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A Generalized Mass Involving Higher Order Symmetric Functions of the Curvature Tensor

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Abstract. We define a generalized mass for asymptotically flat manifolds using some higher order symmetric function of the curvature tensor. This mass is non-negative when the manifold is locally conformally flat and the σ_k curvature vanishes at infinity. In addition, with the above assumptions, if the mass is zero, then, near infinity, the manifold is isometric to a Euclidean end.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold. Assume that (M, g) is asymptotically flat, i.e., there is a compact set $K \subset M$, some $R \geq 1$ and a diffeomorphism $\Phi : M \backslash K \to \mathbb{R}^n \backslash B_R$ such that

$$(\Phi^* g)_{ij}(x) = \delta_{ij} + o(|x|^{-\tau}) \quad \text{as } |x| \to \infty.$$

If $\tau \geq \frac{n-2}{2}$ and the scalar curvature R_g is integrable, then the so-called ADM mass of (M, g) is defined by (see [1,2])

$$m = \int\limits_{S_\infty} (g_{ij,j} - g_{jj,i}) \mathrm{d}S^i,$$

where $g_{ij,k}$ denotes a partial derivative and $\mathrm{d}S^i$ is the normal surface element of S_∞, the sphere at infinity.

That m is a geometric invariant of (M, g) is a consequence of the following expansion of the Hilbert–Einstein action:

$$R_g * 1 = \mathrm{d}(g^{ab}\, {\omega_a}^c \wedge \eta_{cb}) + g^{ab}\, {\omega_d}^c \wedge {\omega_a}^d \wedge \eta_{bc} \tag{1}$$

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where R_g is the scalar curvature, $*1$ is the volume form, ${\omega_a}^b$ is the Levi–Civita connection one-form with respect to a frame $\{e_a\}$ and $\eta_{ab} = (e_a \wedge e_b)\rfloor * 1$. See Bartnik [2] for more details.

Mass and its properties have attracted much attention since it was introduced. One of the reason is its wide range of applications in mathematical relativity and in geometric analysis. For example, consider the Yamabe problem which asks to find on a compact Riemannian manifold a conformal metric with constant scalar curvature. Its solutions are critical points of the Hilbert–Einstein functional in a fixed conformal class. Thus, it is not too surprising that the notion mass is useful in the study of the Yamabe problem. In fact, it is *very important* in the solution of the Yamabe problem [23] as well as in the resolution of compactness issue of the Yamabe problem [8,13,17–20,26].

In recent years, fully nonlinear versions of the Yamabe problem have received much attention after the work of Viaclovsky [30–32] and of Chang, Gursky and Yang [3–6]; see e.g., [9,11,12,14–16,27–29]. For a metric g, let A_g be the Schouten tensor of g, i.e.,

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)}\,R_g\, g\Big),$$

where Ric_g and R_g denote the Ricci curvature and the scalar curvature of g. Let $\lambda(A_g) = (\lambda_1, \ldots, \lambda_n)$ denote the eigenvalues of A_g with respect to g. For $1 \le k \le n$, let $\sigma_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}, \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, denote the kth elementary symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n | \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n | \lambda_1, \ldots, \lambda_n > 0\}$.

Question 1.1. *Let (N, h) be a compact, smooth Riemannian manifold of dimension $n \ge 3$ satisfying $\lambda(A_h) \in \Gamma$ on N and $1 \le k \le n$. Is there a smooth positive function u on N such that $\hat{h} = u^{\frac{4}{n-2}} h$ satisfies*

$$\sigma_k(\lambda(A_{\hat{h}})) = 1, \quad \lambda(A_{\hat{h}}) \in \Gamma_k, \qquad \text{on } N? \tag{2}$$

Equation (2) is a second order fully nonlinear elliptic equation of u. The special case of Question 1.1 for $k = 1$ is the Yamabe problem in the so-called positive case.

The problem is in general not a variational one when $k \ge 3$. Natural variants of Eq. (2) which are of variational form have been introduced by Chang and Fang [7]; see also a subsequent paper of Graham [10] on the algebraic structure of these equations under conformal transformations.

From the discussion on the relation between the ADM mass and the Yamabe problem, it is natural to ask if there is some notion of mass associated with the σ_k Yamabe problem. The main goal of this note is to give some generalization along this line. While we have not been able to identify such a notion that is directly related to the σ_k curvature, we are able to do so for a variant for $2 \le k < \frac{n}{2}$, which coincides with the σ_k curvature when (M, g) is locally conformally flat. This is motivated by a relation between the Pfaffian and the $\sigma_{n/2}$ curvature (see Viaclovsky [30]). See Theorem 3.1 for a precise definition.

We note that a generalization of mass was studied earlier by Michel in [21,22]. In these works, mass is defined with respect to each member of a large class of differential operators. However, the definition of mass therein uses either the linear structure of the corresponding operator or of its linearization. Our approach is very different.

After our work was done, Yanyan Li heard a talk by Guofang Wang in the conference on ‘Geometric PDEs’ in the trimester on ‘Conformal and Kähler Geometry’ at IHP, Paris (held during 5–9 November 2012) in which he announced that he together with Yuxin Ge and Jie Wu had also developed a notion of higher order mass. After the talk, Yanyan informed Guofang that we had defined a mass by using an invariant of the σ_k curvature which agrees with the σ_k curvature when the manifold is locally conformally flat and proved that the mass is non-negative under the assumption that the manifold is locally conformally flat and the σ_k curvature is zero near infinity, together with a rigidity in that case.

The rest of the paper is organized as follows. In Sect. 2, we define a curvature invariant Λ_k which coincides with the σ_k curvature when (M, g) is locally conformally flat. We also provide a decomposition for Λ_k which is a generalization of the decomposition (1) for the Hilbert–Einstein action; see Eq. (3). In Sect. 3, we used the decomposition developed in Sect. 2 to define a mass, called the kth mass. In Sect. 4, we announce a very restrictive version of the positive mass theorem for the kth mass.

2. A Curvature Invariant

Consider an n-dimensional Riemannian manifold (M, g). Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame and $\{\theta^1, \ldots, \theta^n\}$ its dual coframe. Let $({\omega_i}^j)$ be the skew-symmetric matrix of Levi–Civita connection one-forms:

$$\nabla_\xi e_i = {\omega_i}^j(\xi)\, e_j.$$

(Here and below, upper indices label rows while lower indices label columns.) The first structural equations read

$$\mathrm{d}\theta^j = \theta^i \wedge {\omega_i}^j.$$

The curvature tensor is viewed as a skew-symmetric matrix of two-forms,

$${\Omega_i}^j(X, Y) = \theta^j(R(X, Y)e_i).$$

The second structural equations read

$${\Omega_i}^j = \mathrm{d}{\omega_i}^j - {\omega_i}^k \wedge {\omega_k}^j.$$

Also, the first and second Bianchi identities read

$$\theta^i \wedge {\Omega_i}^j = 0 \quad \text{and} \quad \mathrm{d}{\Omega_i}^j = -{\Omega_i}^k \wedge {\omega_k}^j + {\omega_i}^k \wedge {\Omega_k}^j.$$

For a multi-index $I = (i_1, \ldots, i_m)$, let $\theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_m}$ and $\theta^{[I]} = *\theta^I$ where $*$ is the Hodge dual operator.

For $1 \leq k \leq \frac{n}{2}$, define the n-form

$$\Lambda_k = \frac{1}{2^k\, k!} \sum_{I=(i_1,\ldots,i_{2k}) \in S_{n,2k}} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-1}}{}^{i_{2k}} \wedge \theta^{[I]},$$

where

$$S_{n,2k} := \Big\{(i_1, \ldots, i_{2k}) : 1 \leq i_p \leq n, i_p \neq i_q \text{ whenever } p \neq q\Big\}.$$

It is useful to observe that

$$\Lambda_k = \frac{1}{2^k\, k!} \sum_{1 \leq i_1,\ldots,i_{2k} \leq n} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-1}}{}^{i_{2k}} \wedge \theta^{[I]}.$$

For $k = 1, *\Lambda_1$ is half the scalar curvature. For n even and $k = n/2, \Lambda_{n/2}$ is the Pfaffian. Those quantities are frame independent. We claim that this is true for all Λ_ks. Indeed, let P be an orthonormal matrix function, $\tilde{e}_i = P_i{}^j\, e_j, \tilde{\theta}^j = (P^{-1})_i{}^j\, \theta^i$. We have

$$\tilde{\omega}_i{}^j = P_i{}^k\, \omega_k{}^l\, (P^{-1})_l{}^j + P_i^k\, \mathrm{d}(P^{-1})_k{}^j \quad \text{ and } \quad \tilde{\Omega}_i{}^j = P_i{}^k\, \Omega_k{}^l\, (P^{-1})_l{}^j.$$

We then have

$$\begin{aligned}
2^k\, k! \tilde{\Lambda}_k &= \sum_{1 \leq i_1,\ldots,i_{2k} \leq n} \tilde{\Omega}_{i_1}{}^{i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2k-1}}{}^{i_{2k}} \wedge \tilde{\theta}^{[I]} \\
&= \sum_{1 \leq i_1,\ldots,i_{2k} \leq n} \left[P_{i_1}{}^{p_1}\, (P^{-1})_{p_2}{}^{i_2} \Omega_{p_1}{}^{p_2}\right] \\
&\quad \wedge \cdots \wedge \left[P_{i_{2k-1}}{}^{p_{2k-1}}\, (P^{-1})_{p_{2k}}{}^{i_{2k}} \Omega_{p_{2k-1}}{}^{p_{2k}}\right] \\
&\quad \wedge * \left(\left[(P^{-1})_{q_1}{}^{i_1}\, \theta^{q_1}\right] \wedge \cdots \wedge \left[(P^{-1})_{q_{2k}}{}^{i_{2k}}\, \theta^{q_{2k}}\right]\right) \\
&= \sum_{1 \leq i_1,\ldots,i_{2k} \leq n} P_{i_1}{}^{p_1}\, (P^{-1})_{q_1}{}^{i_1}\, (P^{-1})_{p_2}{}^{i_2}\, (P^{-1})_{q_2}{}^{i_2} \ldots \\
&\quad \cdots P^{-1}_{i_{2k-1}}{}^{p_{2k-1}}\, (P^{-1})_{q_{2k-1}}{}^{i_{2k-1}}\, (P^{-1})_{p_{2k}}{}^{i_{2k}}\, (P^{-1})_{q_{2k}}{}^{i_{2k}} \\
&\quad \Omega_{p_1}{}^{p_2} \wedge \cdots \wedge \Omega_{p_{2k-1}}{}^{p_{2k}} \wedge *(\theta^{q_1} \wedge \cdots \wedge \theta^{q_{2k}}) \\
&= \sum_{1 \leq p_1,\ldots,p_{2k} \leq n} \Omega_{p_1}{}^{p_2} \wedge \cdots \wedge \Omega_{p_{2k-1}}{}^{p_{2k}} \wedge *(\theta^{p_1} \wedge \cdots \wedge \theta^{p_{2k}}) \\
&= 2^k\, k! \Lambda_k,
\end{aligned}$$

where in the second-to-last identity we have used the orthogonality of P.

When g is conformally flat, $*\Lambda_k$ is proportional to the σ_k-curvature. This was noticed by Viaclovsky [30] in case $k = n/2$. The argument for general k is similar. We include it here for completeness. Recall that the Riemann curvature tensor Riem admits the decomposition

$$\text{Riem} = W_g + A_g \odot g$$

where $\odot$ is the Kulkarni–Nomizu product and W_g is the Weyl tensor of g. When (M, g) is locally conformally flat, $W_g \equiv 0$. Fix a point $p \in M$. The local orthonormal frame $\{e_1, \ldots, e_n\}$ is chosen so that A_g is diagonalized at p with eigenvalue $\lambda_1, \ldots, \lambda_n$; in particular, $A_i{}^j = \lambda_i\, \delta_{ij}$. We have at p that

$$\begin{aligned}\Omega_i{}^j &= \mathrm{Riem}_i{}^j{}_{kl}\, \theta^k \wedge \theta^l \\ &= (A_g \odot g)_i{}^j{}_{kl}\, \theta^k \wedge \theta^l \\ &= (A_{ik}\, \delta_{jl} - A_{jk}\, \delta_{il} + A_{jl}\, \delta_{ik} - A_{il}\, \delta_{jk})\, \theta^k \wedge \theta^l \\ &= 2(\lambda_i + \lambda_j)\, \theta^i \wedge \theta^j.\end{aligned}$$

It follows that, also at p,

$$\begin{aligned}\Lambda_k &= \frac{1}{2^k\, k!} \sum_{I=(i_1,\ldots,i_{2k}) \in S_{n,2k}} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-1}}{}^{i_{2k-2}} \wedge \theta^{[I]} \\ &= \frac{1}{k!} \sum_{I=(i_1,\ldots,i_{2k}) \in S_{n,2k}} (\lambda_{i_1} + \lambda_{i_2}) \cdots (\lambda_{i_{2k-1}} + i_{2k})\, \underbrace{\theta^I \wedge \theta^{[I]}}_{=\mathrm{d}v_g} \\ &= \frac{1}{k!} \sum_{I=(i_1,\ldots,i_{2k}) \in S_{n,2k}} (\lambda_{i_1} + \lambda_{i_2}) \cdots (\lambda_{i_{2k-1}} + i_{2k}) \mathrm{d}v_g \\ &= \frac{2^k (n-k)!}{k!(n-2k)!} \sum_{J=(j_1,\ldots,j_k) \in S_{n,k}} \lambda_{j_1} \cdots \lambda_{j_k}\, \mathrm{d}v_g \\ &= \frac{2^k (n-k)!}{(n-2k)!} \sigma_k(A_g)\, \mathrm{d}v_g,\end{aligned}$$

i.e., Λ_k is proportional to $\sigma_k(A_g)\, \mathrm{d}v_g$.

To finish this section, we derive a decomposition of Λ_k which we will need later. By the second structural equations,

$$\begin{aligned}2^k\, k! \Lambda_k = &\sum_{1 \le i_1,\ldots,i_{2k} \le n} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}} \wedge \mathrm{d}\omega_{i_{2k-1}}{}^{i_{2k}} \wedge \theta^{[I]} \\ &- \sum_{1 \le i_1,\ldots,i_{2k} \le n} \mathrm{d}\Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}} \wedge \omega_{i_{2k-1}}{}^{p_k} \wedge \omega_{p_k}{}^{i_{2k}} \wedge \theta^{[I]} \\ = &\sum_{1 \le i_1,\ldots,i_{2k} \le n} \mathrm{d}\Big(\Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}} \wedge \omega_{i_{2k-1}}{}^{i_{2k}} \wedge \theta^{[I]}\Big) \\ &- \sum_{1 \le i_1,\ldots,i_{2k} \le n} \mathrm{d}\Big(\Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}}\Big) \wedge \omega_{i_{2k-1}}{}^{i_{2k}} \wedge \theta^{[I]} \\ &+ \sum_{1 \le i_1,\ldots,i_{2k} \le n} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}} \wedge \omega_{i_{2k-1}}{}^{i_{2k}} \wedge \mathrm{d}\theta^{[I]} \\ &- \sum_{1 \le i_1,\ldots,i_{2k} \le n} \Omega_{i_1}{}^{i_2} \wedge \cdots \wedge \Omega_{i_{2k-3}}{}^{i_{2k-2}} \wedge \omega_{i_{2k-1}}{}^{p_k} \wedge \omega_{p_k}{}^{i_{2k}} \wedge \theta^{[I]}.\end{aligned}$$

Note that, by symmetry,

$$\begin{aligned}
&\sum_{1\le i_1,\dots,i_{2k}\le n} \mathrm{d}\Omega_{i_1}{}^{i_2}\wedge\Omega_{i_3}{}^{i_4}\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]}\\
&\quad=\cdots=\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-5}}{}^{i_{2k-4}}\wedge\mathrm{d}\Omega_{i_{2k-3}}{}^{i_{2k-2}}\\
&\qquad\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]},
\end{aligned}$$

which, by the second Bianchi identity and anti-symmetry, is equal to

$$\begin{aligned}
&\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-5}}{}^{i_{2k-4}}\wedge\\
&\quad\wedge(-\Omega_{i_{2k-3}}{}^{r_{k-1}}\wedge\omega_{r_{k-1}}{}^{i_{2k-2}}+\omega_{i_{2k-3}}{}^{r_{k-1}}\wedge\Omega_{r_{k-1}}{}^{i_{2k-2}})\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]}\\
&=-2\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-5}}{}^{i_{2k-4}}\wedge\Omega_{i_{2k-3}}{}^{r_{k-1}}\wedge\omega_{r_{k-1}}{}^{i_{2k-2}}\\
&\quad\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]}.
\end{aligned}$$

Next, we compute $\mathrm{d}\theta^{[I]}$. Assume first that $I=(i_1,\dots,i_{2k})\in S_{n,2k}$. We supplement I with $i_{2k+1},\dots,i_n$ so that $(i_1,\dots,i_n)$ is a permutation of $(1,\dots,n)$. We have

$$\theta^{[I]}=\delta^{i_1\dots i_n}_{1\dots n}\theta^{i_{2k+1}}\wedge\cdots\wedge\theta^{i_n}.$$

In view of the first structural equations, this implies that

$$\begin{aligned}
\mathrm{d}\theta^{[I]}&=\delta^{i_1\dots i_n}_{1\dots n}\sum_{s=2k+1}^{n}\sum_{t=1}^{2k}(-1)^{s-1}\theta^{i_{2k+1}}\wedge\cdots\wedge\theta^{i_{s-1}}\wedge(\theta^{i_t}\wedge\omega_{i_t}{}^{i_s})\\
&\quad\wedge\theta^{i_{s+1}}\wedge\cdots\wedge\theta^{i_n}\\
&=\delta^{i_1\dots i_n}_{1\dots n}\sum_{t=1}^{2k}\sum_{s=2k+1}^{n}(-1)^{s}\omega_{i_t}{}^{i_s}\wedge\theta^{i_t}\wedge\theta^{i_{2k+1}}\wedge\cdots\wedge\theta^{i_{s-1}}\\
&\quad\wedge\theta^{i_{s+1}}\wedge\cdots\wedge\theta^{i_n}\\
&=\sum_{t=1}^{2k}\sum_{s=2k+1}^{n}\omega_{i_t}{}^{i_s}\wedge\theta^{[I:i_t\to i_s]}\\
&=\sum_{t=1}^{2k}\sum_{s=1}^{n}\omega_{i_t}{}^{s}\wedge\theta^{[I:i_t\to s]},
\end{aligned}$$

where $I:i_t\to i_s$ denotes $(i_1,\dots,i_{t-1},i_s,i_{t+1},\dots,i_{2k})$. This continues to hold for general multi-index $I=(i_1,\dots,i_{2k})\in\{1,\dots,n\}^k$. Indeed, if $I\notin S_{n,2k}$, we have $\mathrm{d}\theta^{[I]}=0$ and

$$\begin{aligned}
&\sum_{t=1}^{2k}\sum_{s=1}^{n}\omega_{i_t}{}^{s}\wedge\theta^{[I:i_t\to s]}\\
&\quad=\sum_{\text{special } t's}\sum_{s=1}^{n}\omega_{i_t}{}^{s}\wedge\theta^{[I:i_t\to s]}\\
&\quad=\frac{1}{2}\sum_{\text{special } t's}\sum_{s=1}^{n}(\omega_{i_t}{}^{s}\wedge\theta^{[I:i_t\to s]}+\omega_{i_{\tilde t}}{}^{s}\wedge\theta^{[I:i_{\tilde t}\to s]})\\
&\quad=0.
\end{aligned}$$

where the set of special ts are those such that there is a unique $\tilde t\neq t$ such that $i_t=i_{\tilde t}$. It thus follows that

$$\begin{aligned}
&\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge \mathrm{d}\theta^{[I]}\\
&\quad=\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\sum_{t=1}^{2k}\sum_{s=1}^{n}\omega_{i_t}{}^{s}\wedge\theta^{[I:i_t\to s]}\\
&\quad=2(k-1)\\
&\qquad\sum_{1\le i_1,\dots,i_{2k},s\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\omega_{i_{2k-2}}{}^{s}\wedge\theta^{[I:i_{2k-2}\to s]}\\
&\qquad+2\sum_{1\le i_1,\dots,i_{2k},s\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\omega_{i_{2k}}{}^{s}\wedge\theta^{[I:i_{2k}\to s]}\\
&\quad=-2(k-1)\\
&\qquad\sum_{1\le j_1,\dots,j_{2k},i_{2k-2}\le n}\Omega_{j_1}{}^{j_2}\wedge\cdots\wedge\Omega_{j_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-2}}{}^{j_{2k-2}}\wedge\omega_{j_{2k-1}}{}^{j_{2k}}\wedge\theta^{[J]}\\
&\qquad+2\sum_{1\le j_1,\dots,j_{2k},i_{2k}\le n}\Omega_{j_1}{}^{j_2}\wedge\cdots\wedge\Omega_{j_{2k-3}}{}^{j_{2k-2}}\wedge\omega_{j_{2k-1}}{}^{i_{2k}}\wedge\omega_{i_{2k}}{}^{j_{2k}}\wedge\theta^{[J]}.
\end{aligned}$$

We thus get

$$\begin{aligned}
2^k\,k!\Lambda_k=&\sum_{1\le i_1,\dots,i_{2k}\le n}\mathrm{d}\Big(\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]}\Big)\\
&+\sum_{1\le i_1,\dots,i_{2k},p_k\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{p_k}\wedge\omega_{p_k}{}^{i_{2k}}\wedge\theta^{[I]}.
\end{aligned}$$

Thus, if we set

$$\begin{aligned}
\Lambda_k^1&=\frac{1}{2^k\,k!}\sum_{1\le i_1,\dots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]},\\
\Lambda_k^2&=\frac{1}{2^k\,k!}\sum_{1\le i_1,\dots,i_{2k},p_k\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{p_k}\wedge\omega_{p_k}{}^{i_{2k}}\wedge\theta^{[I]},
\end{aligned}$$

then we obtain the following generalization of (1):

$$\Lambda_k=d\Lambda_k^1+\Lambda_k^2.\tag{3}$$

It should be noted that, unlike Λ_k, Λ_k^1 and Λ_k^2 are frame dependent and only defined over parallelizable subsets of M.

3. Higher Order Mass

Let (M^n, g) be a Riemannian manifold and assume that there is a compact set $K \subset M$ such that $M \backslash K$ has an asymptotic flat structure of order τ: there are some $R \geq 1$ and a diffeomorphism $\Phi : M\backslash K \to \mathbb{R}^n \backslash B_R$ such that

$$(\Phi^* g)_{ij}(x) = \delta_{ij} + o_3(|x|^{-\tau}) \quad \text{as } |x| \to \infty, \tag{4}$$

where $x = (x^1, \ldots, x^n)$ is the coordinate function with respect to Φ and we write $f = o_l(|x|^{-\tau})$ if $\partial_{i_1} \ldots \partial_{i_p} f = o(|x|^{-\tau-p})$ for any $1 \leq p \leq l$. We also assume that

$$\Lambda_k \in L^1(M).$$

For simplicity, we only consider the case where M has one end; the general case requires minor modification.

With respect to the asymptotic structure Φ, let $\hat{e}_i = \partial_{x^i}$ and $\hat{\theta}^i = \mathrm{d}x^i$. The connection one-forms and curvature two-forms are defined by

$$\nabla_X \hat{e}_i = \hat{\omega}_i{}^j(X)\, \hat{e}_j \quad \text{and} \quad \hat{\Omega}_i{}^j(X, Y) = \hat{\theta}^j(R(X,Y)\hat{e}_i).$$

Define the $(n-1)$-form

$$\hat{\Lambda}_k^1(\Phi) = \frac{1}{2^k\, k!} \sum_{1 \leq i_1, \ldots, i_{2k} \leq n} \hat{\Omega}_{i_1}{}^{i_2} \wedge \cdots \wedge \hat{\Omega}_{i_{2k-3}}{}^{i_{2k-2}} \wedge \hat{\omega}_{i_{2k-1}}{}^{i_{2k}} \wedge \hat{\theta}^{[I]}$$

and the associate "kth order mass"

$$m_k(\Phi) = \lim_{R \to \infty} \int_{S_R} (-1)^k\, \hat{\Lambda}_k^1(\Phi).$$

Here, S_R denotes the coordinate sphere of radius R centered at the origin. The factor of $(-1)^k$ is to ensure the positivity of mass in favorable situations.

To show that $m_k(\Phi)$ is well-defined, we use the decomposition of Λ_k which we derived earlier (in an orthonormal frame). We first use the Gram–Schmidt orthogonalization to construct an orthonormal frame:

$$\begin{aligned} e_1 &= \frac{\hat{e}_1}{|\hat{e}_1|}, \\ e_2 &= \frac{\hat{e}_2 - g(\hat{e}_2, e_1)\, e_1}{|\hat{e}_2 - g(\hat{e}_2, e_1)\, e_1|}, \\ &\cdots \\ e_n &= \frac{\hat{e}_n - g(\hat{e}_n, e_1)\, e_1 - \cdots - g(\hat{e}_n, e_{n-1}) e_{n-1}}{|\hat{e}_n - g(\hat{e}_n, e_1)\, e_1 - \cdots - g(\hat{e}_n, e_{n-1}) e_{n-1}|}. \end{aligned}$$

Evidently,

$$e_i = \hat{e}_i + o_3(|x|^{-\tau}).$$

Let $\{\theta^i\}$ be the dual frame to $\{e_i\}$ and define the connection one-forms $\omega_i{}^j$ and curvature two-forms $\Omega_i{}^j$ accordingly. Set

$$\Lambda_k^1 = \frac{1}{2^k\,k!}\sum_{1\le i_1,\ldots,i_{2k}\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{i_{2k}}\wedge\theta^{[I]},$$

$$\Lambda_k^2 = \frac{1}{2^k\,k!}\sum_{1\le i_1,\ldots,i_{2k},p_k\le n}\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\omega_{i_{2k-1}}{}^{p_k}\wedge\omega_{p_k}{}^{i_{2k}}\wedge\theta^{[I]}.$$

Then, by (3), $\Lambda_k = \mathrm{d}\Lambda_k^1 + \Lambda_k^2$.

To relate $\omega_i{}^j$ and $\Omega_i{}^j$ to $\hat\omega_i{}^j$ and $\hat\Omega_i{}^j$, we write

$$e_i = P_i{}^j\,\hat e_j \quad\text{ and }\quad \theta^j = (P^t)_i{}^j\,\hat\theta^i$$

where the matrix P satisfies

$$P_i{}^j = \delta_{ij} + o_1(|x|^{-\tau}).$$

(In fact, $P_i{}^j = \delta_{ij} + o_3(|x|^{-\tau})$, but the above suffices.)

We have

$$\begin{aligned}\nabla_X e_i &= \nabla_X(P_i{}^j\,\hat e_j) = (dP_i{}^j(X) + P_i{}^k\,\hat\omega_k{}^j(X))\hat e_j\\ &= (dP_i{}^j(X) + P_i{}^k\,\hat\omega_k{}^j(X))(P^{-1})_j{}^l\,e_l,\end{aligned}$$

which implies that

$$\omega_i{}^j = \hat\omega_i{}^j + dP_i{}^j + o(|x|^{-2\tau-1}).$$

Likewise

$$\begin{aligned}\Omega_i{}^j &= \theta_j(R(X,Y)e_i) = (P^t)_j{}^k P_i{}^l\,\hat\theta_k(R(X,Y)\hat e_l)\\ &= (P^t)_j{}^k P_i{}^l\,\hat\Omega_k{}^l = \hat\Omega_i{}^j + o_1(|x|^{-2\tau-2}).\end{aligned}$$

From the above computation, we see that

$$\Lambda_k^2 = O(|x|^{-(\tau(k+1)+2k)}) \in L^1(M) \text{ provided } \tau > \frac{n-2k}{k+1}, \tag{5}$$

and

$$\begin{aligned}&\Lambda_k^1 - \hat\Lambda_k^1\\ &\quad= \mathrm{d}\Big(\frac{1}{2^k\,k!}\sum_{1\le i_1,\ldots,i_{2k}\le n}[P_{i_{2k-1}}{}^{i_{2k}} - \delta_{i_{2k-1}i_{2k}}]\hat\Omega_{i_1}{}^{i_2}\wedge\cdots\wedge\hat\Omega_{i_{2k-3}}{}^{i_{2k-2}}\wedge\hat\theta^{[I]}\Big)\\ &\qquad+o(|x|^{-(\tau(k+1)+2k)+1}).\end{aligned}\tag{6}$$

From (6), we obtain for $R \gg N \gg R_0$ that

$$\int_{S_R}\hat\Lambda_k^1 = \int_{S_R}\Lambda_k^1 + o(R^{n-(\tau(k+1)+2k)}), \tag{7}$$

which, in view of (3), leads to

$$\int_{S_R}\hat\Lambda_k^1 = \int_{S_N}\Lambda_k^1 + \int_{B_R\setminus B_N}\Big[\Lambda_k + \Lambda_k^2\Big] + o(R^{n-(\tau(k+1)+2k)}).$$

Recalling (5), we conclude that the mass $m_k(\Phi)$ is well-defined when

$$\tau > \frac{n-2k}{k+1}.$$

In fact, the argument above shows that if D_j is an exhaustion of M by closed sets such that

$$R_j = \inf\{|x| : x \in \partial D_j\} \to \infty \quad \text{and } R_j^{-(n-1)} |\partial D_j| \\ \text{remains bounded as } j \to \infty$$

then

$$m_k(\Phi) = \lim_{j\to\infty} \int_{S_j} (-1)^k \, \hat{\Lambda}_k^1.$$

We would like to show next that $m_k(\Phi)$ is independent of the asymptotic structure Φ. Assume that $\tilde{\Phi}$ is another asymptotic structure of (M, g) and let $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n)$ denote the coordinate function with respect to $\tilde{\Phi}$. To show that $m_k(\Phi) = m_k(\tilde{\Phi})$, we appeal to a theorem of Bartnik [2, Theorem 3.1] to find harmonic coordinates $y = (y^1, \ldots, y^n)$ and $\tilde{y} = (\tilde{y}^1, \ldots, \tilde{y}^n)$ such that

$$\begin{aligned}
&|x^i - y^i| + |\tilde{x}^i - \tilde{y}^i| = o(|x|^{1-\tau}) = o(|\tilde{x}|^{1-\tau}),\\
&|g(\partial_{x^i}, \partial_{x^j}) - g(\partial_{y^i}, \partial_{y^j})| + |g(\partial_{\tilde{x}^i}, \partial_{\tilde{x}^j}) - g(\partial_{\tilde{y}^i}, \partial_{\tilde{y}^j})| = o(|x|^{-\tau}) = o(|\tilde{x}|^{-\tau}),\\
&y^i = A_j{}^i \, \tilde{y}^j + c^i,
\end{aligned}$$

where $A_j{}^i$ and c^i are constants. Note that the second relation and that the metric g is asymptotically flat implies that the matrix $A = (A_j{}^i)$ is orthonormal. We would like to apply the proof of (7) to show that

$$\text{in defining } m_k(\Phi) \\ \text{we can use the coordinate functions } y^i\text{s instead of the } x^i\text{s.} \tag{8}$$

Assuming the correctness of this statement, we proceed as follows. By (8), $m_k(\Phi)$ and $m_k(\tilde{\Phi})$ can be computed using y_is and $\tilde{y}^i$s, respectively. On the other hand, the frames ∂_{y^i} and $\partial_{\tilde{y}^i}$ differ from one another by a rigid rotation: $\partial_{y_i} = A_i{}^j \, \partial_{\tilde{y}^j}$. The argument proving that Λ_k is frame independent applies showing that $\hat{\Lambda}_k^1(\partial_{y^i}) = \hat{\Lambda}_k^1(\partial_{\tilde{y}^i})$. This proves that $m_k = m_k(\Phi)$ is independent of Φ.

We now prove (8). To apply the proof of (7), it is enough to establish

$$\partial_{x_i} = \partial_{y_i} + o_1(|x|^{-\tau}). \tag{9}$$

Recall that y_i is harmonic, i.e., $\Delta_g y_i = 0$. It follows that

$$\Delta_g(x_i - y_i) = o_2(r^{-1-\tau}).$$

Also, $x_i - y_i = o(r^{1-\tau})$. Applying standard elliptic estimates, we obtain $x_i - y_i = o_3(r^{1-\tau})$, which implies (9). (In fact, we have $\partial_{x_i} = \partial_{y_i} + o_2(|x|^{-\tau})$.)

We have thus shown that

Theorem 3.1. *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold which is asymptotically flat of order* τ*, i.e., there is a compact set* K *and a diffeomorphism* $\Phi : M\backslash K \to \mathbb{R}^n\backslash B_{R_0}$ *for some* $R_0 > 0$ *such that in such coordinate system the metric* g *satisfies* $g_{ij} = \delta_{ij} + o_3(|x|^{-\tau})$*. Assume further that the curvature* Λ_k *belongs to* $L^1(M)$*. Fix* $1 \leq k < \frac{n}{2}$*. If* $\tau > \frac{n-2k}{k+1}$*, then the mass*

$$m_k = \lim_{R\to\infty} \int_{S_R} (-1)^k \, \hat{\Lambda}_k^1$$

is well-defined and is independent of the asymptotic structure at infinity.

To finish this section, we give an example. Fix some $1 \leq k < \frac{n}{2}$. Consider an asymptotically flat manifold where the metric takes the following form at infinity

$$g_{ij} = \exp\left(\frac{2m}{r^{\frac{n-2k}{k}}}\right) \delta_{ij} + o_2(r^{-\frac{n-2k}{k}}).$$

The connection one-forms and the curvature two-forms are

$$\begin{aligned}
\omega_i{}^j &= \Gamma_{ik}^j \, \mathrm{d}x^k = -\frac{(n-2k)}{k} \frac{m}{r^{\frac{n}{k}}} (\delta_{ij}\, r \, \mathrm{d}r + x^i \, \mathrm{d}x^j - x^j \, \mathrm{d}x^i) + o(r^{1-\frac{n}{k}}), \\
\Omega_i{}^j &= \mathrm{d}\omega_i{}^j - \omega_i{}^t \wedge \omega_t{}^j \\
&= \frac{(n-2k)n}{k^2} \frac{m}{r^{\frac{n+k}{k}}} \, \mathrm{d}r \wedge (x^i \, \mathrm{d}x^j - x^j \, \mathrm{d}x^i) \\
&\quad - \frac{2(n-2k)}{k} \frac{m}{r^{\frac{n}{k}}} \, \mathrm{d}x^i \wedge \mathrm{d}x^j + o(r^{-\frac{n}{k}}).
\end{aligned}$$

Thus,

$$\Lambda_k^1 \rfloor S_R = c(n,k) \, \frac{(-1)^k \, m^k}{R^n} \sum_{1 \leq i,j \leq n} (x^i \mathrm{d}x^j - x^j \mathrm{d}x^i) \wedge *(\mathrm{d}x^i \wedge \mathrm{d}x^j)$$

and so

$$m_k = c(n,k) \, m^k.$$

where $c(n,k)$ is some positive constant.

The above computation also shows that, for the time symmetric slice of the Schwarzschild spacetime in higher dimensions, only the first mass (which is the same as the ADM mass) is nonzero. All the higher order masses, if well-defined, vanish.

4. On the Non-Negativity of the *k*th Mass

It is of interest to see if the kth mass is non-negative under some assumption on either the Λ_k or the σ_k curvature. We are only able to do so under a very restrictive hypothesis that (M, g) is locally conformally flat and that $\lambda(A_g)$ is asymptotically on the boundary of the Γ_k cone. Of the two assumptions, we believe the local conformal flatness assumption is more severe. The second hypothesis should be compared to the ADM case where, in Schoen and Yau's [24,25] proof of the positive mass theorem, one can assume without loss of generality that the manifold is asymptotically scalar flat (i.e., $\lambda(A_g) \in \partial\Gamma_1$).

Theorem 4.1. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 4$ and $2 \leq k < \frac{n}{2}$. Assume that (M, g) is asymptotically flat of order $\tau > \frac{n-2k}{k+1}$. If, near a given end, g is locally conformally flat, A_g belongs to the $\bar{\Gamma}_k$ cone and the Λ_k curvature vanishes, then the kth mass of that end is non-negative. Furthermore, if the kth mass is zero, then, near that end, (M, g) is isometric to a Euclidean end.*

The proof of this theorem has a different flavor from what is presented in this paper and will be published elsewhere.

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