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A GENERALIZED CONFORMING ISOPARAMETRIC ELEMENT

Long Yu-qiu (龙驭球) Huang Min-feng (黄民丰)

(*Qinghua University, Beijing*)

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Abstract

Based on the generalized compatibility condition under constant and linear stress field, a quadrilateral generalized conforming isoparametric element, GC–Q6, for plane stress analysis, is developed. The element GC–Q6 can be regarded as an improved form of Wilson's non-conforming isoparametric element Q6. GC–Q6 can pass the patch test for arbitrary irregular mesh while Q6 can not. GC–Q6 degenerates to Q6 when it is a parallelogram. Numerical examples show that the GC–Q6 element gives more accurate stress solution than the existing non-conforming elements and is less sensitive to geometric distortion.

I. Introduction

For the two-dimensional four-node isoparametric element (the Q4 element), the interpolation functions for displacements u and v are given by

$$u=\sum_{i=1}^{4}N_i u_i,\quad v=\sum_{i=1}^{4}N_i v_i \tag{1.1}$$

where u_i and v_i ($i=1,2,3,4$) are the nodal displacements, N_i the shape functions which are bilinear functions of natural coordinates ξ,η :

$$N_i=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)\qquad (i=1,2,3,4) \tag{1.2}$$

The Q4 element is very popular. However, for bending problems it gives results with low accuracy.

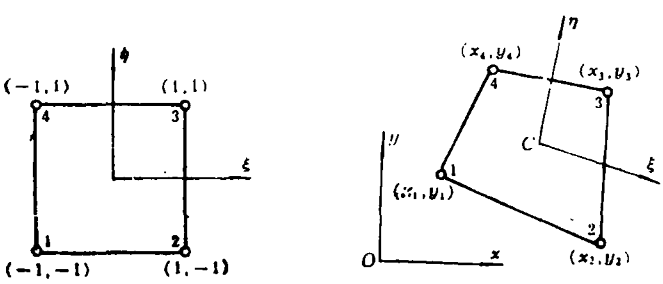

Fig. 1

In order to improve the bending behavior of the Q4 element, Wilson[1] proposed a non-conforming element Q6, in which the displacements are split into compatible and incompatible parts:

$$\{u\}=\{u_q\}+\{u_\lambda\} \tag{1.3}$$

where $\{u_q\}$ is the compatible part given by Eqs. (1.1) and (1.2), and $\{u_\lambda\}$ is the incompatible part given by

$$\{u_\lambda\}=\begin{Bmatrix} u_\lambda \\ v_\lambda \end{Bmatrix}=\begin{bmatrix} 1-\xi^2 & 1-\eta^2 & 0 & 0 \\ 0 & 0 & 1-\xi^2 & 1-\eta^2 \end{bmatrix}\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_1' \\ \lambda_2' \end{Bmatrix} \tag{1.4}$$

Some excellent numerical results are obtained by the Q6 element, however, it can not pass the patch test for irregular mesh.

In references [2] and [3], some treatments were proposed to make the element pass the patch test, then the so called QM6 and QP6 elements were developed.

In reference [4], two kinds of the quadrilateral isoparametric quasi-conforming elements QC5 and QC6 have been derived by the Quasi-conforming Method.

According to a strong form of the low order patch test, reference [5] presented a general formulation of incompatible shape function and then the NQ6 element was obtained.

In this paper, to derive the generalized conforming displacement mode, a general procedure will be presented based on the generalized compatibility condition under constant and linear stress field and a generalized conforming isoparametric element GC – Q6 is developed.

II. Generalized Compatibility Condition under Constant and Linear Stress Field

For conforming element, the displacement field $\{u\}$ of the element must satisfy the compatibility condition along the element boundary ∂A_e :

$$\{u\}-\{\tilde{u}\}=\{0\} \qquad (\text{on}\,\partial A_e) \tag{2.1}$$

where $\{\tilde{u}\}$ denote the boundary displacements of the element.

For generalized conforming element, the compatibility condition (2.1) is relaxed and replaced by the following compatibility condition in the limit of mesh refinement (the stress field of each element tends to be constant):

$$\oint_{\partial A_e}\{T_c\}^T(\{u\}-\{\tilde{u}\})ds=0 \tag{2.2}$$

where $\{T_c\}$ denote the boundary tractions of the constant stress field.

In Ref. [6], during the generalization of the generalized conforming element, the following compatibility condition of the average displacements on each side S_i of the element is used:

$$\int_{S_i}(\{u\}-\{\tilde{u}\})ds=\{0\} \tag{2.3}$$

Obviously, condition (2.3) is a strong form of condition (2.2).

In this paper, a new kind of generalized conforming element is established by using another strong form of condition (2.2), i.e.

$$\oint_{\partial A_e}\{T\}^T(\{u\}-\{\tilde{u}\})ds=0 \tag{2.4}$$

where $\{T\}$ denote the boundary tractions of both the constant and linear stress fields.

Substituting Eq. (1.3) into Eq. (2.4),and applying the following condition satisfied by the conforming displacement $\{u_q\}$

$$\{u_q\}-\{\bar{u}\}=\{0\} \qquad (\text{on}\,\partial A_e)$$

we have

$$\oint_{\partial A_e}\{T\}^T\{u_\lambda\}ds=0 \tag{2.5}$$

Considering the stress field given below:

$$\sigma_x=\beta_1+\beta_4\eta,\ \sigma_y=\beta_2+\beta_5\xi,\ \tau_{xy}=\beta_3 \tag{2.6}$$

we obtain

$$T_x=l\beta_1+m\beta_3+l\eta\beta_4,\ T_y=m\beta_2+l\beta_3+m\xi\beta_5 \tag{2.7}$$

where l and m denote the directional cosines of outward normal to the boundary.

Substituting Eq. (2.7) into Eq. (2.5), we have

$$\oint_{\partial A_e}[\beta_1 lu_\lambda+\beta_2 mv_\lambda+\beta_3(mu_\lambda+lv_\lambda)+\beta_4 l\eta u_\lambda+\beta_5 m\xi v_\lambda]ds=0 \tag{2.8}$$

Since the five parameters β_i are arbitraries, five conditions can be obtained from Eq. (2.8):

$$\left.\begin{aligned}&\oint_{\partial A_e}lu_\lambda\,ds=0,\ \oint_{\partial A_e}mv_\lambda ds=0,\ \oint_{\partial A_e}(mu_\lambda+lv_\lambda)ds=0\\&\oint_{\partial A_e}l\eta u_\lambda ds=0,\ \oint_{\partial A_e}m\xi v_\lambda ds=0\end{aligned}\right\} \tag{2.9}$$

Eq. (2.9) is the generalized compatibility condition under constant and linear stress field.

III. Formulation of Generalized Conforming Displacements $\{u_\lambda\}$

Firstly, the generalized conforming displacements u_λ and v_λ are expressed in a complete quadratic polynomial form:

$$\left.\begin{aligned}u_\lambda&=\lambda_1+\lambda_2\xi+\lambda_3\eta+\lambda_4\xi^2+\lambda_5\xi\eta+\lambda_6\eta^2\\v_\lambda&=\lambda_1'+\lambda_2'\xi+\lambda_3'\eta+\lambda_4'\xi^2+\lambda_5'\xi\eta+\lambda_6'\eta^2\end{aligned}\right\} \tag{3.1}$$

Let the translations and the rotation at the centroid $C(\xi=0,\eta=0)$ of the element be u_c, v_c and ω_C, then

$$u_C=\lambda_1,\ v_C=\lambda_1',\ \omega_C=\frac{1}{2|J|_C}(a_3\lambda_2-a_1\lambda_3+b_3\lambda_2'-b_1\lambda_3') \tag{3.2}$$

where the following notations are used:

$$\left.\begin{aligned}&a_1=\frac{1}{4}(-x_1+x_2+x_3-x_4),\ a_2=\frac{1}{4}(x_1-x_2+x_3-x_4),\ a_3=\frac{1}{4}(-x_1-x_2+x_3+x_4)\\&b_1=\frac{1}{4}(-y_1+y_2+y_3-y_4),\ b_2=\frac{1}{4}(y_1-y_2+y_3-y_4),\ b_3=\frac{1}{4}(-y_1-y_2+y_3+y_4)\\&|J|_C=a_1b_3-a_3b_1\neq0\end{aligned}\right\} \tag{3.3}$$

Secondly, substituting Eq. (3.1) into Eq. (2.9), we obtain

$$\left.\begin{array}{l}3b_3\lambda_2-3b_1\lambda_3+2b_2(\lambda_4-\lambda_6)=0\\3a_3\lambda_2'-3a_1\lambda_3'+2a_2(\lambda_4'-\lambda_6')=0\\ [3a_3\lambda_2-3a_1\lambda_3+2a_2(\lambda_4-\lambda_6)]+[3b_3\lambda_2'-3b_1\lambda_3'+2b_2(\lambda_4'-\lambda_6')]=0\\3b_1\lambda_1+2b_2\lambda_3+b_1\lambda_4-b_3\lambda_5+3b_1\lambda_6=0\\3a_3\lambda_1'+2a_2\lambda_2'+3a_3\lambda_4'-a_1\lambda_5'+a_3\lambda_6'=0\end{array}\right\}\qquad(3.4)$$

From Eqs. (3.4) and (3.2), eight parameters $\lambda_1,\lambda_2,\lambda_3,\lambda_5,\lambda_1',\lambda_2',\lambda_3'$ and λ_5' can be expressed in terms of another four parameters $\lambda_4,\lambda_6,\lambda_4',\lambda_6'$, and three centroid displacements u_C, v_C and ω_C

If we let

$$u_C=-\lambda_4-\lambda_6,\quad v_C=-\lambda_4'-\lambda_6',\quad \omega_C=0 \qquad (3.5)$$

then from Eq. (3.1) we obtain

$$\left\{\begin{array}{c}u_\lambda\\v_\lambda\end{array}\right\}=\begin{bmatrix}\xi^2-1+F_1 & \eta^2-1+F_2 & F_3 & -F_3\\ F_3' & -F_3' & \xi^2-1+F_1' & \eta^2-1+F_2'\end{bmatrix}\left\{\begin{array}{c}\lambda_4\\\lambda_6\\\lambda_4'\\\lambda_6'\end{array}\right\} \qquad (3.6)$$

in which

$$\left.\begin{array}{l}F_1=\dfrac{1}{3|J|_C}\{-(2a_1b_2-a_2b_1)\xi-(2a_3b_2-a_2b_3)\eta+[(3a_1b_1+2a_2b_2)-\dfrac{a_3}{b_3}(3b_1^2+4b_2^2)\\ \qquad -\dfrac{b_1}{b_3}|J|_C]\xi\eta\}\\ F_1'=\dfrac{1}{3|J|_C}\{(2a_2b_1-a_1b_2)\xi+(2a_2b_3-a_3b_2)\eta+[(9a_3b_3-2a_2b_2)+\dfrac{b_1}{a_1}(4a_2^2-9a_3^2)\\ \qquad -\dfrac{a_3}{a_1}|J|_C]\xi\eta\}\\ F_2=-F_1-\dfrac{2b_1}{b_3}\xi\eta,\quad F_2'=-F_1'-\dfrac{2a_3}{a_1}\xi\eta\\ F_3=\dfrac{-1}{3|J|_C}[b_1b_2\xi+b_2b_3\eta+2b_2^2\xi\eta],\quad F_3'=\dfrac{1}{3|J|_C}[a_1a_2\xi+a_2a_3\eta+2a_2^2\xi\eta]\end{array}\right\}(3.7)$$

Eq. (3.6), involving four internal displacement parameters $\lambda_4,\lambda_6,\lambda_4'$ and λ_6', represents the required generalized conforming displacement mode which satisfies condition (2.9).

If the element is a parallelogram, Eq. (3.6) degenerates to Eq. (1.4).

IV. The Stiffness Matrix of the Generalized Conforming Element GC—Q6

As soon as the generalized conforming displacement mode (3.6) is determined, the stiffness matrix may be derived by the conventional procedure.

Substituting Eqs. (1.1) and (3.6) into Eq. (1.3), element displacement may be written as:

$$\{u\}=\{u_q\}+\{u_\lambda\}=[N]\{q\}^e+[N_\lambda]\cdot\{\lambda\} \qquad (4.1)$$

Element strain may be expressed as

$$\{\varepsilon\}=[B]\{q\}^e+[B_\lambda]\{\lambda\} \tag{4.2}$$

Element strain energy

$$\begin{aligned} U&=\frac{t}{2}\iint_{A_e}\{\varepsilon\}^T[D]\{\varepsilon\}dA \\ &=\frac{1}{2}\{q\}^{eT}[K_{qq}]\{q\}^e+\frac{1}{2}\{\lambda\}^T[K_{\lambda\lambda}]\{\lambda\}+\{\lambda\}^T[K_{\lambda q}]\{q\}^e \end{aligned} \tag{4.3}$$

in which

$$\left.\begin{aligned} [K_{qq}]&=t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|\,d\xi d\eta \\ [K_{\lambda\lambda}]&=t\int_{-1}^{1}\int_{-1}^{1}[B_\lambda]^T[D][B_\lambda]|J|\,d\xi d\eta \\ [K_{\lambda q}]&=t\int_{-1}^{1}\int_{-1}^{1}[B_\lambda]^T[D][B]|J|\,d\xi d\eta \end{aligned}\right\} \tag{4.4}$$

and $[J]$ is the determinant of the Jacobian matrix. $[D]$ is the matrix of elasticity coefficients.

From $\partial U/\partial\{\lambda\}=\{0\}$, we obtain

$$\{\lambda\}=-[K_{\lambda\lambda}]^{-1}[K_{\lambda q}]\{q\}^e \tag{4.5}$$

and finally the element stiffness matrix

$$[K]=[K_{qq}]-[K_{\lambda q}]^T[K_{\lambda\lambda}]^{-1}[K_{\lambda q}] \tag{4.6}$$

V. Examples

Example 1 Analysis of a tension plate using irregular mesh (Fig.2). Two loading cases are considered: uniform tension under load 1 (an experiment problem for patch test) and pure bending under load 2. Owing to symmetry of the plate, only one-quarter of the plate is modelled. Irregular mesh as shown in Fig. 2 is used.

Results of seven different types of elements are listed in table 1.

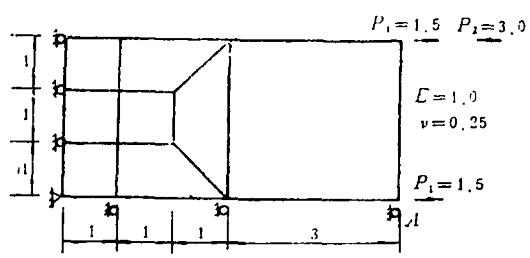

Fig. 2

Table 1 Comparison of results for example 1

Element	Load 1 (uniform tension)		Load 2 (bending)
	u_A	Patch test	v_A
Q4 (isoparametric)	6.00	can pass	−17.00
Q6 (Wilson[1])	6.70	can't pass	−19.66
QM6(Taylor[2])	6.00	can pass	−17.61
QP6(Wachspress[3])	6.00	can pass	−17.61
QC6 (Quasi-conf [4])	6.00	can pass	−17.61
NQ6(Pian[5])	6.00	can pass	−17.61
GC−Q6 (This paper)	6.00	can pass	−17.62
Exact	6.00		−18.00

The results in table 1 show conclusively the failure of the Q6 element to pass the path test, while the other six elements satisfy the test.

Example 2 Analysis of a cantilever beam using irregular mesh (Fig.3). Two loading cases are considered: pure bending under load 1 and transverse bending under load 2.

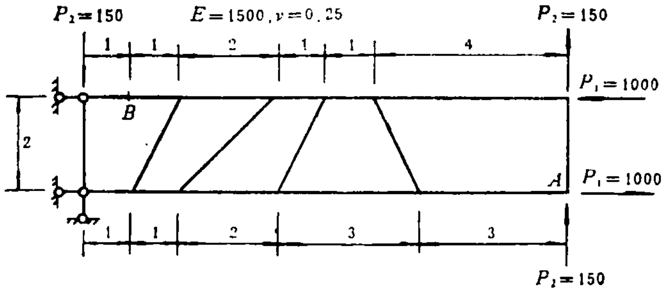

Fig. 3

Table 2 Comparison of results for example 2

Element	Load 1 (pure bending)		Load 2 (transverse bending)	
	v_A	σ_{xB}	v_A	σ_{xB}
Q4 (isoparametric)	45.7	−1761	50.7	−2448
Q6 (Wilson[1])	98.4	−2428	100.4	−3354
QC6(quasi-conf[4])	96.1	−2439	98.1	−3339
NQ6(Pian[5])	96.1	−2439	98.0	−3294
GC − Q6 (This paper)	95.0	−3036*	96.1	−4182*
Analytic solution	100	−3000	102.6	−4050

*Stress at B is computed by extrapolation from the stresses at the 2×2 Gauss quadrature points.

From table 2 we realize that for bending problems the accuracy of conforming element Q4 is very poor, but the non-conforming elements Q6 and NQ6, the quasi-conforming element QC6 and the generalized conforming element GC − Q6 can provide good accuracy even for an irregular mesh, especially the GC − Q6 element which gives the most accurate stress solution.

Example 3 Analysis of a tapered and swept panel with unit load uniformly distributed along right edge (Fig.4).

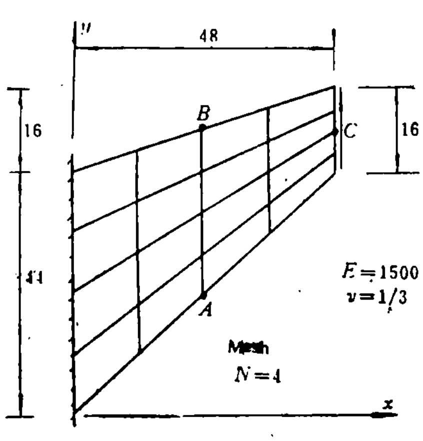

Fig. 4

Table 3 Comparison of results for example 3

(a) Displacement V_c

Element	$N=2$	$N=4$	$N=8$	$N=16$
HL(Cook[7])	18.17	22.03	23.39	23.81
Q4 (Isoparameteric)	11.85	18.30	22.08	—
Q6 (Wilson[1])	22.94	23.48	—	—
QM6 (Taylor[2])	21.05	23.02	—	—
GC – Q6 (This paper)	27.61	24.31	23.99	—

(b) Stress σ_{Bmin}

Element	$N=2$	$N=4$	$N=8$	$N=16$
HL(Cook[7])	−0.1335	−0.1700	−0.1931	−0.2005
Q4 (Isoparameteric)	−0.0916	−0.1510	−0.1866	—
Q6 (Wilson[1])*	−0.1734	−0.1915	—	—
QM6 (Taylor[2])*	−0.1580	−0.1856	—	—
GC – Q6 (This paper)*	−0.1688	−0.1930	−0.1965	—

(c) Stress σ_{Amax}

Element	$N=2$	$N=4$	$N=8$	$N=16$
HL(Cook[7])	0.1582	0.1980	0.2205	0.2294
Q4 (Isoparameteric)	0.1281	0.1905	0.2201	—
Q6 (Wilson[1])*	0.2029	0.2258	—	—
QM6 (Taylor[2])*	0.1928	0.2243	—	—
GC – Q6 (This paper)*	0.2538	0.2349	0.2318	—

* Nodal stresess are computed by extrapolation from the stresses at 2 × 2 Gauss quadrature points and nodal stresses of neighboring element are averaged.

This example was used by Cook[7] for testing the sensitivities of finite elements to geometric distortions. If the solution obtained by element HL using a 16 × 16 mesh[7] is used as a reference, it is seen the present element gives more accurate results than the HL element for coarser meshes.

VI. Conclusions

The present element GC – Q6 shows a superior performance of a generalized conforming element. By satisfying the generalized compatibility condition under constant stress field, it can always pass the path test. By satisfying the generalized compatibility condition under linear stress field, it provides accurate results for both displacements and stresses, and is insensitive to geometric distortions. Moreover, being a displacement-based element, it can be derived and implemented in a routine manner.

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