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A generalized contraction principle

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This paper presents an extension of Banach's contraction mapping principle to Hausdorff spaces, in fact to the larger class of topological spaces in which convergent sequences have unique limits. This is achieved by considering topologies on X generated by families of quasi-pseudo-metrics on X. An extension of the concept of Cauchy sequence to this non-metric setting is given.

The classic contraction mapping principle of Banach [1, p. 160] states that a self-mapping f of a complete metric space (X, d) which satisfies the condition that $d(f(x), f(y)) \le cd(x, y)$ for all $x, y \in X$ and some c such that $0 \le c < 1$, has a unique fixed point which can be realized as the limit of the sequence of Picard iterates $\{f^n(x)\}$ for each x in X. Several authors have considered contractive mappings in the more general setting of uniform spaces, for example Brown and Comfort [2], Edelstein [4], Kammerer and Kasriel [5], Knill [7] and Naimpally [8], and various extensions of Banach's Theorem have been obtained. The purpose of this note is to present an extension of the theorem to arbitrary Hausdorff spaces. The machinery required is the concept of a quasi-gauge structure for topological space. By a quasi-pseudo-metric on a set X we mean a non-negative real valued function on $X \times X$ which vanishes on the diagonal and satisfies the triangle inequality.

DEFINITION 1. A quasi-gauge structure for a topological space (X, T) is a family P of quasi-pseudo-metrics on X such that T has as a subbase the family $\{B(x, p, \varepsilon) : x \in X, p \in P, \varepsilon > 0\}$ where $B(x, p, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$. If (X, T) has a quasi-gauge

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structure P it is a quasi-gauge space and is denoted by (X, P) .

This notion is a generalization of the concept of gauge space which has been discussed by Dugundji [3], and is considered in detail in [9].

If (X, T) is a topological space and $G \in T$, then the function g defined by

$$g(x, y) = \begin{cases} 1 & \text{if } x \in G,\ y \notin G \\ 0 & \text{otherwise} \end{cases}$$

is a quasi-pseudo-metric on X . Moreover, $P = \{g : G \in T\}$ is a quasi-gauge structure for (X, T) . Thus we have the following result.

THEOREM 1. *Every topological space is a quasi-gauge space.*

DEFINITION 2. A self-mapping f on the quasi-gauge space (X, P) is contractive if for each $p \in P$ there is a real number c with $0 \le c < 1$ such that

$$p\big(f(x), f(y)\big) \le cp(x, y) \quad \text{for all } x, y \in X .$$

It is clear that if f is contractive with respect to P then f is continuous.

A major difficulty is to find a suitable definition of Cauchy sequence in this setting. As Kelly [6, Example 5.8] has shown, if one uses the usual metric definition of Cauchy sequence for a quasi-pseudo-metric space then a convergent sequence need not be Cauchy.

DEFINITION 3. If (X, P) is a quasi-gauge space then the sequence $\{x_n\}$ in X is P-Cauchy if for each $p \in P$ and each $\varepsilon > 0$ there is a point $x \in X$ and an integer k such that $p\left(x, x_m\right) < \varepsilon$ for all $m > k$.

It is easy to show that if $\{x_n\}$ is a convergent sequence in X then it is P-Cauchy, but as Example 1 shows the converse is false. Also, if P consists of a single metric then Definition 3 agrees with the usual definition.

EXAMPLE 1. Let X be the interval $(0, 1]$ of real numbers, and define the quasi-pseudo-metric p on X by

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$$p(x, y) = \begin{cases} x-y & \text{if } x \geq y , \\ 1 & \text{if } x < y . \end{cases}$$

For suitable $\varepsilon > 0$, $B(x, p, \varepsilon) = (x-\varepsilon, x]$. Then the sequence $\{x_n\}$ where $x_n = 1/n$ is Cauchy in the sense of Definition 3. For if $\varepsilon > 0$ there is an integer n such that $1/n < \varepsilon$. Then if $x = 1/n$ and $k = n$ we have $p(x, x_m) = 1/n - 1/m < \varepsilon$ for all $m > k$. But $\{x_n\}$ is not convergent in X.

DEFINITION 4. A quasi-gauge space (X, P) is sequentially complete if every P-Cauchy sequence in X converges to some point in X.

We are now in a position to state the generalized contraction principle.

THEOREM 2. *Every contractive self-mapping of a Hausdorff sequentially complete quasi-gauge space has a unique fixed point.*

The proof parallels that of the classical Banach Theorem, see Dugundji [3, p. 305] for example. For any point x one shows that the sequence $\{f^n(x)\}$ of Picard iterates is Cauchy in the sense of Definition 3. The sequential completeness ensures that this sequence has a limit and the Hausdorff condition ensures that the limit is unique. In fact, we can weaken the Hausdorff requirement to the condition that convergent sequences have unique limits, that is, that the space be a *US* space in the sense of Wilansky [10]. Example 2 shows that Theorem 2 is a genuine extension of the Banach Theorem.

EXAMPLE 2. Let X be the set of all ordinal numbers less than or equal to the first uncountable ordinal Ω. Let ω be any limit ordinal other than Ω, and denote by $s(\omega)$ the immediate successor of ω, so that $s^k(\omega)$ is the k-th successor of ω.

Let

$$S_\omega = \{s^k(\omega) : k = 0, 1, 2, \ldots\}$$
$$= \{\omega, s(\omega), s^2(\omega), \ldots\} .$$

We define a non-negative real valued function d_ω on $X \times X$ as follows:

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$$d_\omega(x, y) = \begin{cases} 0 & \text{if } x, y \notin S_\omega , \\ 2^{-k} & \text{if } x \notin S_\omega, y = s^k(\omega) , \\ 2^{-k} & \text{if } x = s^k(\omega), y \notin S_\omega , \\ |2^{-k}-2^{-m}| & \text{if } x = s^k(\omega), y = s^m(\omega) . \end{cases}$$

A discussion of cases shows that d_ω satisfies the triangle inequality. Thus d_ω is a quasi-pseudo-metric on X, indeed a pseudo-metric. Let $P = \{d_\omega : \omega$ is a limit ordinal other than $\Omega\}$. Now if $x \notin S_\omega$, $\varepsilon > 0$, and k is such that $2^{-k} < \varepsilon < 2^{-k+1}$, then

$$B(x, d_\omega, \varepsilon) = X - \{\omega, s(\omega), \ldots, s^{k-1}(\omega)\} .$$

If $x \in S_\omega$, say $x = s^m(\omega)$, then $B\left(x, d_\omega, 2^{-m-1}\right) = \{x\}$. Thus the topology T induced on X by the quasi-gauge structure P is $T = \{G \subset X : \Omega \notin G$, or $\Omega \in G$ and $X-G$ is finite$\}$. Let $\{x_n\}$ be a P-Cauchy sequence in X; then either it is eventually constant or it converges to Ω. For each point different from Ω has an open ball which contains only the point itself, and hence no such point can occur frequently in the sequence $\{x_n\}$ because this would violate the P-Cauchyness. Any open neighbourhood G of Ω contains all but a finite number of points of X, so that if $\{x_n\}$ is not eventually constant it is eventually in G. Thus (X, T) is sequentially complete. It is easily seen to be Hausdorff. Define the self-mapping f on X as follows: $f(x) = s(x)$ for $x \neq \Omega$, and $f(\Omega) = \Omega$. A discussion of cases shows that $d_\omega(f(x), f(y)) \leq \frac{1}{2}d_\omega(x, y)$ for $x, y \in X$ and each $d_\omega \in P$. Theorem 2 yields the existence of the unique fixed point Ω of f. The classical theorem of Banach does not apply because (X, T) is not metrizable. For every countable intersection of open sets containing Ω contains all but a countable number of points of X. But X is uncountable, so that $\{\Omega\}$ is a closed set which is not a G_δ.

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