A Generalized Decision Logic in Interval-set-valued Information Tables*

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Abstract. A generalized decision logic in interval-set-valued information tables is introduced, which is an extension of decision logic studied by Pawlak. Each object in an interval-set-valued information table takes an interval set of values. Consequently, two types of satisfiabilities of a formula are introduced. Truth values of formulas are defined to be interval-valued, instead of single-valued. A semantics model of the proposed logic language is studied.

1 Introduction

The theory of rough sets is commonly developed and interpreted through the use of information tables, in which a finite set of objects are described by a finite number of attributes [10, 11]. A decision logic, called DL-language by Pawlak [11], has been studied by many authors for reasoning about knowledge represented by information tables [8, 11]. It is essentially formulated based on the classical two-valued logic. The semantics of the DL-language is defined in Tarski's style through the notions of a model and satisfiability in the context of information tables. A strong assumption is made about information tables, i.e., each object takes exactly one value with respect to an attribute. In some situations, this assumption may be too restrictive to be applicable in practice. Several proposals have been suggested using much weaker assumptions. More specifically, the notion of set-based information tables (also known as incomplete or nondeterministic information tables) has been introduced and studied, in which an object can take a subset of values for each attribute [3, 14, 16, 20]. Based on the results from those studies, the main objective of this paper is to introduce the notion of interval-set-valued information tables by incorporating results from studies of interval-set algebra [17, 19]. A generalized decision logic GDL is proposed, which is similar to modal logic, but has a different semantics interpretation.

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This paper reports some of our preliminary results. In Section 2, we first briefly review Pawlak's decision logic DL, and then introduce the notions of α-degree truth and α-level truth. In Section 3, the notion of interval-set-valued information tables is introduced. A generalized decision logic DGL is proposed and interpreted based on two types of satisfiabilities. The concepts of interval-degree truth and interval-level truth are proposed and studied. Inference rules are discussed. In Section 4, two related studies are commented.

2 A Decision Logic in Information Tables

The notion of an information table, studied by many authors [3, 10, 11, 16, 21], is formally defined by a quadruple:

$$S = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

where

U is a finite nonempty set of objects,
At is a finite nonempty set of attributes,
V_a is a nonempty set of values for $a \in At$,
$I_a : U \longrightarrow V_a$ is an information function.

Each information function I_a is a total function that maps an object of U to exactly one value in V_a. Similar representation schemes can be found in many fields, such as decision theory, pattern recognition, machine learning, data analysis, data mining, and cluster analysis [11].

With an information table, a decision logic language (DL-language) can be introduced [11]. In the DL-language, an atomic formula is given by (a, v), where $a \in At$ and $v \in V_a$. If ϕ and ψ are formulas in the DL-language, then so are $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\phi \equiv \psi$. The semantics of the DL-language can be defined in Tarski's style through the notions of a model and satisfiability. The model is an information table S, which provides interpretation for symbols and formulas of the DL-language. The satisfiability of a formula ϕ by an object x, written $x \models_S \phi$ or in short $x \models \phi$ if S is understood, is given by the following conditions:

(a1). $x \models (a, v)$ iff $I_a(x) = v$,
(a2). $x \models \neg\phi$ iff not $x \models \phi$,
(a3). $x \models \phi \wedge \psi$ iff $x \models \phi$ and $x \models \psi$,
(a4). $x \models \phi \vee \psi$ iff $x \models \phi$ or $x \models \psi$,
(a5). $x \models \phi \rightarrow \psi$ iff $x \models \neg\phi \vee \psi$,
(a6). $x \models \phi \equiv \psi$ iff $x \models \phi \rightarrow \psi$ and $x \models \psi \rightarrow \phi$.

For a formula ϕ, the set $m_S(\phi)$ defined by:

$$m_S(\phi) = \{x \in U \mid x \models \phi\}, \tag{1}$$

is called the meaning of the formula ϕ in S. If S is understood, we simply write $m(\phi)$. Obviously, the following properties hold [8, 11]:

$$\begin{aligned}
&\text{(b1).} && m(a, v) = \{x \in U \mid I_a(x) = v\},\\
&\text{(b2).} && m(\neg\phi) = -m(\phi),\\
&\text{(b3).} && m(\phi \wedge \psi) = m(\phi) \cap m(\psi),\\
&\text{(b4).} && m(\phi \vee \psi) = m(\phi) \cup m(\psi),\\
&\text{(b5).} && m(\phi \to \psi) = -m(\phi) \cup m(\psi),\\
&\text{(b6).} && m(\phi \equiv \psi) = (m(\phi) \cap m(\psi)) \cup (-m(\phi) \cap -m(\psi)).
\end{aligned}$$

The meaning of a formula ϕ is therefore the set of all objects having the property expressed by the formula ϕ. In other words, ϕ can be viewed as the description of the set of objects $m(\phi)$. Thus, a connection between formulas of the DL-language and subsets of U is established.

A formula ϕ is said to be true in an information table S, written $\models_S \phi$ or $\models \phi$ for short when S is clear from the context, if and only if $m(\phi) = U$. That is, ϕ is satisfied by all objects in the universe. Two formulas ϕ and ψ are equivalent in S if and only if $m(\phi) = m(\psi)$. By definition, the following properties hold [11]:

$$\begin{aligned}
&\text{(c1).} && \models \phi \text{ iff } m(\phi) = U,\\
&\text{(c2).} && \models \neg\phi \text{ iff } m(\phi) = \emptyset,\\
&\text{(c3).} && \models \phi \to \psi \text{ iff } m(\phi) \subseteq m(\psi),\\
&\text{(c4).} && \models \phi \equiv \psi \text{ iff } m(\phi) = m(\psi).
\end{aligned}$$

Thus, we can study the relationships between concepts described by formulas of the DL-language based on the relationships between their corresponding sets of objects.

The previous interpretation of DL-language is essentially based on classical two-valued logic. One may generalize it to a many-valued logic by introducing the notion of degrees of truth [4, 5]. For a formula ϕ, its truth value is defined by [4, 5]:

$$v(\phi) = \frac{|m(\phi)|}{|U|}, \tag{2}$$

where $|\cdot|$ denotes the cardinality of a set. This definition of truth value is probabilistic in natural. Thus, the generalized logic is in fact a probabilistic logic [7]. When $v(\phi) = \alpha \in [0, 1]$, we say that the formula ϕ is α-degree true. By definition, we immediately have the properties:

$$\begin{aligned}
&\text{(d1).} && \models \phi \text{ iff } v(\phi) = 1,\\
&\text{(d2).} && \models \neg\phi \text{ iff } v(\phi) = 0,\\
&\text{(d3).} && v(\neg\phi) = 1 - v(\phi),\\
&\text{(d4).} && v(\phi \wedge \psi) \leq \min(v(\phi), v(\psi)),\\
&\text{(d5).} && v(\phi \vee \psi) \geq \max(v(\phi), v(\psi)),\\
&\text{(d6).} && v(\phi \vee \psi) = v(\phi) + v(\psi) - v(\phi \wedge \psi).
\end{aligned}$$

Properties (d3)-(d6) follow from the probabilistic interpretation of truth value. Similar to the definitions of α-cuts in the theory of fuzzy sets [2], we define α-level truth. For $\alpha \in [0,1]$, a formula ϕ is said to be α-level true, written $\models_\alpha \phi$, if $v(\phi) \geq \alpha$, and ϕ is strong α-level true, written $\models_{\alpha^+}$, if $v(\phi) > \alpha$. From (d1)-(d6), for $0 \leq \alpha \leq \beta \leq 1$ and $\gamma \in [0,1]$ we have:

(e1). $\models_0 \phi$,
(e2). If $\models_\beta \phi$, then $\models_\alpha \phi$,
(e3). $\models_\alpha \neg\phi$ iff not $\models_{(1-\alpha)^+} \phi$,
(e4). If $\models_\alpha \phi \wedge \psi$, then $\models_\alpha \phi$ and $\models_\alpha \psi$,
(e5). If $\models_\alpha \phi$, then $\models_\alpha \phi \vee \psi$,
(e6). If $\models_\alpha \phi$ and $\models_\gamma \psi$, then $\models_{\max(\alpha,\gamma)} \phi \vee \psi$.

Property (e5) is implied by properties (e2) and (e6).

With the concept of α-level truth, we have the probabilistic *modus ponens* rule [15]:

$$\frac{\begin{array}{l}\models_\alpha \phi \rightarrow \psi \\ \models_\beta \phi\end{array}}{\models_{\max(0,\alpha+\beta-1)} \psi} \qquad \frac{\begin{array}{l}v(\phi \rightarrow \psi) \geq \alpha \\ v(\phi) \geq \beta\end{array}}{v(\psi) \geq \max(0, \alpha+\beta-1)}.$$

Given conditions $v(\phi \rightarrow \psi) \geq \alpha$ and $v(\phi) \geq \beta$, from properties (d3) and (d6), we have:

$$\begin{array}{ll} & v(\phi \rightarrow \psi) \geq \alpha \\ \Longrightarrow & v(\neg\phi \vee \psi) \geq \alpha \\ \Longrightarrow & v(\neg\phi) + v(\psi) - v(\neg\phi \wedge \psi) \geq \alpha \\ \Longrightarrow & (1 - v(\phi)) + v(\psi) \geq \alpha \\ \Longrightarrow & v(\psi) \geq \alpha + v(\phi) - 1 \\ \Longrightarrow & v(\psi) \geq \alpha + \beta - 1. \end{array}$$

Since the value $v(\psi)$ must be non-negative, we can conclude that the proposed *modus ponens* rule is correct. Similar properties and rules can be expressed in terms of strong α-level truth.

3 A Generalized Decision Logic

Let $\mathcal{X}$ be a finite set and $2^{\mathcal{X}}$ be its power set. A subset of $2^{\mathcal{X}}$ of the form:

$$\mathcal{A} = [A_1, A_2] = \{X \in 2^{\mathcal{X}} \mid A_1 \subseteq X \subseteq A_2\} \tag{3}$$

is called a closed interval set, where it is assumed $A_1 \subseteq A_2$. The set of all closed interval sets is denoted by $I(\mathcal{X})$. Degenerate interval sets of the form $[A, A]$ are equivalent to ordinary sets. Thus, interval sets may be considered as an extension of standard sets. In fact, interval-set algebra may be considered as a set-theoretic counterpart of interval-number algebra [6]. A detailed study of interval-set algebra can be found in papers by Yao [17, 19].

An interval-set-valued information table generalizes a standard information table by allowing each object to take interval sets as its values. Formally, this can be described by information functions:

$$I_a : U \longrightarrow I(V_a). \tag{4}$$

For an object $x \in U$, its value on an attribute $a \in At$ is an interval set $I_a(x) = [I_{a*}(x), {I_a}^*(x)]$. The object x *definitely* has properties in $I_{a*}(x)$, and *possibly* has properties in ${I_a}^*(x)$. With the introduction of interval-set-valued information tables, a generalized decision logic language, called *GDL*-language, can be established. The symbols and formulas of the *GDL*-language is the same as that of the *DL*-language. The semantics of the *GDL*-language can be defined similarly in Tarski's style using the notions of a model and two types of satisfiabilities, one for necessity and the other for possibility. If an object x *necessarily* satisfies formula ϕ, we write $x \models_* \phi$, and if x *possibly* satisfies ϕ, we write $x \models^* \phi$. The semantics of $\models_*$ and $\models^*$ are defined as follows:

$$\begin{array}{ll}
\text{(f1).} & x \models_* (a,v) \text{ iff } v \in I_{a*}(x), \\
 & x \models^* (a,v) \text{ iff } v \in {I_a}^*(x), \\
\text{(f2).} & x \models_* \neg\phi \text{ iff not } x \models^* \phi, \\
 & x \models^* \neg\phi \text{ iff not } x \models_* \phi, \\
\text{(f3).} & x \models_* \phi \wedge \psi \text{ iff } x \models_* \phi \text{ and } x \models_* \psi, \\
 & x \models^* \phi \wedge \psi \text{ iff } x \models^* \phi \text{ and } x \models^* \psi, \\
\text{(f4).} & x \models_* \phi \vee \psi \text{ iff } x \models_* \phi \text{ or } x \models_* \psi, \\
 & x \models^* \phi \vee \psi \text{ iff } x \models^* \phi \text{ or } x \models^* \psi, \\
\text{(f5).} & x \models_* \phi \rightarrow \psi \text{ iff } x \models_* \neg\phi \vee \psi, \\
 & x \models^* \phi \rightarrow \psi \text{ iff } x \models^* \neg\phi \vee \psi, \\
\text{(f6).} & x \models_* \phi \equiv \psi \text{ iff } x \models_* \phi \rightarrow \psi \text{ and } x \models_* \psi \rightarrow \phi, \\
 & x \models^* \phi \equiv \psi \text{ iff } x \models^* \phi \rightarrow \psi \text{ and } x \models^* \psi \rightarrow \phi,
\end{array}$$

The following property follows immediately from definition:

$$\text{(g1).} \quad \text{If } x \models_* \phi, \text{ then } x \models^* \phi.$$

Although the introduced notions of necessity and possibility are similar in nature to the notions in modal logic [1], our semantics interpretation is different. There is a close connection between the above formulation and three-valued logic [19].

In *GDL*, with respect to an interval-set-valued information system S, the meaning of a formula ϕ is the interval set $m(\phi)$ defined by:

$$m(\phi) = [\{x \in U \mid x \models_* \phi\}, \{x \in U \mid x \models^* \phi\}] = [m_*(\phi), m^*(\phi)]. \tag{5}$$

It can be verified that the following properties hold:

$$\begin{array}{ll} \text{(h1).} & m(a,v) = [\{x \in U \mid x \models_* \phi\}, \{x \in U \mid x \models^* \phi\}], \\ \text{(h2).} & m(\neg\phi) = \backslash m(\phi), \\ \text{(h3).} & m(\phi \wedge \psi) = m(\phi) \sqcap m(\psi), \\ \text{(h4).} & m(\phi \vee \psi) = m(\phi) \sqcup m(\psi), \\ \text{(h5).} & m(\phi \rightarrow \psi) = \backslash m(\phi) \sqcup m(\psi), \\ \text{(h6).} & m(\phi \equiv \psi) = (\backslash m(\phi) \sqcup m(\psi)) \sqcap (m(\phi) \sqcup \backslash m(\psi)), \end{array}$$

where $\backslash$, $\sqcap$, and $\sqcup$ are the interval-set complement, intersection, and union given by [17]: for two interval sets $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$,

$$\begin{aligned} \backslash\, \mathcal{A} &= \{-X \mid X \in \mathcal{A}\} = [-A_2, -A_1], \\ \mathcal{A} \sqcap \mathcal{B} &= \{X \cap Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\} = [A_1 \cap B_1, A_2 \cap B_2], \\ \mathcal{A} \sqcup \mathcal{B} &= \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\} = [A_1 \cup B_1, A_2 \cup B_2]. \end{aligned} \tag{6}$$

The meaning of a formula ϕ is therefore the interval set of objects, representing those that definitely have the properties expressed by the formula ϕ, and those that possibly have the properties.

Given the meaning of formulas in terms of interval sets, we define the interval-valued truth for a formula ϕ by extending equation (2):

$$v(\phi) = \left[\frac{|m_*(\phi)|}{|U|}, \frac{|m^*(\phi)|}{|U|} \right] = [v_*(\phi), v^*(\phi)]. \tag{7}$$

Both lower and upper bounds of $[v_*(\phi), v^*(\phi)]$ have probabilistic interpretation, hence we have a probability related interval-valued logic [18]. Properties corresponding to (d3)-(d6) are given by:

$$\begin{array}{ll} \text{(i1).} & v_*(\neg\phi) = 1 - v^*(\phi), \\ & v^*(\neg\phi) = 1 - v_*(\phi), \\ \text{(i2).} & v_*(\phi \wedge \psi) \leq \min(v_*(\phi), v_*(\psi)), \\ & v^*(\phi \wedge \psi) \leq \min(v^*(\phi), v^*(\psi)), \\ \text{(i3).} & v_*(\phi \vee \psi) \geq \max(v_*(\phi), v_*(\psi)), \\ & v^*(\phi \vee \psi) \geq \max(v^*(\phi), v^*(\psi)), \\ \text{(i4).} & v_*(\phi \vee \psi) = v_*(\phi) + v_*(\psi) - v_*(\phi \wedge \psi), \\ & v^*(\phi \vee \psi) = v^*(\phi) + v^*(\psi) - v^*(\phi \wedge \psi). \end{array}$$

The formula ϕ is said to be $[v_*(\phi), v^*(\phi)]$-degree true. For a sub-interval $[\alpha_*, \alpha^*]$ of the unit interval $[0, 1]$, a formula ϕ is $[\alpha_*, \alpha^*]$-level true, written $\models_{[\alpha_*, \alpha^*]} \phi$, if $\alpha_* \leq v_*(\phi) \leq v^*(\phi) \leq \alpha^*$, and ϕ is strong $[\alpha_*, \alpha^*]$-level true, written $\models_{[\alpha_*, \alpha^*]^+} \phi$, if $\alpha_* < v_*(\phi) \le v^*(\phi) < \alpha^*$. For sub-intervals $[\alpha_*, \alpha^*] \subseteq [\beta_*, \beta^*] \subseteq [0,1]$ and $[\gamma_*, \gamma^*] \subseteq [0,1]$, the following properties hold:

(j1). $\models_{[0,1]} \phi$,

(j2). If $\models_{[\alpha*,\alpha^*]} \phi$, then $\models_{[\beta*,\beta^*]} \phi$,

(j3). $\models_{[\alpha_*,\alpha^*]} \neg\phi$ iff not $\models_{[1-\alpha^*,1-\alpha_*]^+} \phi$,

(j4). If $\models_{[\alpha_*,\alpha^*]} \phi \wedge \psi$, then $\models_{[\alpha_*,1]} \phi$ and $\models_{[\alpha_*,1]} \psi$,

(j5). If $\models_{[\alpha_*,\alpha^*]} \phi$, then $\models_{[\alpha_*,1]} \phi \vee \psi$,

(j6). If $\models_{[\alpha_*,\alpha^*]} \phi$ and $\models_{[\gamma_*,\gamma^*]} \psi$, then $\models_{[\max(\alpha_*,\gamma_*),1]} \phi \vee \psi$,

(j7). If $\models_{[\alpha_*,\alpha^*]} \phi \vee \psi$, then $\models_{[0,\alpha^*]} \phi$ and $\models_{[0,\alpha^*]} \psi$,

(j8). If $\models_{[\alpha_*,\alpha^*]} \phi$, then $\models_{[0,\alpha^*]} \phi \wedge \psi$,

(j9). If $\models_{[\alpha_*,\alpha^*]} \phi$ and $\models_{[\gamma_*,\gamma^*]} \psi$, then $\models_{[0,\min(\alpha^*,\gamma^*)]} \phi \wedge \psi$.

They follow from (i2) and (i3). In fact, properties (j4)-(j6) are the properties (e4)-(e6) of the DL-language. Properties (j4)-(j6) show the characteristics of the lower bound, while (j7)-(j9) state the characteristics of the upper bound.

The generalized interval-based *modus ponens* rule is given by:

$$\frac{\begin{array}{l}\models_{[\alpha_*,\alpha^*]} \phi \to \psi \\ \models_{[\beta_*,\beta^*]} \phi\end{array}}{\models_{[\max(0,\alpha_*+\beta_*-1),\alpha^*]} \psi} \qquad \frac{\begin{array}{l}\alpha_* \le v_*(\phi \to \psi) \le v^*(\phi \to \psi) \le \alpha^* \\ \beta_* \le v_*(\phi) \le v^*(\phi) \le \beta^*\end{array}}{\max(0, \alpha_* + \beta_* - 1) \le v_*(\psi) \le v^*(\psi) \le \alpha^*}.$$

The part concerning the lower bound is in fact the probabilistic modus ponens rule introduced in Section 2. The upper bound can be seen as follows. From $v^*(\phi \to \psi) \le \alpha^*$ and (i3), we can conclude that:

$$v^*(\psi) \le v^*(\neg\phi \vee \psi) = v^*(\phi \to \psi) \le \alpha^*.$$

Thus, the interval-based *modus ponens* rule is correct. Finally, it should be pointed out that the logic of Section 2 is a special case of interval-valued logic. More specifically, α-level truth can be translated into the $[\alpha, 1]$-level truth.

4 Comments on Related Studies

An interval-valued logic can also be introduced in the standard information tables through the use of lower and upper approximations of the rough set theory [5, 9]. For each subset of the attributes, one can define an equivalence relation on the set of objects in an information table. An arbitrary set is approximated by equivalence classes as follows: the lower approximation is the union of those equivalence classes that are included in the set, while the upper approximation is the union of those equivalence classes that have an nonempty intersection with the set. Thus, for a formula ϕ with interpretation $m(\phi)$, we have a pair of lower and upper approximations $\underline{apr}(m(\phi))$ and $\overline{apr}(m(\phi))$. An interval-valued truth can be defined as:

$$v(\phi) = \left[\frac{|\underline{apr}(m(\phi))|}{|U|}, \frac{|\overline{apr}(m(\phi))|}{|U|}\right] = [v_*(\phi), v^*(\phi)]. \tag{8}$$

Based on this interpretation of interval-valued truth, Parsons *et al.* [9] introduced a logic system RL for rough reasoning. Their inference rules are related to, but different from, the inference rules introduced in this paper. A problem with RL is that the interpretation of the rough measure is not entirely clear. The measure is not fully consistent with the definition of truth value given by equation (8). It may be interesting to have an in-depth investigation of the interval-valued logic based on equation (8). An important feature of such a logic is its non-truth-functional logic connectives. This makes it different from the interval set algebra related systems GDL and RL.

In a recent paper, Pawlak [12] introduced the notion of *rough modus ponens* in information tables. The logical formula $\phi \rightarrow \psi$ is interpreted as a decision rule. A certainty factor is associated with $\phi \rightarrow \psi$ as follows:

$$\mu_S(\phi, \psi) = \frac{|m(\phi) \cap m(\psi)|}{|m(\phi)|}. \tag{9}$$

It can in fact be interpreted as a conditional probability. The rough modus ponens rule is given by:

$$\frac{\begin{array}{l}\phi \rightarrow \psi : \mu_S(\phi, \psi) \\ \phi : v(\phi)\end{array}}{\psi : v(\neg\phi \wedge \psi) + v(\phi)\mu_S(\phi, \psi)}.$$

This rule is closely related to Bayes' theorem [13]. One may easily generalize the rough modus ponens if α-level truth values are used. The main difference between two modus ponens rules stems from the distinct interpretations of the logical formula $\phi \rightarrow \psi$.

5 Conclusion

Two generalizations of Pawlak's information table based decision logic DL are introduced and examined. One generalization is based on the notion of degree of truth, which extend DL from two-valued logic to many-valued logic. The other generalization relies on interval-set-based information tables. In this case, two types of satisfiabilities are used, in a similar spirit of modal logic. They lead to interval-set interpretation of formulas. Consequently, interval-degree truth and interval-level truth are introduced as a generalization of single-valued degree of truth. The truth values of formulas are associated with probabilistic interpretations. The derived logic systems are essentially related to probabilistic reasoning. In particularly, probabilistic *modus ponens* rules are studied.

In this paper, we only presented the basic formulation and interpretation of the generalized decision logic. As pointed out by an anonymous referee of the paper, a formal proving system is needed and applications need to be explored. It may also be intersting to analyze other non-probabilistic interpretations of truth values.

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