X \leq x,\ Y \leq y\}$.

The survival probability $\overline{F}(x,y;\delta)$ is clearly decreasing in $\delta \geq 0$ and increasing in $\delta \leq 0$. Since $\overline{F}(x,y;\delta)$ has marginal survival probabilities

$$P\{X > x\} = \overline{F}(x,0;\delta) = e^{-(\lambda_1+\lambda_{12})x}$$
$$P\{Y > y\} = \overline{F}(0,y;\delta) = e^{-(\lambda_2+\lambda_{12})y}$$

that are independent of δ,

$$\overline{F}(x,y;\delta_1) - \overline{F}(x,y;\delta_2) = F(x,y;\delta_1) - F(x,y;\delta_2).$$

Thus, $F(x,y;\delta)$ is also decreasing (increasing) in $\delta \geq 0$ $(\delta \leq 0)$. If $\overline{F}(x,y;\delta)$ is the survival probability of a pair of items, it follows that the probability both items survive and the probability neither item survives both reach a maximum when $\delta = 0$ (the case of (1.1)), and a minimum when $\delta = \pm\infty$ (the case of independence).

Various properties of the distribution $F(x,y;\delta)$ are discussed in §3, generalizations are obtained in §4, and a brief discussion of the multivariate case is given in §5.

2. Derivations.

2.1. The bivariate Poisson process.

The univariate Poisson distribution can, of course, be obtained as a limit from the binomial distribution, which in turn arises via convolutions from the Bernoulli distribution. This derivation can be repeated in the bivariate case to obtain a bivariate Poisson distribution, because there is a unique bivariate distribution with Bernoulli marginals. That distribution places mass only at (0,0), (0,1), (1,0) and (1,1). By convolving this distribution and taking limits, Teicher (1954) obtained the bivariate Poisson distribution

$$(2.1)\quad P\{Z_1 = x,\ Z_2 = y\} = e^{-\lambda(p_{00}+p_{10}+p_{01})} \sum_{\alpha=0}^{\min(x,y)} \frac{(\lambda p_{00})^{\alpha}(\lambda p_{01})^{x-\alpha}(\lambda p_{10})^{y-\alpha}}{\alpha!(x-\alpha)!(y-\alpha)!}.$$

Earlier, this distribution was obtained by McKendrick (1926), who obtained (2.1) as the solution to a difference-differential equation. Starting with the bivariate Bernoulli distribution, Campbell (1934) obtained (2.1) via generating functions.

Dwass and Teicher (1957) have shown that this distribution is the only bivariate distribution with Poisson marginals that is infinitely divisible. Now if $\{Z_1(t),\ t \geq 0\}$ and $\{Z_2(t),\ t \geq 0\}$ are Poisson processes and if $\{(Z_1(t),\ Z_2(t)),\ t \geq 0\}$ has stationary and independent increments, then the increments must have an infinitely divisible distribution, hence the distribution given by (2.1). Because of this uniqueness, we call the process *the* bivariate Poisson process.

Dwass and Teicher have derived the bivariate Poisson process $\{(Z_1(t), Z_2(t)), t \geq 0\}$ directly from a one-dimensional Poisson process $\{Z^*(t), t \geq 0\}$ as follows: If $Z^*(t)$ jumps at time τ, then $Z(t) = (Z_1(t), Z_2(t))$ jumps at τ from $Z(\tau-)$ to $Z(\tau-) + (i,j)$ where $P\{(i,j) = (0,0)\} = p_{11}$, $P\{(i,j) = (0,1)\} = p_{10}$, $P\{(i,j) = (1,0)\} = p_{01}$, and $P\{(i,j) = (1,1)\} = p_{00}$. The increments (i,j) at each jump are independent and $\{Z^*(t), t \geq 0\}$ has parameter λ, so that

$$P\{Z_1(t) = x, Z_2(t) = y\} = \sum_{\alpha=0}^{\min(x,y)} \sum_{k=x+y-\alpha}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \frac{k! p_{00}{}^{\alpha} p_{01}{}^{x-\alpha} p_{10}{}^{y-\alpha} p_{11}{}^{k-x-y+\alpha}}{\alpha!(x-\alpha)!(y-\alpha)!(k-x-y+\alpha)!}$$

$$= e^{-\lambda t(p_{00}+p_{10}+p_{01})} \sum_{\alpha=0}^{\min(x,y)} \frac{(\lambda t p_{00})^{\alpha}(\lambda t p_{01})^{x-\alpha}(\lambda t p_{10})^{y-\alpha}}{\alpha!(x-\alpha)!(y-\alpha)!} .$$

Directly from the model, it is easily seen that

$$P\{Z_1(t) = x\} = \frac{e^{-\lambda t(p_{00}+p_{01})}}{x!} [\lambda t(p_{00}+p_{01})]^x .$$

In their derivation, Dwass and Teicher take $p_{11} = 0$, but this results in no loss of generality.

One can also obtain the bivariate Poisson process from three independent one-dimensional Poisson processes $\{Z_1^*(t), t \geq 0\}$, $\{Z_2^*(t), t \geq 0\}$ and $\{Z_{12}^*(t), t \geq 0\}$ with respective parameters λ_1, λ_2 and λ_{12}. In fact, $\{(Z_1^*(t) + Z_{12}^*(t), Z_2^*(t) + Z_{12}^*(t)), t \geq 0\}$ is a bivariate Poisson process. This observation is utilized in §2.3.

2.2. <u>The joint waiting time</u>.

Choose s_1 and s_2, and consider the time X from s_1 to the next event in the process $\{Z_1(t),\ t \geq 0\}$ together with the time Y from s_2 to the next event in the process $\{Z_2(t),\ t \geq 0\}$. To find the joint survival probability $P\{X > x,\ Y > y\}$, it is convenient to assume first that $\delta = s_2 - s_1 \geq 0$. Let $t_1 - s_1 = x$ and $t_2 - s_2 = y$. The three cases (i) $x \leq \delta$, (ii) $\delta \leq x \leq y + \delta$, and (iii) $y + \delta \leq x$ (indicated in Figure 1) must be treated separately.

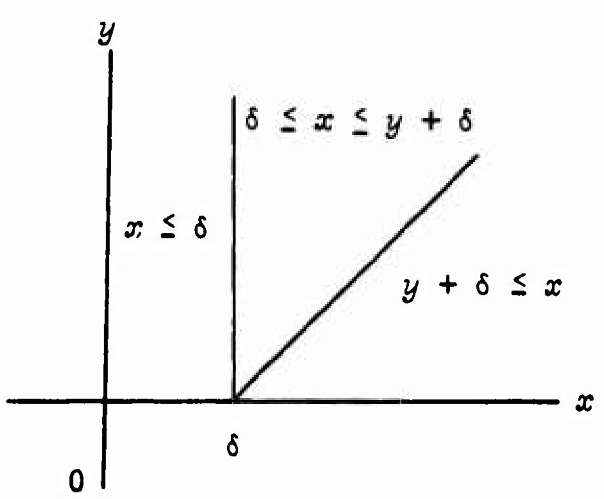

Fig. 1

<u>Case (i)</u>: $x \leq \delta$ $(s_1 \leq t_1 \leq s_2 \leq t_2)$.

$$P\{X > x,\ Y > y\} =$$

$$= \sum_{i,j} e^{-\lambda(t_1-s_1)} \frac{[\lambda(t_1-s_1)]^i}{i!} (p_{11}+p_{10})^i e^{-\lambda(t_2-s_2)} \frac{[\lambda(t_2-s_2)]^j}{j!} (p_{11}+p_{01})^j$$

$$= e^{-\lambda(p_{00}+p_{01})x-\lambda(p_{00}+p_{10})y} .$$

<u>Case *(ii)*</u>: $\delta \le x \le y + \delta$ $(s_1 \le s_2 \le t_1 \le t_2)$.

$$P\{X > x,\ Y > y\} = \sum_{i,j,k} e^{-\lambda(s_2-s_1)}\frac{[\lambda(s_2-s_1)]^i}{i!}(p_{11}+p_{10})^i \cdot$$
$$\cdot e^{-\lambda(t_1-s_2)}\frac{[\lambda(t_1-s_2)]^j}{j!}p_{11}^j e^{-\lambda(t_2-t_1)}\frac{[\lambda(t_2-t_1)]^k}{k!}(p_{11}+p_{01})^k$$

$$= e^{-\lambda p_{00}\delta-\lambda p_{01}x-\lambda(p_{00}+p_{10})y}.$$

<u>Case *(iii)*</u>: $y + \delta \le x$ $(s_1 \le s_2 \le t_2 \le t_1)$.

$$P\{X > x,\ Y > y\} = \sum_{i,j,k} e^{-\lambda(s_2-s_1)}\frac{[\lambda(s_2-s_1)]^i}{i!}(p_{11}+p_{10})^i \cdot$$
$$\cdot e^{-\lambda(t_2-s_2)}\frac{[\lambda(t_2-s_2)]^j}{j!}p_{11}^j e^{-\lambda(t_1-t_2)}\frac{[\lambda(t_1-t_2)]^k}{k!}(p_{11}+p_{10})^k$$

$$= e^{-\lambda(p_{00}+p_{01})x-\lambda p_{10}y}.$$

Letting $\lambda_1 = \lambda p_{01}$, $\lambda_2 = \lambda p_{10}$, and $\lambda_{12} = \lambda p_{00}$, we obtain from these three cases the survival probability for $\delta \ge 0$. If $\delta \le 0$, the result is then obtained by interchanging λ_1 and λ_2, x and y, δ and $-\delta$. Together, this yields

$$(2.2)\quad P\{X > x,\ Y > y\} = \begin{cases} e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(x+y)}, & \delta \le -y \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(x-\delta)}, & -y \le \delta \le \min(x-y,0) \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}y}, & \min(x-y,0) \le \delta \le 0 \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}x}, & 0 \le \delta \le \max(x-y,0) \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(y+\delta)}, & \max(x-y,0) \le \delta \le x \\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(x+y)}, & \delta \ge x, \end{cases}$$

which can be rewritten as (1.2).

2.3. A shock model derivation.

We have previously us both fatal and non-fatal shock models to derive the BVE$(\lambda_1,\lambda_2,\lambda_{12};0)$ (Marshall & Olkin, 1966). Such models can be modified to yield the BVE$(\lambda_1,\lambda_2,\lambda_{12};\delta)$ for any δ; only the fatal shock case is considered here.

Suppose that a device is placed in an environment where it is subject to fatal shocks from two sources, governed by Poisson processes $Z_1(t)$, $Z_{12}(t)$ with parameters λ_1, λ_{12}. After a time δ, a second device is placed in a similar environment where fatal shocks are governed by the processes $Z_2(t)$, $Z_{12}(t)$ with parameters λ_2, λ_{12}. Suppose, in addition, that the three processes are independent.

Let X (Y) be the age, or length of service, of the first (second) device at the time of death. To obtain the joint survival probability $P\{X > x,\ Y > y\}$, we again consider the cases represented by the three regions in Figure 1.

Case (i): $x \leq \delta$.

$$P\{X > x,\ Y > y\} = e^{-(\lambda_1+\lambda_{12})x-(\lambda_2+\lambda_{12})y}$$

Case (ii): $\delta \leq x \leq y + \delta$.

$$P\{X > x,\ Y > y\} = e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(y+\delta)}$$

Case (iii): $y + \delta \leq x$.

$$P\{X > x,\ Y > y\} = e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}x}.$$

Thus (X,Y) has the joint distribution given by (1.2).

3. Properties of the distribution $F(x,y;\delta)$.

3.1. The distribution function.

The BVE distribution $(\delta=0)$ has both an absolutely continuous and a singular part, with positive mass on the line $x = y$. From the shock model of §2.3, we see that if (X,Y) is BVE(δ), then in case an event in the process $Z_{12}(t)$ causes both devices to fail, $X = Y + \delta$. Thus the BVE(δ) distribution has a singular part, with positive mass on the line $x = y + \delta$.

Theorem 3.1. If $F(x,y)$ is BVE$(\lambda_1,\lambda_2,\lambda_{12};\delta)$, $\delta \geq 0$, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, then

$$\bar{F}(x,y) = \alpha\bar{F}_a(x,y) + \bar{\alpha}\bar{F}_s(x,y),$$

where

$$\alpha = 1 - \frac{\lambda_{12}}{\lambda} e^{-(\lambda_1+\lambda_{12})\delta}, \quad \bar{\alpha} = 1 - \alpha,$$

$$\alpha\bar{F}_a(x,y) = e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max[x,y+\min(x,\delta)]} - \frac{\lambda_{12}}{\lambda} e^{\lambda_2\delta-\lambda\max(x,y+\delta)}$$

is absolutely continuous, and

$$\bar{F}_s(x,y) = e^{-\lambda[\max(x,y+\delta)-\delta]}$$

is a singular distribution.

Proof. Let $\theta_1 = \lambda_1 + \lambda_{12}$, $\theta_2 = \lambda_2 + \lambda_{12}$. Since

$$\bar{F}(u,v) = \begin{cases} e^{-\theta_1 u-\theta_2 v}, & u \leq \delta \\ e^{-\lambda_1 u-\theta_2 v-\lambda_{12}\delta}, & \delta \leq u \leq \delta + v \\ e^{-\theta_1 u-\lambda_2 v}, & \delta + v \leq u, \end{cases}$$

it follows by differentiation that

$$\alpha f_a(u,v) = \begin{cases} \theta_1\theta_2 e^{-\theta_1 u-\theta_2 v}, & u < \delta \\ \lambda_1\theta_2 e^{-\lambda_1 u-\theta_2 v-\lambda_{12}\delta}, & \delta < u < \delta + v \\ \theta_1\lambda_2 e^{-\theta_1 u-\lambda_2 v}, & \delta + v < u. \end{cases}$$

By computing $\alpha\overline{F}_a(x,y) = \int_x^\infty du \int_y^\infty dv\ \alpha f_a(u,v)$, we can obtain α from the condition $\overline{F}_a(0,0) = 1$. $\overline{\alpha F}_s(x,y)$ is obtained by subtraction: $\overline{\alpha F}_s(x,y) = \overline{F}(x,y) - \alpha\overline{F}_a(x,y)$.

In order to carry out this program, we begin by computing three integrals which can be combined to give $\alpha\overline{F}_a(x,y)$. These integrals again correspond to the regions shown in Figure 1.

Case (i): $x \le \delta$.

Let $A(x,y)$ be the integral of αf_a over the region $\{(u,v)\colon x < u < \delta,\ v > y\}$. Then

$$A(x,y) = \int_x^\delta du \int_y^\infty dv\quad \alpha f_a(u,v) = e^{-\theta_2 y}\left[e^{-\theta_1 x} - e^{-\theta_1\delta}\right]$$

Case (ii): $\delta \le x \le y + \delta$.

Let $B(x,y)$ be the integral of αf_a over the region $\{(u,v)\colon x < u \le v + \delta,\ v \ge y\}$. Then

$$B(x,y) = \int_y^\infty dv \int_x^{v+\delta} du\ \alpha f_a(u,v) = e^{-\lambda_1 x-\lambda_{12}\delta-\theta_2 y} - \frac{\theta_2}{\lambda}\, e^{-\theta_1\delta-\lambda y}.$$

When $x = y + \delta$,

$$B(y+\delta,y) \equiv \frac{\lambda_1}{\lambda} e^{-\theta_1\delta-\lambda y};$$

when $x = \delta$,

$$B(\delta,y) = e^{-\theta_1\delta-\theta_2 y} - \frac{\theta_2}{\lambda} e^{-\theta_1\delta-\lambda y}.$$

<u>Case (*iii*):</u> $x \geq y + \delta$.

Let $C(x,y)$ be the integral of αf_a over the region $\{(u,v)\colon u > x,\ y < v < u - \delta\}$. Then

$$C(x,y) = \int_x^\infty du \int_y^{u-\delta} dv\ \alpha f_a(u,v) = e^{-\lambda_2 y-\theta_1 x} - \frac{\theta_1}{\lambda} e^{\lambda_2\delta-\lambda x}.$$

When $x = y + \delta$,

$$C(y+\delta,y) = \frac{\lambda_2}{\lambda} e^{-\lambda y-\theta_1\delta}.$$

By combining the various integrals, we obtain $\alpha\bar{F}_a$:

<u>Case (*i*):</u> $x \leq \delta$.

$$\alpha\bar{F}_a(x,y) = A(x,y) + B(\delta,y) + C(y+\delta,y) = e^{-\theta_1 x-\theta_2 y} - \frac{\lambda_{12}}{\lambda} e^{-\theta_1\delta-\lambda y}.$$

<u>Case (*ii*):</u> $\delta \leq x \leq y + \delta$.

$$\alpha\bar{F}_a(x,y) = B(x,y) + C(y+\delta,y) = e^{-\lambda_1 x-\lambda_{12}\delta-\theta_2 y} - \frac{\lambda_{12}}{\lambda} e^{-\theta_1\delta-\lambda y}.$$

<u>Case (*iii*):</u> $x \geq y + \delta$.

$$\alpha\bar{F}_a(x,y) = B(x,x-\delta) + C(x,y) = e^{-\lambda_2 y-\theta_1 x} - \frac{\lambda_{12}}{\lambda} e^{-\lambda x+\lambda_2\delta}.$$

This completes the derivation of $\bar{F}_a$.

Note that

$$\alpha = \alpha\bar{F}_a(0,0) = 1 - \frac{\lambda_{12}}{\lambda} e^{-(\lambda_1+\lambda_{12})\delta}.$$

It is now easy to obtain

$$(1-\alpha)\bar{F}_s(x,y) = \bar{F}(x,y) - \alpha\bar{F}_a(x,y) = \frac{\lambda_{12}}{\lambda} e^{-\lambda \max(x,y+\delta) + \lambda_2\delta},$$

so that

$$\bar{F}_s(x,y) = e^{-\lambda[\max(x,y+\delta)-\delta]}. \quad ||$$

3.2. <u>The moment generating function.</u>

As in the case of the BVE distribution $(\delta=0)$, direct computation of an integral of the form $\int_0^\infty\int_0^\infty G(x,y)dF(x,y)$ requires separate consideration of the absolutely continuous and singular parts of F. However, this can be avoided by choosing a kernel G such that $G(0,y) \equiv 0 \equiv G(x,0)$ where G is of bounded variation on finite intervals. It then follows from integration by parts (Young, 1917) that

$$\int_0^\infty\int_0^\infty G(x,y)dF(x,y) = \int_0^\infty\int_0^\infty \bar{F}(x,y)dG(x,y).$$

The kernel $G(x,y) = (1-e^{-sx})(1-e^{-ty})$ has the required property and in addition, the integral $\iint GdF$ is a moment generating function. Thus we compute (again, take $\delta \geq 0$)

$$\phi(s,t;\delta) \equiv \int_0^\infty\int_0^\infty (1-e^{-sx})(1-e^{-ty})dF(x,y;\delta) = st\int_0^\infty\int_0^\infty \bar{F}(x,y)e^{-sx-ty}dxdy$$

by breaking up the integral into three parts.

$$\int_0^\delta dx\int_0^\infty dy\ \overline{F}(x,y)e^{-sx-ty} = \int_0^\delta dx\int_0^\infty dy\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(x+y)-sx-ty}$$

$$= \frac{1-e^{-(\lambda_1+\lambda_{12}+s)\delta}}{(\lambda_1+\lambda_{12}+s)(\lambda_2+\lambda_{12}+t)} .$$

$$\int_0^\infty dy\int_\delta^{y+\delta} dx\ \overline{F}(x,y)e^{-sx-ty} = \int_0^\infty dy\int_\delta^{y+\delta} dx\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}(\delta+y)-sx-ty}$$

$$= \frac{e^{-(\lambda_1+\lambda_{12}+s)\delta}}{(\lambda_2+\lambda_{12}+t)(\lambda_1+\lambda_2+\lambda_{12}+s+t)} .$$

$$\int_0^\infty dy\int_{y+\delta}^\infty dx\ \overline{F}(x,y)e^{-sx-ty} = \int_0^\infty dy\int_{y+\delta}^\infty dx\ e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}x-sx-ty}$$

$$= \frac{e^{-(\lambda_1+\lambda_{12}+s)\delta}}{(\lambda_1+\lambda_{12}+s)(\lambda_1+\lambda_2+\lambda_{12}+s+t)} .$$

Summing these three integrals yields, for $\delta \geq 0$,

$$\text{(3.1)}\qquad \frac{\phi(s,t;\delta)}{st} = \frac{1}{(\lambda_1+\lambda_{12}+s)(\lambda_2+\lambda_{12}+t)}\left[1+\frac{\lambda_{12}}{\lambda+s+t}\,e^{-(\lambda_1+\lambda_{12}+s)\delta}\right].$$

Interchange of s and t, λ_1 and λ_2, δ and $-\delta$ yields $\phi(s,t)/st$ for $\delta < 0$.

To obtain the moments of the BVE(δ), we note first that since the marginal distributions do not depend on δ,

$$EX = \frac{1}{\lambda_1+\lambda_{12}},\quad Var\ X = \frac{1}{(\lambda_1+\lambda_{12})^2}$$

$$EY = \frac{1}{\lambda_2+\lambda_{12}},\quad Var\ Y = \frac{1}{(\lambda_2+\lambda_{12})^2} .$$

We also have, for $\delta \geq 0$,

$$EXY = \lim_{s,t\to 0} \frac{\psi(s,t)}{st} = \frac{1}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} + \frac{\lambda_{12}e^{-(\lambda_1+\lambda_{12})\delta}}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})\lambda},$$

$$\text{(3.2)} \qquad Cov(X,Y) = \frac{\lambda_{12}e^{-(\lambda_1+\lambda_{12})\delta}}{\lambda(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})}$$

and the correlation

$$\text{(3.3)} \qquad Corr(X,Y) \equiv \rho(X,Y) = \frac{\lambda_{12}}{\lambda} e^{-(\lambda_1+\lambda_{12})\delta}.$$

Note that $0 \leq \rho(X,Y) \leq 1$, and that X and Y are independent when $\rho(X,Y) = 0$.

3.3. Representation in terms of independent random variables.

Theorem 3.2. For $\delta \geq 0$, (X,Y) is BVE$(\lambda_1,\lambda_2,\lambda_{12};\delta)$ if and only if there exist independent exponential random variables U_1,U_2,U_3 and U_4 with respective parameters $\lambda_1,\lambda_2,\lambda_{12},\lambda_{12}$ such that

$$X = \min(U_1,U_3),$$

$$Y = \begin{cases} \min(U_2,U_3-\delta), & \text{if } U_3 \geq \delta, \\ \min(U_2,U_4), & \text{if } U_3 < \delta. \end{cases}$$

This theorem can be obtained directly from the shock model of §2.3, or it can be verified formally from the relation

$$P\{X > x,\ Y > y\} = P\{U_1 > x,\ U_2 > y\}\Big[P\{U_3 > x,\ U_3 > y + \delta | U_3 > \delta\}P\{U_3 > \delta\}$$
$$+ P\{U_4 > y\}P\{U_3 > x | U_3 < \delta\}P\{U_3 < \delta\}\Big].$$

This characterization may be of some interest for $\delta > 0$, even though the simplicity of the case $\delta = 0$ is lost.

In case $\delta = 0$ the representation $X = \min(U_1,U_3)$, $Y = \min(U_2,U_3)$ immediately yields the fact that $\min(X,Y) = \min(U_1,U_2,U_3)$ is exponentially distributed. However, if (X,Y) is BVE(δ) and $\delta > 0$, then m (X,Y) is not exponentially distributed, but

$$P\{\min(X,Y) > w\} = e^{-\lambda w-\lambda_{12}\min(w,\delta)},$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. This distribution is piecewise exponential, and has a decreasing hazard rate.

4. More general bivariate exponential distributions.

In §2.2 we considered the joint distribution $F(x,y;\delta)$ of waiting times X (Y) from s_1 (s_2) to the next event in the Z_1 (Z_2) process. As mentioned in the introduction, s_1 and s_2 can be replaced by random variables (stopping times) S_1 and S_2, so long as $S_2 - S_1 = \delta$ is retained as a fixed number. That the resulting waiting time distribution is unchanged can be seen in a variety of ways, e.g. by observing that $F(x,y;\delta)$ depends on s_1 and s_2 only through δ.

If, on the other hand, δ is replaced by a random variable Δ, then the joint waiting time distribution is a mixture over δ of distributions $F(x,y;\delta)$ (mixed according to the distribution of Δ). Of course, such a distribution has exponential marginals, because the marginals of $F(x,y;\delta)$ are independent of δ. If $\Delta = S_2 - S_1$ has the right continuous distribution G and X (Y) is the waiting time to the next event after S_1 (S_2) in the Z_1 (Z_2) process, then we obtain from (2.2) that

$$(4.1)\quad \frac{P\{X > x,\ Y > y\}}{e^{-\lambda_1 x-\lambda_2 y}} = \int_{-\infty}^{\infty} \frac{\bar{F}(x,y;\delta)}{e^{-\lambda_1 x-\lambda_2 y}}\, dG(\delta) =$$

$$= e^{-\lambda_{12}(x+y)} G(-y) + e^{-\lambda_{12}x}\int_{-y}^{\min(x-y,0)+} e^{\lambda_{12}\delta} dG(\delta)$$

$$+ e^{-\lambda_{12}y}[G(0) - G(\min(x-y,0))] + e^{-\lambda_{12}x}[G(\max(x-y,0)) - G(0)]$$

$$+ e^{-\lambda_{12}y}\int_{\max(x-y,0)}^{x+} e^{-\lambda_{12}\delta} dG(\delta) + e^{-\lambda_{12}(x+y)}\bar{G}(x).$$

An interesting special case of (4.1) is obtained with S_1 (S_2) the time of first event in the Z_1 (Z_2) process. These are natural times to initiate waiting periods, and one might hope that for this case (4.1) would take a relatively simple form. Unfortunately, this is not the case. To see this, we note first that (S_1,S_2) have the joint distribution $F(x,y;0)$. Thus $\Delta = S_2 - S_1$ has the distribution G which with $\theta_1 = \lambda_1 + \lambda_{12}$, $\theta_2 = \lambda_2 + \lambda_{12}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ is given by

$$P\{S_2 - S_1 > \delta\} = \int_0^{\infty} \lambda_1 e^{-\theta_2\delta-\lambda s_1} ds_1 = \lambda_1 e^{-\theta_2\delta}/\lambda,\ \delta > 0,$$

$$P\{S_2-S_1 < \delta\} = P\{S_1 - S_2 > -\delta\} = \lambda_2 e^{\theta_1\delta}/\lambda,\ \delta < 0,$$

$$P\{S_2 - S_1 = 0\} = \lambda_{12}/\lambda.$$

It follows from this and from (4.1) that if X (Y) is the waiting time between the first and second events in the Z_1 (Z_2) process, then for $x \le y$,

$$\frac{P\{X > x, Y > y\}}{e^{-\lambda_1 x - \lambda_2 y}} = \frac{e^{-\lambda_{12} y}}{\lambda}\left\{\left[\frac{\theta_2(\lambda+\lambda_{12})}{\theta_2+\lambda_{12}} + \frac{\lambda_1\lambda_{12}}{\theta_2+\lambda_{12}} e^{-(\theta_2+\lambda_{12})x}\right]\right.$$

$$\left. - \frac{\lambda_2\lambda_{12}}{\theta_1+\lambda_{12}} e^{-\theta_1(y-x)}\left[1 - e^{-(\theta_1+\lambda_{12})x}\right]\right\} .$$

5. Multivariate exponential distributions.

The extension of the BVE(δ) distribution to higher dimensions is best understood by reference to the shock model derivation of §2.3. Suppose that device i is placed in service at time δ_i. For each non-empty subset of the variables, there is a Poisson process governing shocks to members of the subset.

Let $S = \{s = (s_1,\ldots,s_n) : s_i = 0 \text{ or } 1, \text{ but } s \neq (0,\ldots,0)\}$. For each $s \in S$, let Z_s be a Poisson process with parameter λ_s, and suppose that an event in this process corresponds to a fatal shock to component i if and only if $s_i = 1$. Suppose that the processes Z_s are mutually independent. Let X_i represent the life of the i^{th} component, and let I_i be the interval $[\delta_i, x_i + \delta_i]$. Then if μ is Lebesgue measure,

$$(5.1) \qquad P\{X_1 > x_1,\ldots,X_n > x_n\} = e^{-\sum_{s\in S}\lambda_s\mu[\cup_{\{i:s_i=1\}} I_i]} .$$

One can also write (5.1) expli tly, giving its form for various regions defined by inequalities on the δ_i and x_i. Assuming $\delta_1 = 0 < \delta_2 < \cdots < \delta_n$, there are $3\cdot5\cdot7 \cdots \cdot(2n-1)$ such regions, so this is prohibitive even in three dimensions. (5.1) can also be written using maxima and minima, but this also is unattractive even in three dimensions. On the other hand (5.1) in its present form is both compact and easily evaluated for any given δ_i and x_i.

REFERENCES

[1] Campbell, J. T. (1934). The Poisson correlation function. *Proc. Edinburgh Math. Soc.*, Series 2, **4** 18-26.

[2] Dwass, Meyer and Teicher, Henry (1957). On infinitely divisible random vectors. *Ann. Math. Statist.* **28** 461-470.

[3] Marshall, Albert W. and Olkin, Ingram (1966). A multivariate exponential distribution. Boeing Scientific Research Laboratories Document D1-82-0505. To appear in *J. Amer. Statist. Soc.*

[4] McKendrick, A. G. (1926). Applications of mathematics to medical problems. *Proc. Edinburgh Math. Soc.* **44** 98-130.

[5] Teicher, Henry (1954). On the multivariate Poisson distribution. *Skand. Aktuarietidskr.* **37** 1-9.

[6] Young, W. H. (1917). On multiple integration by parts and the second theorem of the mean. *Proc. London Math. Soc.*, Series 2, **16** 237-293.