# A GENERALIZED IGUSA LOCAL ZETA FUNCTION AND LOCAL DENSITIES OF QUADRATIC FORMS 

Hidenori Katsurada<br>(Received March 11, 1991, revised August 8, 1991)


#### Abstract

A certain formal power series attached to local densities of quadratic forms is defined. It is shown that this series can be realized as a coefficient of the Laurent expansion of a generalized Igusa local zeta function.


1. Introduction. Böcherer and Sato [BS] found a relation between the $p$-adic integrals defined by Denef [D1] and certain formal power series attached to local densities of quadratic forms. In this paper we consider another type of relation between the $p$-adic integrals defined by Igusa [I] and Deshommes [D2] and similar formal power series.

To be more precise, let $A$ and $B$ be non-degenerate symmetric matrices of degrees $m$ and $n$, respectively, with entries in the ring $Z_{p}$ of $p$-adic integers. Let $R=\boldsymbol{Z}_{p}\left[x_{i j}(1 \leq i \leq m, 1 \leq j \leq n), x_{i}(1 \leq i \leq n)\right]$ be the polynomial ring over $\boldsymbol{Z}_{p}$. We simply write $X=\left(x_{i j}\right)_{1 \leq i \leq m, 1 \leq j \leq n}$, and $x=\left(x_{1}, \ldots, x_{n}\right)$. Let $\left(g_{i j}\right)$ be the symmetric matrix of degree $n$ with entries in $R$ defined by

$$
\left(g_{i j}\right)=A[X]-B\left[\operatorname{diag}\left(x_{1}, \ldots, x_{n}\right)\right],
$$

where for an $(m, m)$-matrix $U$ and an $(m, n)$-matrix $V$ we write $U[V]=^{t} V U V$, and for square matrices $A_{1}, \ldots, A_{r}$ we often simply write

$$
\operatorname{diag}\left(A_{1}, \ldots, A_{r}\right)=\left(\begin{array}{ccc}
A_{1} & 0 & 0 \\
0 & \cdot & 0 \\
0 & 0 & A_{r}
\end{array}\right) .
$$

For a set $S$ and non-negative integers $n$ and $s$, put $\langle n\rangle=n(n+1) / 2$, and

$$
S^{\langle n\rangle+s}=\left\{\left(\left(a_{i j}\right)_{1 \leq i \leq j \leq n}, a_{1}, \ldots, a_{s}\right) ; a_{i j}, a_{k} \in S\right\} .
$$

We often write an element $\left(\left(a_{i j}\right)_{1 \leq i \leq j \leq n},\left(a_{i}\right)_{1 \leq i \leq s}\right)$ of $S^{\langle n\rangle+s}$ as $\left(\left(a_{i j}\right)\right.$, $\left.\left(a_{i}\right)\right)$ if no confusion arises. Further for an element $\left(a_{i j}\right)$ of $S^{\langle n\rangle}$ with $S$ a commutative ring, we often write $\sum_{1 \leq i \leq j \leq n} a_{i j}$, and $\prod_{1 \leq i \leq j \leq n} a_{i j}$ as $\sum a_{i j}$, and $\prod a_{i j}$, respectively. Let $C$ be the field of complex numbers, and $\boldsymbol{Z}$ the ring of rational integers. Further let ord ${ }_{p}$ be the normalized additive valuation of the field $\boldsymbol{Q}_{p}$ of $p$-adic numbers, and put $|v|_{p}=p^{-\operatorname{ord}_{p}(v)}$ for $v \in \boldsymbol{Q}_{p}$.

We then define a function $Z\left(B, A ;\left(s_{i j}\right), s_{1}, \ldots, s_{n}\right)$ on the set

$$
\boldsymbol{C}_{+}^{\langle n\rangle+n}=\left\{\left(\left(s_{i j}\right), s_{1}, \ldots, s_{n}\right) \in \boldsymbol{C}^{\langle n\rangle+n} ; \operatorname{Re} s_{i j}>0, \operatorname{Re} s_{i}>0\right\}
$$

by

$$
Z\left(B, A ;\left(s_{i j}\right), s_{1}, \ldots, s_{n}\right)=\int_{\mathbf{Z}_{p}^{n} \times M_{m n}\left(\mathbf{Z}_{p}\right)} \prod\left|g_{i j}\right|_{p}^{s_{i j}} \prod_{i=1}^{n}\left|x_{i}\right|_{p}^{s_{i}} d x d X
$$

where $M_{m n}\left(Z_{p}\right)$ denotes the ring of ( $m, n$ )-matrices with entries in $\boldsymbol{Z}_{p}$ (for the precise definition, see Section 2). We call this function a generalized Igusa local zeta function attached to $A$ and $B$. Put $z_{i j}=p^{-s_{i j}}(1 \leq i \leq j \leq n), z_{i}=p^{-s_{i}}(1 \leq i \leq n)$. We often write $Z=\left(z_{i j}\right), w=\left(z_{i}\right)$. Then $Z\left(B, A ;\left(s_{i j}\right), s_{1}, \ldots, s_{n}\right)$ can be regarded as a function of $Z$ and $w$. Thus we write $\zeta(B, A ; Z, w)=Z\left(B, A ;\left(s_{i j}\right), s_{1}, \ldots, s_{n}\right)$.

On the other hand, define a local density $\alpha_{p}(B, A)$ by

$$
\alpha_{p}(B, A)=\lim _{e \rightarrow \infty} p^{(-m n+\langle n\rangle) e} \#\left\{\bar{X} \in M_{m n}\left(Z_{p}\right) / p^{e} M_{m n}\left(Z_{p}\right) ; A[X] \equiv B \bmod p^{e}\right\}
$$

Then we define a formal power series $P\left(B, A ; x_{1}, \ldots, x_{n}\right)$ by

$$
P\left(B, A ; x_{1}, \ldots, x_{n}\right)=\sum_{r_{1}, \cdots, r_{n}=0}^{\infty} \alpha_{p}\left(B\left[\operatorname{diag}\left(p^{r_{1}}, \ldots, p^{r_{n}}\right)\right], A\right) x_{1}^{r_{1}} \cdots x_{n}^{r_{n}} .
$$

See Section 2 for the relation between our formal power series and the one defined by Böcherer and Sato [BS].

The main purpose of the present paper is to show that the series $P\left(B, A ; x_{1}, \ldots, x_{n}\right)$ can be realized as a coefficient of the Laurent expansion of $\zeta\left(B, A ; Z, p x_{1} \Pi\left(p^{-1} z_{i j}\right)^{-4}\right.$, $\left.\ldots, p x_{n} \Pi\left(p^{-1} z_{i j}\right)^{-4}\right)$ with respect to $\left(z_{i j}\right)$ :

Theorem 1.1. Put

$$
N(B)=\left\{\left(k_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle} ; \min k_{i j} \geq 2 \operatorname{ord}_{p}(2 \operatorname{det} B)+1\right\} .
$$

Then in the region

$$
E=\left\{\left(\left(z_{i j}\right), x_{1}, \ldots, x_{n}\right) \in \boldsymbol{C}^{\langle n\rangle+n} ; 0<\left|z_{i j}\right|<1,0<\left|p x_{i} \prod\left(p^{-1} z_{i j}\right)^{-4}\right|<1\right\},
$$

we have

$$
\begin{aligned}
& \zeta(B, A\left.; Z, p x_{1} \prod\left(p^{-1} z_{i j}\right)^{-4}, \ldots, p x_{n} \prod\left(p^{-1} z_{i j}\right)^{-4}\right) \\
&= \sum_{\left(k_{i j}\right) \in \mathbf{Z}^{(n)} \backslash N(B)} P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{n}\right) \Pi\left(p^{-1} z_{i j}\right)^{k_{i j}} \\
& \quad+\left(1-p^{-1}\right)^{\langle n\rangle+n} \sum_{\left(k_{i j}\right) \in N(B)} P\left(B, A ; x_{1}, \ldots, x_{n}\right) \prod\left(p^{-1} z_{i j}\right)^{k_{i j}},
\end{aligned}
$$

where $P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{n}\right)$ is a convergent power series of $x_{1}, \ldots, x_{n}$ for each $\left(k_{i j}\right)$ in $\boldsymbol{Z}^{\langle n\rangle} \backslash N(B)$.

In Section 2, we treat a more general case (cf. Theorem 2.4). Using our arguments
we can prove the rationality of $P\left(B, A ; x_{1}, \ldots, x_{n}\right)$ and calculate its denominator explicitly. The details will be published in a subsequent paper [K].
2. Generalized Igusa local zeta functions and the proof of the main result. Let $R=\boldsymbol{Z}_{p}\left[\left[x_{1}, \ldots, x_{s}\right]\right]$ be a formal power series ring, and

$$
R_{c}=\left\{f\left(x_{1}, \ldots, x_{s}\right) \in R ; f\left(a_{1}, \ldots, a_{s}\right) \text { converges for any }\left(a_{1}, \ldots, a_{s}\right) \in \boldsymbol{Z}_{p}^{s}\right\}
$$

For two sets $S$ and $\Lambda$, we put

$$
S^{\Lambda}=\prod_{\lambda \in \Lambda} S_{\lambda}
$$

with $S_{\lambda}=S$. Let $\Lambda$ be a finite set. Put

$$
C_{+}^{\Lambda}=\left\{\left(s_{\lambda}\right) \in C^{\Lambda} ; \operatorname{Re} s_{\lambda}>0\right\}
$$

For a subset $\left\{f_{\lambda}\right\}_{\lambda \in \Lambda}$ of $R_{c}$, and $\left(s_{\lambda}\right) \in \boldsymbol{C}_{+}^{\Lambda}$ define $Z\left(\left\{f_{\lambda}\right\} ;\left(s_{\lambda}\right)\right)$ by

$$
Z\left(\left\{f_{\lambda}\right\} ;\left(s_{\lambda}\right)\right)=\int_{Z_{p}^{s}} \prod_{\lambda \in \Lambda}\left|f_{\lambda}\left(x_{1}, \ldots, x_{s}\right)\right|_{p}^{s_{\lambda}} d x
$$

where $d x$ is the Haar measure of $\boldsymbol{Q}_{p}^{s}$ so normalized that

$$
\int_{\mathbf{Z}_{p}^{s}} d x=1
$$

This function was studied by Igusa [I] when \# $\Lambda=1$, and was generalized by Deshommes to the case where \# $\Lambda$ is arbitrary. So we call this function a generalized Igusa local zeta function attached to $\left\{f_{\lambda}\right\}$. The function $Z\left(\left\{f_{\lambda}\right\} ;\left(s_{\lambda}\right)\right)$ is holomorphic on $\boldsymbol{C}_{+}^{A}$. Put $z_{\lambda}=p^{-s_{\lambda}}$. Then $Z\left(\left\{f_{\lambda}\right\} ;\left(s_{\lambda}\right)\right)$ can be regarded as a function of $\left(z_{\lambda}\right)$. So we put $\zeta\left(\left\{f_{\lambda}\right\} ;\left(z_{\lambda}\right)\right)=$ $Z\left(\left\{f_{\lambda}\right\} ;\left(s_{\lambda}\right)\right)$.

Let $m$ and $n$ be non-negative integers such that $m \geq n \geq 1$. Let $A$ and $B$ be non-degenerate symmetric matrices of degrees $m$ and $n$, respectively, with entries in $Z_{p}$. For a subset $I$ of $I_{m n}=\{(i, j) ; 1 \leq i \leq m, 1 \leq j \leq n\}$, put

$$
\begin{aligned}
\alpha_{p}(B, A, I)=p^{\# I} \lim _{e \rightarrow \infty} p^{(-m n+\langle n\rangle)} \#\left\{\overline{\left(x_{i j}\right)} \in M_{m n}\left(\boldsymbol{Z}_{p}\right) / p^{e} M_{m n}\left(\boldsymbol{Z}_{p}\right)\right. \\
\left.A\left[\left(x_{i j}\right)\right] \equiv B \bmod p^{e}, \text { and } x_{i j} \equiv 0 \bmod p \text { for any }(i, j) \in I\right\} .
\end{aligned}
$$

We note that $\alpha_{p}(B, A, I)=\alpha_{p}(B, A)$ if $I=\varnothing$. Now let $m, n, l$, and $n_{1}, \ldots, n_{s}, n_{s+1}, \ldots, n_{s+t}$ be non-negative integers such that $m \geq n \geq 1$, and $m \geq l, n_{1}, \ldots, n_{s+t} \geq 1, n_{1}+\cdots+n_{s}=n$, and $n_{s+1}+\cdots+n_{s+t}=l$. Let $A$ and $B$ be non-degenerate symmetric matrices of degrees $m$ and $n$, respectively, with entries in $\boldsymbol{Z}_{p}$, and $I$ be a subset of $I_{m, n}$. Define a formal power series $P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+i} ; x_{1}, \ldots, x_{s+t}\right)$ by

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\(P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; x_{1}, \ldots, x_{s+t}\right)\)
    \(=\sum_{r_{1}, \cdots, r_{s+t}=0}^{\infty} \alpha_{p}\left(B\left[\operatorname{diag}\left(p^{r_{1}} E_{n_{1}}, \ldots, p^{r_{s}} E_{n_{s}}\right)\right], A\left[\operatorname{diag}\left(E_{m-l}, p^{r_{s+1}} E_{n_{s+1}}, \ldots, p^{r_{s+t}} E_{n_{s+t}}\right)\right], I\right)\)
\[
\times x_{1}^{r_{1}} \cdots x_{s+t}^{r_{s}+t},
\]
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where $E_{k}$ denotes the unit matrix of degree $k$. We note that the formal power series $P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; x_{1}, \ldots, x_{s+t}\right)$ coincides with $P\left(B, A ; x_{1}, \ldots, x_{n}\right)$ in Section 1 if $l=0, s=n$, and $I=\varnothing$. Further if $l=0, I=\varnothing$, and $B=\operatorname{diag}\left(b_{1} E_{n_{1}}, \ldots, b_{s} E_{n_{s}}\right)$, the formal power series

$$
\sum_{e_{1}, \cdots, e_{s}=0,1} x_{1}^{e_{1}} \ldots x_{s}^{e_{s}} P\left(\operatorname{diag}\left(p^{e_{1}} b_{1} E_{n_{1}}, \ldots, p^{e_{s}} b_{s} E_{n_{s}}\right), A, I ; 0 ; n_{1}, \ldots, n_{s} ; x_{1}^{2}, \ldots, x_{s}^{2}\right)
$$

coincides with the $P\left(B, A, x_{1}, \ldots, x_{s}\right)$ defined in 1.2 of [BS]. In this section we show that $P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; x_{1}, \ldots, x_{s+t}\right)$ can be realized as a coefficient of the Laurent expansion of a certain generalized Igusa local zeta function.

For this, we give some preliminaries. For a commutative ring $R$, let $\operatorname{Sym}(k ; R)$ denote the set of symmetric matrices of degree $k$ with entries in $R$. Let $U=\left(u_{i j}\right) \in \operatorname{Sym}\left(m, \boldsymbol{Z}_{p}\right)$, and $V=\left(v_{i j}\right) \in \operatorname{Sym}\left(n, \boldsymbol{Z}_{p}\right)$, and $I$ be a subset of $\{(i, j) ; 1 \leq i \leq m$, $1 \leq j \leq n\}$. For each $\left(e_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle}$, put $M\left(\left(e_{i j}\right)\right)=\max _{i j}\left(e_{i j}\right)$, and

$$
\begin{aligned}
& A\left(\left(e_{i j}\right) ; V, U, I\right)=\left\{\overline{\left\{\left(x_{\alpha i}\right)\right.} \in M_{m n}\left(Z_{p}\right) / p^{M\left(\left(e_{i j}\right)\right)} M_{m n}\left(Z_{p}\right) ;\right. \\
& \left.\quad \sum_{1 \leq \alpha, \beta \leq m} u_{\alpha \beta} x_{\alpha i} x_{\beta j} \equiv v_{i j} \bmod p^{e_{i j}} \text { for any } i, j \text { and } x_{\alpha i} \equiv 0 \bmod p \text { for any }(\alpha, i) \in I\right\},
\end{aligned}
$$

and

$$
a\left(\left(e_{i j}\right) ; V, U, I\right)=\# A\left(\left(e_{i j}\right) ; V, U, I\right)
$$

If $e_{i j}=e$ for all $i, j$, we simply write $A\left(\left(e_{i j}\right) ; V, U, I\right)=A(e ; V, U, I)$. The following lemma is well known:

Lemma 2.1 (cf. Siegel [S, Hilfssatz 13]). In addition to the above notation and assumptions, assume that $U$ and $V$ are non-degenerate, and put $e_{0}=2 \operatorname{ord}_{p}(2 \operatorname{det} V)+1$. Then for any integer $e \geq e_{0}$, we have

$$
a(e+1 ; V, U, I)=p^{m n-\langle n\rangle} a(e ; V, U, I) .
$$

The following is essential to proving Theorem 2.4.
Proposition 2.2. Let the assumptions and notation be as above. Then for any $\left(e_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle}$ such that $\min _{i j} e_{i j} \geq e_{0}$, we have

$$
p^{\left.-M\left(\left(e_{i}\right)\right)\right) m n+\sum e_{i j}} a\left(\left(e_{i j}\right) ; V, U, I\right)=p^{e_{0}(-m n+\langle n\rangle)} a\left(e_{0} ; V, U, I\right) .
$$

Proof. Put $e=M\left(\left(e_{i j}\right)\right)$. For an element $\left(u_{i j}\right)_{1 \leq i \leq j \leq n}$ of $\boldsymbol{Z}^{\langle n\rangle}$, we define an element
$\left(u_{i j}^{*}\right)_{1 \leq i, j \leq n}$ of $\operatorname{Sym}\left(n, \boldsymbol{Z}_{p}\right)$ by $u_{i j}^{*}=u_{i j}$ or $=u_{j i}$ according as $i \leq j$ or not. Then we have

$$
\begin{equation*}
a\left(\left(e_{i j}\right) ; V, U, I\right)=\sum_{\left(c_{i j}\right)} a\left(e ; V+\left(\left(p^{e_{i j}} c_{i j}\right)^{*}\right), U, I\right), \tag{2.1}
\end{equation*}
$$

where $\left(c_{i j}\right)$ runs through all representatives of the direct product $\Pi \boldsymbol{Z}_{p} / p^{e-e_{i j}} \boldsymbol{Z}_{\boldsymbol{p}}$ of $\left\{\boldsymbol{Z}_{p} / p^{e-e_{i j}} \boldsymbol{Z}_{p}\right\}_{1 \leq i \leq j \leq n}$. On the other hand, for any ( $c_{i j}$ ), we have

$$
a\left(e ; V+\left(\left(p^{e_{i j}} c_{i j}\right)^{*}\right), U, I\right)=p^{\left(e-e_{0}\right)(m n-\langle n\rangle)} a\left(e_{0} ; V, U, I\right) .
$$

Thus the right-hand side of (2.1) is equal to

$$
\begin{aligned}
p^{\left(e-e_{0}\right)(m n-\langle n\rangle)} \# \prod Z_{p} / p^{e-e_{i j}} Z_{p} a\left(e_{0} ; V, U, I\right) & =p^{\left(e-e_{0}\right)(m n-\langle n\rangle} \prod p^{e-e_{i j}} a\left(e_{0} ; V, U, I\right) \\
& =p^{e m n-\sum e_{i j}} p^{e_{0}(-m n+\langle n\rangle)} a\left(e_{0} ; V, U, I\right) .
\end{aligned}
$$

Now let $A, B$ and $I$ be as above. Define an element $\left(g_{i j}\right)=\left(g_{i j}(X)\right)$ of $\operatorname{Sym}\left(n, Z_{p}\left[x_{\alpha k}(1 \leq \alpha \leq m, 1 \leq k \leq n)\right]\right.$ by

$$
\left(g_{i j}\right)=A\left[\left(p^{e(\alpha, k ; i)} x_{\alpha k}\right)\right]-B,
$$

where $e(\alpha, k ; I)=1$ or $=0$ according as $(\alpha, k) \in I$ or not. Further for each $\left(e_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle}$ put

$$
E\left(\left(e_{i j} ; B, A, I\right)=\left\{X \in M_{m n}\left(Z_{p}\right) ; g_{i j}(X) \in p^{e_{i j}} Z_{p} \text { for any } 1 \leq i \leq j \leq n\right\},\right.
$$

and let $\tilde{v}\left(\left(e_{i j}\right) ; B, A, I\right)$ be the volume of $E\left(\left(e_{i j}\right) ; B, A, I\right)$. Then we have:
Proposition 2.3. For each $\left(e_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle}$ such that $\min _{i j} e_{i j} \geq e_{0}$, we have

$$
\tilde{v}\left(\left(e_{i j}\right), B, A, I\right) p^{\Sigma e_{i j}}=\alpha_{p}(B, A ; I)
$$

Proof. We have

$$
\tilde{v}\left(\left(e_{i j}\right), B, A, I\right)=p^{-M\left(\left(e_{i j}\right)\right) m n} p^{\# I} a\left(\left(e_{i j}\right) ; B, A, I\right) .
$$

Thus the assertion follows from Proposition 2.2.
Corollary. Let $v\left(\left(e_{i j}\right), B, A, I\right)$ be the volume of the set

$$
\left\{(X) \in M_{m n}\left(\left(Z_{p}\right) ; g_{i j}(X) \in p^{e_{i j}} Z_{p}^{*} \text { for any } 1 \leq i \leq j \leq n\right\},\right.
$$

where $Z_{p}^{*}$ denotes the unit group of $\boldsymbol{Z}_{p}$. Then for each $\left(e_{i j}\right) \in \boldsymbol{Z}^{\langle n\rangle}$ such that $\min _{i j} e_{i j} \geq e_{0}$, we have

$$
v\left(\left(e_{i j}\right) ; B, A, I\right) p^{\sum e_{i j}}=\left(1-p^{-1}\right)^{\langle n\rangle} \alpha_{p}(B, A ; I) .
$$

Proof. We simply write $v\left(\left(e_{i j}\right)\right)=v\left(\left(e_{i j}\right) ; B, A, I\right)$. We arrange the quantities $\left\{e_{i j}\right\}$ indexed by the set $\{(i, j) ; 1 \leq i \leq j \leq n\}$ in the lexicographic order, and put $e_{1}=$ $e_{11}, e_{2}=e_{12}, \ldots, e_{n}=e_{1 n}, \ldots, e_{\langle n\rangle}=e_{n n}$, and $\sum e_{i}=\sum_{1 \leq i \leq\langle n\rangle} e_{i}$. Then we have

$$
v\left(\left(e_{i}\right)\right)=\sum_{j=0}^{\langle n\rangle}(-1)^{j} \sum_{1 \leq i_{1}<\cdots<i_{j} \leq\langle n\rangle} \tilde{v}\left(e_{1}, \ldots, e_{i_{1}}+1, \ldots, e_{i_{j}}+1, \ldots, e_{\langle n\rangle}\right) .
$$

Then by Proposition 2.3, for any $\left(e_{i}\right)$ such that $\min e_{i} \geq e_{0}$ we have

$$
\tilde{v}\left(e_{1}, \ldots, e_{i_{1}}+1, \ldots, e_{i_{j}}+1, \ldots, e_{\langle n\rangle}\right) p^{\sum e_{i}}=p^{-j} \alpha_{p}(B, A, I) .
$$

Thus the assertion holds.
Now let $x_{i j}(1 \leq i \leq m, 1 \leq j \leq n), x_{1}, \ldots, x_{s+t}$ be variables over $\boldsymbol{Z}_{p}$, and put $R=\boldsymbol{Z}_{p}\left[x_{i j}(1 \leq i \leq m, 1 \leq j \leq n), x_{1}, \ldots, x_{s+t}\right]$. Let $A, B, I, l$, and the others be as above. Define elements $y_{n+1}, \ldots, y_{m+n}$ of $R$ by

$$
\operatorname{diag}\left(y_{n+1}, \ldots, y_{n+m}\right)=\operatorname{diag}\left(E_{m-l}, x_{s+1} E_{n_{s+1}}, \ldots, x_{s+t} E_{n_{s+t}}\right) .
$$

Define an element $\left(h_{i j}(x, X)\right)_{1 \leq i, j \leq n}$ of $\operatorname{Sym}(n, R)$ by

$$
\left(h_{i j}(x, X)\right)=A\left[\left(y_{n+\alpha} p^{e(\alpha, k ; I)} x_{\alpha k}\right)_{1 \leq \alpha \leq m, 1 \leq k \leq n}\right]-B\left[\operatorname{diag}\left(x_{1} E_{n_{1}}, \ldots, x_{s} E_{n_{s}}\right)\right] .
$$

Now let $\Lambda=\{(i, j) ; 1 \leq i \leq j \leq n\} \cup\{i ; 1 \leq i \leq s+t\}$, and define a subset $\left\{h_{\lambda}\right\}_{\lambda \in \Lambda}$ of $R$ indexed by $\Lambda$ by

$$
h_{\lambda}= \begin{cases}h_{i j} & \text { if } \lambda=(i, j) \\ x_{i} & \text { if } \lambda=i,\end{cases}
$$

and $\zeta\left(B, A, I ; l, n_{1}, \ldots, n_{s+i} ;\left(z_{i j}\right)_{1 \leq i \leq j \leq n}, z_{1}, \ldots, z_{s+t}\right)$ by

$$
\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+i} ;\left(z_{i j}\right)_{1 \leq i \leq j \leq n}, z_{1}, \ldots, z_{s+t}\right)=\zeta\left(\left\{h_{\lambda}\right\} ;\left(z_{\lambda}\right)\right) .
$$

We write $\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ;\left(z_{i j}\right)_{1 \leq i \leq j \leq n}, z_{1}, \ldots, z_{s+t}\right)=\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t}\right.$; $Z, w)$ as in Section 1. Then $\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+i} ; Z, w\right)$ can be expressed as

$$
\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+i} ; Z, w\right)=\int_{\boldsymbol{Z}_{p}^{s+i} \times M_{m n}\left(\boldsymbol{Z}_{p}\right)} \prod\left|h_{i j}\right|_{p}^{s_{i j} j} \prod_{k=1}^{s+t}\left|x_{k}\right|_{p}^{s_{k}} d x d X,
$$

where $d x$ (resp. $d X$ ) denotes the Haar measure of $\boldsymbol{Q}_{p}^{s+t}\left(\right.$ resp. $\left.M_{m n}\left(\boldsymbol{Q}_{p}\right)\right)$ so normalized that

$$
\int_{\mathbf{Z}_{P_{-}}^{s+t}} d x=1 \quad\left(\text { resp. } \int_{M_{m n}\left(\boldsymbol{Z}_{P}\right)} d X=1\right) .
$$

We note that $\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; Z, w\right)$ coincides with $\zeta(B, A ; Z, w)$ if $l=0, s=n$, and $I=\varnothing$. Thus Theorem 1 is a special case of the following:

## Theorem 2.4. In the region

$$
E=\left\{\left(\left(z_{i j}\right), x_{1}, \ldots, x_{s+t}\right) \in C^{\langle n\rangle+s+t} ; 0<\left|z_{i j}\right|<1,0<\left|p x_{i} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{i}}\right|<1\right\}
$$

we have

$$
\begin{aligned}
& \zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; Z, p x_{1} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{1}}, \ldots, p x_{s+t} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{s+t}}\right) \\
& \quad=\sum_{\left(k_{i j}\right) \in \mathbf{Z}^{\langle n \backslash \backslash N(B)}} P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right) \prod\left(p^{-1} z_{i j}\right)^{k_{i j}} \\
& \quad+\left(1-p^{-1}\right)^{\langle n\rangle+s+t} \sum_{\left(k_{i j}\right) \in N(B)} P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; x_{1}, \ldots, x_{s+t}\right) \prod\left(p^{-1} z_{i j}\right)^{k_{i j}}
\end{aligned}
$$

where $P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right)$ is a convergent power series of $x_{1}, \ldots, x_{s+t}$ for each $\left(k_{i j}\right)$ in $\boldsymbol{Z}^{\langle n\rangle} \backslash N(B)$.

Proof. Put $\zeta(Z, w)=\zeta\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t}, Z, w\right)$. Then we have

$$
\begin{aligned}
\zeta(Z, w) & =\int_{Z_{p}^{s+t} \times M_{m n}\left(\mathbf{Z}_{p}\right)} \prod\left|h_{i j}(x, X)\right|_{p}^{s_{i j}} \prod_{k=1}^{s+t}\left|x_{k}\right|_{p}^{s_{k}} d x d X \\
& =\sum_{r_{1}, \cdots, r_{s+t}=0}^{\infty} \int_{X_{0}^{\prime}\left(r_{1}, \cdots, r_{s+t}\right)} \prod\left|h_{i j}(x, X)\right|_{p}^{s_{j}} \prod_{i=1}^{s+t}\left|x_{i}\right|_{p}^{s_{i}} d x d X,
\end{aligned}
$$

where $X_{0}^{\prime}\left(r_{1}, \ldots, r_{s+t}\right)=\left\{(x, X) \in \boldsymbol{Z}_{p}^{s+t} \times M_{m n}\left(Z_{p}\right) ;\left|x_{i}\right|_{p}=p^{-r_{i}}\right\}$. Thus we have $\zeta(Z, w)$

$$
=\sum_{r_{1}, \ldots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t}\left(p^{-1} p^{\left.-s_{i}\right)^{r_{i}}} \int_{\boldsymbol{Z}_{p}^{t_{s+1}} \times M_{m n}\left(\boldsymbol{Z}_{p}\right)} \prod\left|h_{i j}\left(p^{r_{1}} x_{1}, \ldots, p^{r_{s+t}} x_{s+t}, X\right)\right|_{p}^{s_{i j} j} d x d X .\right.
$$

Define elements $y_{1}, \ldots, y_{n+m}$ of $R$ by

$$
\operatorname{diag}\left(y_{1}, \ldots, y_{n}\right)=\operatorname{diag}\left(x_{1} E_{n_{1}}, \ldots, x_{s} E_{n_{s}}\right)
$$

and

$$
\operatorname{diag}\left(y_{n+1}, \ldots, y_{n+m}\right)=\operatorname{diag}\left(E_{m-l}, x_{s+1} E_{n_{s+1}}, \ldots, x_{s+t} E_{n_{s+t}}\right)
$$

We change the variables as follows:

$$
x_{\alpha j} \longmapsto x_{\alpha j} y_{j} y_{n+\alpha}^{-1}(1 \leq \alpha \leq m, 1 \leq j \leq n), \quad x_{j} \longmapsto x_{j}(1 \leq j \leq s+t) .
$$

Then we have

$$
\begin{aligned}
\zeta(Z, w) & =\sum_{r_{1}, \ldots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t}\left(p^{-1} z_{i}\right)^{r_{i}} \int_{\boldsymbol{Z}_{p}^{s_{s}+t} \times M_{m n}\left(\mathbf{Z}_{p}\right)} \prod\left|h_{i j}\left(p^{r_{1}}, \ldots, p^{r_{s+t}}, X\right)\right|_{p}^{s_{i j}} d x d X \\
& =\left(1-p^{-1}\right)^{s+t} \sum_{r_{1}, \ldots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t}\left(p^{-1} z_{i}\right)^{r_{i}} \int_{M_{m n}\left(\mathbf{Z}_{p}\right)} \prod\left|h_{i j}\left(p^{r_{1}}, \ldots, p^{r_{s+t}}, X\right)\right|_{p}^{s_{i j}} d X .
\end{aligned}
$$

Now for non-negative integers $r_{1}, \ldots, r_{s+t}, e_{i j}(1 \leq i \leq j \leq n)$, put
$v\left(r_{1}, \ldots, r_{s+t},\left(e_{i j}\right)\right)=v\left(\left\{X \in M_{m, n}\left(Z_{p}\right) ; h_{i j}\left(p^{r_{1}}, \ldots, p^{r_{r}+s}, X\right) \in p^{e_{i j}} Z_{p}^{*}\right.\right.$ for any $\left.\left.1 \leq i \leq j \leq n\right\}\right)$.
Then we have

$$
\zeta(Z, w)=\sum_{r_{1}, \ldots, r_{s+t, e_{i j}=0}^{\infty}} v\left(r_{1}, \ldots, r_{s+t},\left(e_{i j}\right)\right) \prod z_{i j}^{e_{i j}} \prod_{i=1}^{s+t}\left(p^{-1} z_{i}\right)^{r_{i}} .
$$

We write $(r)=\left(r_{1}, \ldots, r_{s+t}\right)$ and $s((r))=4\left(n_{1} r_{1}+\cdots+n_{s+t} r_{s+t}\right)$. Define

$$
\tilde{\zeta}\left(Z, x_{1}, \ldots, x_{s+t}\right)=\zeta\left(Z, p x_{1} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{1}}, \ldots, p x_{s+t} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{s+t}}\right) .
$$

Then we have

$$
\begin{aligned}
\tilde{\zeta}\left(Z, x_{1}, \ldots, x_{s+t}\right)= & \sum_{r_{1}, \ldots, r_{s+t}=0}^{\infty} \sum_{\left(k_{i j}\right)} v\left(r_{1}, \ldots, r_{s+t},\left(s((r))+k_{i j}\right)\right) p^{\sum\left(s(r(r))+k_{i j}\right)} \\
& \times \prod\left(p^{-1} z_{i j}\right)^{k_{i j}} \prod_{i=1}^{s+t} x_{i}^{r_{i}}
\end{aligned}
$$

where $\left(k_{i j}\right)$ runs through all elements of $\boldsymbol{Z}^{\langle n\rangle}$ such that $k_{i j} \geq-s((r))$ for any $1 \leq i \leq j \leq n$. For each ( $k_{i j}$ ), define a formal power series $P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right)$ by

$$
P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right)=\sum_{r_{1}, \cdots, r_{s+t}} v\left(r_{1}, \ldots, r_{s+t},\left(s((r))+k_{i j}\right)\right) p^{\sum\left(s(r(r))+k_{i j}\right)} x_{1}^{r_{1} \cdots x_{s+t}^{r_{s}+t}},
$$

where $r_{1}, \ldots, r_{s+t}$ run through all non-negative integers such that $r_{1}, \ldots, r_{s+t} \geq 0$, and $s((r)) \geq-k_{i j}$ for any $1 \leq i \leq j \leq n$. Since the right-hand side of $\tilde{\zeta}\left(Z, x_{1}, \ldots, x_{s+t}\right)$ is absolutely convergent in the region $0<\left|z_{i j}\right|<1,0<\left|p x_{i} \prod\left(p^{-1} z_{i j}\right)^{-4 n_{i}}\right|<1$, the formal power series $P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right)$ is a convergent power series, and we have

$$
\tilde{\zeta}\left(Z, x_{1}, \ldots, x_{s+t}\right)=\left(1-p^{-1}\right)^{s+t} \sum_{\left(k_{i j}\right)} P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right) \prod\left(p^{-1} z_{i j}\right)^{k_{i j}} .
$$

Further by Corollary to Proposition 2.3, for any $\left(k_{i j}\right) \in N(B)$ we have

$$
P\left(\left(k_{i j}\right) ; x_{1}, \ldots, x_{s+t}\right)=\left(1-p^{-1}\right)^{\langle n\rangle} P\left(B, A, I ; l ; n_{1}, \ldots, n_{s+t} ; x_{1}, \ldots, x_{s+t}\right) .
$$

Thus the assertion holds.

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Muroran Institute of Technology
27-1 Mizumoto
Muroran 050
JAPAN

