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出版情報：Progress in mathematics. 129, pp.165-172, 1995. Birkhäuser
バージョン：
権利関係：

A GENERALIZED JACOBI THETA FUNCTION AND QUASIMODULAR FORMS

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In this note we give a direct proof using the theory of modular forms of a beautiful fact explained in the preceding paper by Robbert Dijkgraaf [1, Theorem 2 and Corollary]. Let $\widetilde{M}_*(\Gamma_1)$ denote the graded ring of *quasimodular forms* on the full modular group $\Gamma_1 = PSL(2,\mathbb{Z})$. This is the ring generated by G_2, G_4, G_6, and graded by assigning to each G_k the weight k, where

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\Bigl(\sum_{d|n} d^{k-1}\Bigr) q^n \qquad (k = 2,\, 4,\, 6, \ldots\,, \quad B_k = k\text{th Bernoulli number})$$

are the classical Eisenstein series, all of which except G_2 are modular. (See §1 for a more general and more intrinsic definition of quasi-modular.) We define a generalization of the classical Jacobi theta function by the triple product

$$\Theta(X,q,\zeta) = \prod_{n>0}(1-q^n)\prod_{\substack{n>0\\ n\text{ odd}}}\bigl(1-e^{n^2X/8}\,q^{n/2}\,\zeta\bigr)\bigl(1-e^{-n^2X/8}\,q^{n/2}\,\zeta^{-1}\bigr)\,, \tag{1}$$

considered as a formal power series in X and $q^{1/2}$ with coefficients in $\mathbb{Q}[\zeta,\zeta^{-1}]$. (We can also consider q and ζ as complex numbers, in which case the coefficient of X^n is a holomorphic function of these variables for each n, but we cannot consider the product as a holomorphic function of the third variable X because it diverges rapidly for any X with non-zero real part.) Let $\Theta_0(X,q) \in \mathbb{Q}[[q,X]]$ denote the coefficient of ζ^0 in $\Theta(X,q,\zeta)$, considered as a Laurent series in ζ, and expand Θ_0 as a Taylor series

$$\Theta_0(X,q) = \sum_{n=0}^{\infty} A_n(q)\,X^{2n}\,, \qquad A_n(q) \in \mathbb{Q}[[q]] \tag{2}$$

in X. (It is easy to see that there are no odd powers of X in this expansion.) The result in question is then

Theorem 1. *$A_n \in \widetilde{M}_{6n}(\Gamma_1)$ for all $n \geq 0$.*

The coefficient of X^{2g-2} in $\log\Theta_0$, which as explained in [1] is the generating function counting maps of curves of genus $g > 1$ to a curve of genus 1, is then also quasi-modular of weight $6g-6$, but we will not discuss this connection further.

The proof of Theorem 1 will be given in §2. In §3 and §4 we compute the "highest degree term" (coefficient of G_2^{3n}) in A_n and comment on the relationship to Jacobi forms.

§1. Quasimodular forms. We denote by $\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ the complex upper half-plane and for $\tau \in \mathcal{H}$ write $q = e^{2\pi i\tau}$ and $Y = 4\pi\Im(\tau)$, while D denotes the differential operator $D = \dfrac{1}{2\pi i}\,\dfrac{d}{d\tau} = q\,\dfrac{d}{dq}$. (The factors 4π and $2\pi i$ are included for convenience and to avoid unnecessary irrationalities later.) Recall that a *modular form of weight k* on a subgroup Γ of finite index of Γ_1 is a holomorphic function f on $\mathcal{H}$ satisfying

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k\,f(\tau) \qquad \forall\tau\in\mathcal{H},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and growing at most polynomially in $1/Y$ as $Y \to 0$. If $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, then these conditions imply that f has a convergent Fourier series expansion $f(\tau) = \sum_{n=0}^{\infty} a(n)\,q^{n/\lambda}$ at infinity. The space of holomorphic modular forms of weight k on Γ is denoted by $M_k(\Gamma)$ and the graded ring $\bigoplus_k M_k(\Gamma)$ by $M_*(\Gamma)$.

As well as the holomorphic modular forms, there are also functions $F(\tau)$ which satisfy the same transformation properties and growth conditions as before but which belong to $\mathbb{C}[[q^{1/\lambda}]]\,[Y^{-1}]$ instead of $\mathbb{C}[[q^{1/\lambda}]]$, i.e. which have the form

$$F(\tau) = \sum_{m=0}^{M} f_m(\tau)\,Y^{-m} \qquad (f_m(\tau) \text{ holomorphic for } m = 0, 1, \ldots, M) \tag{3}$$

for some integer $M \geq 0$ (and necessarily $\leq k/2$). We call such a function an *almost-holomorphic modular form of weight k* and denote the vector space of them by $\widehat{M}_k(\Gamma)$, while the holomorphic function $f_0(\tau)$ obtained formally as the "constant term with respect to $1/Y$" of f will be called a *quasi-modular form of weight k* and the space of such functions denoted by $\widetilde{M}_k(\Gamma)$. It is clear that the spaces $\widehat{M}_*(\Gamma) = \bigoplus \widehat{M}_k(\Gamma)$ and $\widetilde{M}_*(\Gamma) = \bigoplus \widetilde{M}_k(\Gamma)$ are graded rings and the map $\widehat{M}_*(\Gamma) \to \widetilde{M}_*(\Gamma)$ is a ring homomorphism.

As the basic example, if $\Gamma = \Gamma_1$ and we think of the power series $G_k \in \mathbb{Q}[[q]]$ defined in the introduction as functions of $\tau \in \mathcal{H}$, then $G_k(\tau)$ is a holomorphic modular form of weight k for $k > 2$ but $G_2(\tau)$ satisfies instead

$$G_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2\,G_2(\tau) - \frac{c(c\tau+d)}{4\pi i} \qquad \forall\tau\in\mathcal{H},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\,. \tag{4}$$

(A standard proof is to notice that G_2 is a multiple of the logarithmic derivative of the Ramanujan function $\Delta(\tau) = q\prod(1-q^n)^{24}$, which is a modular form of weight 12.) It is easy to check that (4) is equivalent to the assertion that the function $G_2^*(\tau) = G_2(\tau) + 1/2Y$ is an almost-holomorphic modular form of weight 2, so G_2 itself is indeed quasi-modular. Another easy consequence of (4) is that the expressions $D(G_2) + 2G_2^2$ and $D(f) + 2kG_2 f$ ($f \in M_k$) are holomorphic modular forms (of weights 4 and $k+2$, respectively), from which it follows that the ring of quasi-modular forms is closed under differentiation and, in particular, that all derivatives of holomorphic modular forms or of G_2 are quasi-modular. A converse to this and some other simple properties of quasi-modular forms are contained in the following proposition, whose proof (by induction on the degree of almost-holomorphic forms with respect to $1/Y$) we omit.

Proposition 1. *Let $\Gamma \subset \Gamma_1$ be a subgroup of finite index of the full modular group. Then:*

(a) *The "constant term map" $\widehat{M}_*(\Gamma) \to \widetilde{M}_*(\Gamma)$ is an isomorphism of rings;*

(b) $\widetilde{M}_*(\Gamma) = M_*(\Gamma) \otimes \mathbb{C}[G_2]$, *i.e. every quasi-modular form on Γ can be written uniquely as a polynomial in G_2 with coefficients which are modular forms on Γ;*

(c) *For (even) $k > 0$ we have $\widetilde{M}_k(\Gamma) = \bigoplus_{0 \le i \le k/2} D^i M_{k-2i}(\Gamma) \oplus \langle D^{k/2-1} G_2 \rangle$, i.e., any quasi-modular form has a unique representation as a sum of derivatives of modular forms and of G_2.*

§**2. Expansions of $\Theta(X, q, \zeta)$.** As well as the variables X, q, and ζ, we introduce further variables w and Z defined by $w = e^X$, $\zeta = e^Z$. We will also follow the physicists' practice of using the same letter to denote a function expressed in terms of different independent variables, denoting for instance the Eisenstein series G_k by either $G_k(\tau)$ or $G_k(q)$ and the Dedekind eta-function $q^{1/24} \prod(1 - q^n)$ by either $\eta(\tau)$ or $\eta(q)$, and writing $\Theta(X, \tau, Z)$ for the function defined by (1). Finally, we denote by Γ_2 the group (usually denoted $\Gamma^0(2)$) of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ with b even and by $\theta(\tau)$ the theta-series $\sum_r (-1)^r q^{r^2/2}$, a modular form of weight $1/2$ on Γ_2. The first result is:

Proposition 2. *The function $\Theta(X, \tau, Z)$ has an expansion of the form*

$$\Theta(X, \tau, Z) = \theta(\tau) \sum\nolimits_{j,\, l \ge 0} H_{j,l}(\tau) \, \frac{X^j}{j!} \, \frac{Z^l}{l!} \tag{5}$$

where $H_{0,0}(\tau) = 1$ and each $H_{j,l}(\tau)$ is quasimodular of weight $3j + l$ on Γ_2.

Proof. From (1) and the identity $\theta(\tau) = \eta(\tau/2)^2/\eta(\tau)$ we find

$$\begin{aligned} \log\left(\frac{\Theta(X, \tau, Z)}{\theta(\tau)}\right) &= -\sum_{\substack{n,\, r > 0 \\ n \text{ odd}}} \frac{1}{r} \left(e^{n^2 r X/8} \zeta^r - 2 + e^{-n^2 r X/8} \zeta^{-r}\right) q^{nr/2} \\ &= -2 \sum_{\substack{j,\, l \ge 0 \\ j \equiv l \pmod 2 \\ j + l > 0}} \phi_{j,l}(\tau) \, \frac{(X/8)^j}{j!} \, \frac{Z^l}{l!} \end{aligned} \tag{6}$$

with

$$\phi_{j,l}(\tau) = \sum_{\substack{n,\, r > 0 \\ n \text{ odd}}} r^{l+j-1} \, n^{2j} \, q^{nr/2} = \begin{cases} 2^{2j} D^{2j} F^{(1)}_{l-j}(\tau) & \text{if } l > j, \\ 2^{j+l-1} D^{j+l-1} F^{(2)}_{j-l+2}(\tau) & \text{if } j \ge l, \end{cases} \tag{7}$$

where $F^{(1)}_k$ and $F^{(2)}_k$ ($k = 2, 4, \ldots$) are the two Eisenstein series

$$F^{(1)}_k(\tau) = G_k\left(\frac{\tau}{2}\right) - G_k(\tau) \;=\; \sum_{n=1}^{\infty} \Biggl(\sum_{d | n,\, 2 \nmid d} (n/d)^{k-1} \Biggr) q^{n/2}\,,$$

$$F^{(2)}_k(\tau) = G_k\left(\frac{\tau}{2}\right) - 2^{k-1} G_k(\tau) \;=\; (2^{k-1} - 1)\, \frac{B_k}{2k} + \sum_{n=1}^{\infty} \Biggl(\sum_{d | n,\, 2 \nmid d} d^{k-1} \Biggr) q^{n/2}$$

of weight k on Γ_2. Since each of these is quasi-modular (indeed, actually modular except for $F^{(1)}_2$) of weight k on Γ_2, and since the mth derivative of a quasi-modular form of weight k is quasi-modular of weight $k + 2m$, it follows that in both cases $\phi_{j,l} \in \widetilde{M}_{3j+l}(\Gamma_2)$. The result now follows by exponentiating, since quasi-modular forms form a graded ring.

The next identity is an analogue of the Jacobi triple product identity. Set

$$H(w,q,\zeta)=q^{-1/24}\prod_{n>0,\ n\text{ odd}}\bigl(1-w^{n^2/8}q^{n/2}\zeta\bigr)\bigl(1-w^{-n^2/8}q^{n/2}\zeta^{-1}\bigr)$$

and denote by $H_0(w,q)$ the coefficient of ζ^0 in $H(w,q,\zeta)$ as a Laurent series in ζ.

Proposition 3. *The function $H(w,q,\zeta)$ has the expansion*

$$H(w,q,\zeta)=\sum\nolimits_{r\in Z}(-1)^r\,H_0(w,w^rq)\,w^{r^3/6}\,q^{r^2/2}\,\zeta^r\,.$$

Proof. From the product expansion of H we find

$$\begin{aligned}H(w,\,wq,\,w^{1/2}q\zeta)\;&=\;(wq)^{-\frac{1}{24}}\prod_{n\text{ odd}}\bigl(1-w^{\frac{(n+2)^2}{8}}\,q^{\frac{n+2}{2}}\,\zeta\bigr)\bigl(1-w^{-\frac{(n-2)^2}{8}}\,q^{\frac{n-2}{2}}\,\zeta^{-1}\bigr)\\&=\;w^{-1/24}\,\frac{1-w^{-1/8}q^{-1/2}\zeta^{-1}}{1-w^{1/8}\,q^{1/2}\,\zeta}\,H(w,q,\zeta)\\&=\;-w^{-1/6}\,q^{-1/2}\,\zeta^{-1}\,H(w,q,\zeta)\,,\end{aligned}$$

and this means that if we write the Laurent expansion of H with respect to ζ in the form

$$H(w,q,\zeta)=\sum\nolimits_{r\in Z}(-1)^r\,H_r(w,q)\,w^{r^3/6}\,q^{r^2/2}\,\zeta^r\,,$$

then $H_{r+1}(w,q)=H_r(w,wq)$ and hence by induction $H_r(w,q)=H_0(w,w^rq)$ for all r.

Proof of Theorem 1. From Proposition 2 we have

$$H(X,\tau,Z)=\frac{1}{\eta(\tau)}\,\Theta(X,\tau,Z)=\frac{\theta(\tau)}{\eta(\tau)}\sum_{j,\,l\geq 0}H_{j,l}(\tau)\,\frac{X^j}{j!}\,\frac{Z^l}{l!}\,.$$

(Recall that $w=e^X$.) On the other hand, Proposition 3 can be written in the form

$$H(X,\tau,Z)=\sum_{r\in\mathbb{Z}}(-1)^r\,e^{r^3X/6+rZ}\,H_0\bigl(X,\,\tau+\frac{rX}{2\pi i}\bigr)\,q^{r^2/2}\,,$$

while by (2) and the definitions of $H_0(X,\tau)$ and $\Theta_0(X,\tau)$ we have

$$H_0(X,\tau)=\frac{1}{\eta(\tau)}\,\Theta_0(X,\tau)=\sum_{n=0}^{\infty}\frac{A_n(\tau)}{\eta(\tau)}\,X^{2n}\,.$$

Substituting the Taylor series expansions

$$e^{r^3X/6+rZ}\;=\sum_{p,l\geq 0}\frac{r^{3p+l}}{6^p p!\,l!}\,X^p\,Z^l,\qquad \frac{A_n}{\eta}\bigl(\tau+\frac{rX}{2\pi i}\bigr)\;=\sum_{m\geq 0}\frac{r^m}{m!}\,D^m\bigl(\frac{A_n}{\eta}(\tau)\bigr)\,X^m\,,$$

and comparing the coefficients of X^jZ^l in the two expansions of $H(X,\tau,Z)$, we obtain

$$\begin{aligned}\frac{\theta(\tau)}{\eta(\tau)}\,H_{j,l}(\tau)\;&=\sum_{\substack{m,\,n,\,p\geq 0\\ p+2n+m=j}}\frac{j!}{6^p p!\,m!}\,D^m\bigl(\frac{A_n(\tau)}{\eta(\tau)}\bigr)\sum_{r\in\mathbb{Z}}(-1)^r\,r^{3p+l+m}\,q^{r^2/2}\\&=\sum_{\substack{m,\,n,\,s\geq 0\\ 2m+2s+6n=3j+l}}\frac{2^s\,(2n)!}{6^{j-2n-m}}\binom{2n+m}{m}\binom{j}{2n+m}\,D^m\bigl(\frac{A_n(\tau)}{\eta(\tau)}\bigr)\,D^s\theta(\tau)\,.\end{aligned}$$

Now the fact that θ and η are modular (with character) and $H_{j,l}$ quasi-modular of weight $3j+l$ on Γ_2, together with the fact that the operator D preserves the property of quasi-modularity and raises weights by 2, implies by induction that A_n is quasi-modular of weight $6n$ on Γ_2 for all n. But Γ_1 is generated by Γ_2 and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, so a modular or quasi-modular form on Γ_2 which has a Fourier expansion with only integral powers of q is automatically modular or quasi-modular on Γ_1. This completes the proof of Theorem 1.

§**3. Highest order terms.** As we discussed in §1, there is an isomorphism $\widehat{M_k}(\Gamma)\to\widetilde{M_k}(\Gamma)$ obtained by sending an almost-holomorphic modular form $F(\tau)$ to the first term $f_0(\tau)$ of its expansion (3). The map in the other direction sends a quasi-modular form $f(\tau)$ to the function $f^*(\tau)$ obtained by writing $f(\tau)$ as a polynomial in G_2 with modular coefficients and then replacing G_2 by $G_2^*(\tau)=G_2(\tau)+1/2Y$. In particular, we can define the "leading coefficient" $L[f]$ of $f\in\widetilde{M_k}(\Gamma)$ by

$$f(\tau)=2^{k/2}L[f]\,G_2(\tau)^{k/2}+\cdots\,,\qquad f^*(\tau)=\frac{L[f]}{Y^{k/2}}+\cdots\,,$$

where "$\cdots$" denotes terms of smaller degree in G_2 or in $1/Y$. Equivalently, $L[f]$ is the coefficient $f_{k/2}(\tau)$ in the expansion (3), which is a constant if $M=k/2$ and zero otherwise (in general, the term $f_M(\tau)$ in (3) is a modular form of weight $k-2M$). The map $L:\widetilde{M_k}(\Gamma)\to\mathbb{C}$ for $k>0$ is also proportional to the projection onto the final summand $\langle D^{k/2-1}G_2\rangle$ in the direct sum decomposition of Proposition 1 (c). We wish to compute its value for the quasi-modular form $A_n\in\widetilde{M}_{6n}(\Gamma_1)$ of Theorem 1.

Theorem 2. $\quad L[A_n]=\dfrac{(1/72)^n}{1-6n}\,\dfrac{(6n)!}{(3n)!\,(2n)!}$ *for all* $n\ge 0$.

Sketch of proof. Note first that the map $L:\widetilde{M}_*(\Gamma)\to\mathbb{C}$ is a ring homomorphism and annihilates all modular forms of positive weight. (It is simply the map sending $P(G_2,G_4,G_6)$ to $P(\frac12,0,0)$.) It also has the property $L[D^nf]=(-1)^n\frac{\Gamma(k/2+n)}{\Gamma(k/2)}\,L[f]$ for $f\in\widetilde{M_k}(\Gamma)$ with $k>0$, because $D(G_2)=-2G_2^2+\frac56G_4$ and D is a derivation. Hence from (7) and the fact that all $F_k^{(2)}$ and all $F_k^{(1)}$ except for $F_2^{(1)}=F_2^{(2)}+G_2$ are modular we have

$$L[\phi_{j,l}]=\begin{cases}2^{2j-1}(2j)! & \text{if } l=j+2,\ j\ge0,\\ 0 & \text{otherwise.}\end{cases}$$

Inserting this into (6) gives

$$L\Big[\frac{\Theta(X,\tau,Z)}{\theta(\tau)}\Big]=\exp\big(-Z^2\Phi\big(\frac{XZ}{2}\big)\big)\,,\quad \Phi(t):=\sum_{j=0}^{\infty}\frac{(2j)!\,t^j}{j!\,(j+2)!}=\frac{(1-4t)^{3/2}-(1-6t)}{12t^2}\,.$$

On the other hand, induction on m and s gives

$$L\big[\eta(\tau)\,D^m\big(\frac{A_n(\tau)}{\eta(\tau)}\big)\big]=(-1)^m\frac{\Gamma(3n+m-\frac12)}{\Gamma(3n-\frac12)}\,L[A_n]\,,\qquad L\big[\frac{D^s\theta(\tau)}{\theta(\tau)}\big]=(-1)^s\,\frac{(2s)!}{2^{2s}\,s!}\,,$$

so the calculation in the proof of Theorem 1 leads to the identity

$$\exp\big(-Z^2\Phi\big(\frac{XZ}{2}\big)\big)=\sum_{m,n,p,l\ge0}(-1)^m\frac{\kappa(3p+l+m)}{6^p p!\,l!}\binom{3n+m-\frac32}{m}\,L[A_n]\,X^{p+2n+m}\,Z^l$$

where $\kappa(n)$ is $(-\frac{1}{2})^{n/2}\frac{n!}{(n/2)!}$ for n even and 0 for n odd. This generating series identity overdetermines the numbers $L[A_n]$, and even its specialization to $Z=0$, for which the left-hand side equals 1, yields a system of linear equations which determines them uniquely. The solution of this system is as given in the theorem, although the proof of this is not easy. One could in principle continue in a similar way and find the next terms (coefficients of $G_2^{3n-2}G_4$, $G_2^{3n-3}G_6$, etc.) in the expansion of A_n, but the calculations rapidly become unmanageable.

§**4. Quasi-Jacobi forms.** Finally, we discuss the nature of the "function" $\Theta(X,\tau,Z)$ (which is, of course, not a function of X at all, but just a formal power series). To simplify the comparison with Jacobi forms, we change variables again to $x=X/2\pi i$, $z=Z/2\pi i$. Recall [2] that a holomorphic Jacobi form on Γ is a holomorphic function ϕ of two variables $\tau\in\mathcal{H}$ and $z\in\mathbb{C}$ satisfying three properties: a "modular" transformation property with respect to $(\tau,z)\mapsto\bigl(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\bigr)$ for $\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)\in\Gamma$, an "elliptic" transformation property with respect to $z\mapsto z+\lambda\tau+\mu$ for (λ,μ) in some lattice in $\mathbb{Q}^2$, and a "holomorphic at infinity" property which says that the Fourier expansion of the function has only terms $q^n\zeta^r$ with $r^2\le 4nm$ for some rational number m, called the index of the form. All three properties are reflected by the function $\Theta(x,\tau,z)$, but in somewhat modified form. Specifically, Proposition 2 tells us that $\Theta(x,\tau,z)$ is the holomorphic part of a function $\Theta^*(x,\tau,z)$ (obtained by replacing each $H_{j,l}$ in (5) by $H^*_{j,l}$) which is invariant under $(x,\tau,z)\mapsto\bigl(\frac{x}{(c\tau+d)^3},\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\bigr)$ for all $\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)\in\Gamma_2$; Proposition 3 tells us that the function $H(x,\tau,z)=\eta(\tau)^{-1}\Theta(x,\tau,z)$ is multiplied by a simple factor under the substitution $(x,\tau,z)\mapsto(x,\tau+\lambda x,\,z+\lambda\tau+\frac{1}{2}\lambda^2x+\mu)$ for λ and μ in $\mathbb{Z}$; and Proposition 3 also implies that the Fourier expansion of $H(x,\tau,z)$ contains only terms $q^n\zeta^r$ with $n\ge r^2/2$. This suggests that there should be an analogue of the theory of Jacobi forms involving three variables z, τ and x of degrees 1, 2, and 3 rather than just two variables of degrees 1 and 2 as in the usual Jacobi case, where by "degree" we mean that under a modular transformation $\tau\mapsto\frac{a\tau+b}{c\tau+d}$ the variables z, τ and x change to variables z^*, τ^* and x^* with $\partial z^*/\partial z=(c\tau+d)^{-1}$, $\partial\tau^*/\partial\tau=(c\tau+d)^{-2}$, and $\partial x^*/\partial x=(c\tau+d)^{-3}$. It would be interesting to see whether such a theory can be worked out and whether there are further extensions containing other (presumably infinitesimal) variables of yet higher degree.

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