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A Generalized Mixed Topology on Orlicz Spaces

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ABSTRACT. Let L^φ be an Orlicz space defined by an arbitrary Orlicz function φ over a positive measure space (Ω, Σ, μ) and provided with its usual F-norm $\|\cdot\|_\varphi$. In L^φ a natural convergence can be defined as follows: a sequence (x_n) in L^φ is said to be γ_φ-convergent to $x \in L^\varphi$ whenever $x_n \to x$ $(\mu - \Omega)$ and $\sup \|x_n\|_\varphi < \infty$. In this paper we examine some kind of generalized inductive-limit topology (in the sense of Turpin) $\mathcal{J}_I^\varphi$ in L^φ that generates our γ_φ-convergence in L^φ. The main aim of the paper is to obtain a description of the topology $\mathcal{J}_I^\varphi$ in terms of some family of F-norms defined by other Orlicz functions. As an application we obtain a topological characterization of the γ_φ -convergence in L^φ.

1. INTRODUCTION AND PRELIMINARIES

Every Orlicz space L^φ defined by an Orlicz function φ (not necessarily convex) over a measure space (Ω, Σ, μ) can be equipped with two

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F-norms: $||\cdot||_\varphi$- the usual F-norm on L^φ and $||\cdot||_\mu$- the F-norm of convergence in measure (on Ω) restricted to L^φ. Thus a natural sequential convergence in L^φ can be defined as follows: a sequence (x_n) in L^φ is said to be γ_φ-convergent to $x \in L^\varphi$, in symbols $x_n \stackrel{\gamma_\varphi}{\to} x$, whenever

$$x_n \to x(\mu - \Omega) \ \text{(i.e., } ||x_n - x||_\mu \to 0 \text{) and } \sup_n ||x_n||_\varphi < \infty.$$

When we replace in the above definition the condition: $\sup ||x_n||_\varphi < \infty$ with the boundedness of the set $\{x_n : n \geq 0\}$ for the topology $\mathcal{J}_{||\cdot||_\varphi}$, then this new convergence comes under the definition of the so-called two-norm convergence or γ-convergence in the sense of Alexiewicz ([1,1954]). The general theory of two-norm convergence has been extensively developed by A.Alexiewicz [1], W. Orlicz [19], A. Alexiewicz and Z. Semadeni [2], A. Wiweger [23], [24], [25].

It is well known that the theory of two-norm convergence is closely related to the Wiweger's theory of mixed topologies [23], [24]. Indeed, in case when $||\cdot||$ is a homogenuous norm and $||\cdot||^*$ is an F-norm on a linear space X and $||x_n - x||^* \to 0$ implies $\liminf ||x_n|| \geq ||x||$, then the sequential γ-convergence in X is generated by the so-called mixed topology $\gamma[\mathcal{J}_{||\cdot||}, \mathcal{J}_{||\cdot||^*}]$.

The notion of the mixed topology was a starting point for the theory of generalized inductive-limit topologies. There are many kinds of such topologies introduced for different reasons by A. Persson [21], D.J.H. Garling [7], J.B. Cooper [3], P. Turpin [22] and others.

The question arises whether our γ_φ-convergence in L^φ is topologized by some linear topology. It turns out that there is a positive answer to this question when we take into account an appropriate generalized inductive-limit topology in the sense of Turpin. This topology will be called here a generalized mixed topology and denoted by $\mathcal{J}_I^\varphi$. This term is justified by the fact that $\mathcal{J}_I^\varphi$ coincides with the usual mixed topology $\gamma[\mathcal{J}_{||\cdot||_\varphi}, \mathcal{J}_{\mu|_{L^\varphi}}]$ (in the sense of Wiweger) when the space $(L^\varphi, \mathcal{J}_{||\cdot||_\varphi})$ is locally bounded.

In this paper we investigate the generalized mixed topology $\mathcal{J}_I^\varphi$. Our main aim is to obtain a description of $\mathcal{J}_I^\varphi$ in terms of some family of F-norms defined by other Orlicz functions. As application we obtain a topological characterization of our γ_φ-convergence in L^φ. Moreover, for φ being a convex Orlicz function we establish the general form of $\mathcal{J}_I^\varphi$-continuous linear functionals on L^φ.

In some special cases the topology $\mathcal{J}_I^\varphi$ was examined by P. Turpin [22] and the author [14], [15], [16].

Given a linear topological space (X,ξ) by $Bd(X,\xi)$ we will denote the collection of all ξ-bounded subsets of X. As usual $\mathcal{N}$ stands for the set of all natural numbers. We assume that $0 \cdot \infty = 0$.

Now we recall some notation and terminology concerning Orlicz spaces (see [9], [11], [12], [22] for more details).

By an Orlicz function we mean a function $\varphi : [0,\infty) \to [0,\infty]$ which is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$, and not identically equal to 0.

An Orlicz function φ is called convex whenever $\varphi(\alpha u + \beta v) \leq \alpha\varphi(u) + \beta\varphi(v)$ for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $u, v \geq 0$. A convex Orlicz function is usually called a Young function.

For a Young function φ we denote by φ^* the function complementary to φ in the sense of Young, i.e.,

$$\varphi^*(v) = \sup\{uv - \varphi(u) : \ u \geq 0\} \text{ for } v \geq 0.$$

For a set Ψ of Young functions we will write: $\Psi^* = \{\psi^* : \ \psi \in \Psi\}$.

Let φ and ψ be a pair of Orlicz functions vanishing only at zero (resp. taking only finite values). We say that φ increases essentially more rapidly than ψ for small u (resp. for large u) in symbols $\psi \overset{s}{\prec\!\!\prec} \varphi$ (resp. $\psi \overset{l}{\prec\!\!\prec} \varphi$) whenever for any $c > 0$, $\psi(cu)/\varphi(u) \to 0$ as $u \to 0$ (resp. $u \to \infty$).

We will write $\psi \overset{a}{\prec\!\!\prec} \varphi$ when $\psi \overset{s}{\prec\!\!\prec} \varphi$ and $\psi \overset{l}{\prec\!\!\prec} \varphi$ hold.

For φ and ψ being Young functions the condition $\psi \overset{s}{\prec\!\!\prec} \varphi$ (resp. $\psi \overset{l}{\prec\!\!\prec} \varphi$) implies $\varphi^* \overset{s}{\prec\!\!\prec} \psi^*$ (resp. $\varphi^* \overset{l}{\prec\!\!\prec} \psi^*$) (see [9, Lemma 13.1]).

Let (Ω, Σ, μ) be a positive measure space, and let L^0 denote the set of equivalence classes of all real valued measurable functions defined and finite a.e. on Ω. For a subset A of Ω and $x \in L^0$ we will write $x_A = x \cdot \chi_A$, where χ_A stands for the characteristic function of A.

An Orlicz function φ determines a functional $m_\varphi : L^0 \to [0,\infty]$ by

$$m_\varphi(x) = \int_\Omega \varphi(|x(t)|)d\mu.$$

The Orlicz space generated by φ is the ideal of L^0 defined by

$$L^\varphi = \{x \in L^0 : \ m_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The functional m_φ restricted to L^φ is an orthogonally additive semi-modular.

L^φ can be equipped with the complete metrizable topology $\mathcal{J}_\varphi$ of the F-norm

$$||x||_\varphi = inf\{\lambda > 0 : \ m_\varphi(x/\lambda) \leq \lambda\}.$$

Moreover, if φ is a Young function, then the topology $\mathcal{J}_\varphi$ can be generated by the Luxemburg norm

$$|||x|||_\varphi = \ inf\{\lambda > 0 : \ m_\varphi(x/\lambda) \leq 1\}.$$

For $r > 0$ let

$$B_\varphi(r) = \{x \in L^\varphi : \ ||x||_\varphi \leq r\}$$

and let

$$B_{(\varphi)}(r) = \{x \in L^\varphi : \ |||x|||_\varphi \leq r\}$$

whenever φ is a Young function.

We shall need the following lemma.

Lemma 1.1. *Let φ_1, φ_2 be Orlicz functions, and let $\varphi(u) = \varphi_1(u) \vee \varphi_2(u)$ for $u \geq 0$. Then φ is an Orlicz function and the following statements hold:*

(i) $L^{\varphi} = L^{\varphi_1} \cap L^{\varphi_2}$.

(ii) $\|x\|_{\varphi_1} \vee \|x\|_{\varphi_2} \le \|x\|_{\varphi} \le \|x\|_{\varphi_1} + \|x\|_{\varphi_2}$ *for* $x \in L^{\varphi}$.

(iii) $\mathcal{J}_{\varphi} = \mathcal{J}_{\varphi_1|L^{\varphi}} \vee \mathcal{J}_{\varphi_2|L^{\varphi}}$

and $Bd(L^{\varphi}, \mathcal{J}_{\varphi}) = Bd(L^{\varphi}, \mathcal{J}_{\varphi_1|L^{\varphi}}) \cap Bd(L^{\varphi}, \mathcal{J}_{\varphi_2|L^{\varphi}})$.

Proof. (i) See [8, Theorem 1].

(ii) It follows from the definition of $\|\cdot\|_{\varphi}$.

(iii) It follows from (ii).

Let

$$E^{\varphi} = \{x \in L^0 : \ m_{\varphi}(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

It is known that $L^{\varphi} = E^{\varphi}$ whenever φ satisfies the Δ_2- condition, i.e., lim sup $\varphi(2u)/\varphi(u) < \infty$ as $u \to 0$ and $u \to \infty$.

Let

$$\varphi_0(u) = \begin{cases} 0 & \text{for} \quad 0 \le u \le 1, \\ 1 & \text{for} \quad u > 1. \end{cases}$$

It is known that L^{φ_0} is the largest Orlicz space and consists of all those $x \in L^0$ that are bounded outside of some set of finite measure, and

$$\|x\|_{\varphi_0} = \inf\{\lambda > 0 : \ \mu(\{t \in \Omega : \ |x(t)| > \lambda\}) \le \lambda\}.$$

It is seen that $\|x_n - x\|_{\varphi_0} \to 0$ in L^{φ_0} iff $x_n \to x$ in measure on Ω (in symbols $x_n \to x$ $(\mu - \Omega)$). Therefore we will write $\|\cdot\|_{\mu}$ instead of $\|\cdot\|_{\varphi_0}$, and by $\mathcal{J}_{\mu}$ we will denote the topology of the F-norm $\|\cdot\|_{\varphi_0}$.

For $\varepsilon > 0$ let

$$B_{\mu}(\varepsilon) = \{x \in L^{\varphi_0} : \ \|x\|_{\mu} \le \varepsilon\}.$$

We shall need the following lemma.

Lemma 1.2. *Let φ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. Then for $r > 0$, $B_{\varphi}(r) \in Bd(L^{\varphi}, \mathcal{J}_{\mu|L^{\varphi}})$.*

Proof. Let $x_n \in B_\varphi(r)$ $(n = 1,2,\ldots)$ and let $\lambda_n \to 0$. For $\varepsilon > 0$ let $\Omega_n(\varepsilon) = \{t \in \Omega : |\lambda_n x_n(t)| > \epsilon\}$. Then

$$\mu(\Omega_n(\varepsilon))\varphi\left(\frac{\varepsilon}{r|\lambda_n|}\right) \leq \int_{\Omega_n(\varepsilon)} \varphi\left(\frac{|x_n(t)|}{r}\right) d\mu$$
$$\leq m_\varphi\left(\frac{x_n}{r}\right) \leq r.$$

Since $\varphi(u) \to \infty$ as $u \to \infty$ we get $\mu(\Omega_n(\varepsilon)) \to 0$, and this means that $||\lambda_n x_n||_\mu \to 0$.

2. A GENERALIZED MIXED TOPOLOGY ON L^φ - GENERAL PROPERTIES

In this section we consider some kind of generalized inductive limit topology on L^φ.

Let φ be an arbitrary Orlicz function, and let

$$F_n = B_\varphi(2^n) \text{ and } \mathcal{J}_n = \mathcal{J}_{\mu|F_n} \text{ for } n \geq 0.$$

Then the family $\mathcal{B}_\varphi = \{F_n : n \geq 0\}$ forms a base of metric bounded sets in $(L^\varphi, ||\cdot||_\varphi)$.

Moreover, the sequence $(F_n, \mathcal{J}_n)$ $(n \geq 0)$ of balanced topological spaces satisfies the following conditions:

(i) $L^\varphi = \bigcup_{n\geq 0} F_n$.

(ii) $F_n + F_n \subset F_{n+1}$, and the function

$$F_n \times F_n \ni (x,y) \to x + y \in F_{n+1}$$

is continuous.

(iii) The function $[-1,1] \times F_n \ni (\lambda, x) \mapsto \lambda \cdot x \in F_n$ is continuous.

(iv) $\mathcal{J}_{n+1|F_n} = \mathcal{J}_n$ for $n \geq 0$.

Thus the space L^φ with the system $\{(F_n, \mathcal{J}_n) : n \geq o\}$ comes under the conditions of the strict inductive limit of balanced topological spaces in the sense of P. Turpin [22, Ch. I].

Definition 2.1. *The family of all sets of the form*

$$\bigcup_{N=0}^{\infty} \Big(\sum_{n=0}^{N} (B_\varphi(2^n) \cap B_\mu(\varepsilon_n))\Big) \tag{2.1}$$

where $(\varepsilon_n : n \geq 0)$ is a sequence of positive numbers, forms a base of neighbourhoods of zero for a linear topology on L^φ (in the sense of Turpin) which will be denoted by $\mathcal{J}_I^\varphi$.

According to [22, Theorem 1.1.6] $\mathcal{J}_I^\varphi$ in the finest of all linear topologies ξ on L^φ which satisfy the conditions:

$$\xi|_{F_n} \subset \mathcal{J}_{\mu|F_n} \text{ for } n \geq 0. \tag{2.2}$$

Moreover, in view of [22, Theorem 1.1.8] we have

$$\mathcal{J}^\varphi_{I|F_n} = \mathcal{J}_{\mu|F_n} \text{ for } n \geq 0. \tag{2.3}$$

Since $\mathcal{J}_{\mu|_{L^\varphi}} \subset \mathcal{J}_\varphi$ we have $\mathcal{J}_I^\varphi \subset \mathcal{J}_\varphi$; hence $\mathcal{J}_{\mu|_{L^\varphi}} \subset \mathcal{J}_I^\varphi \subset \mathcal{J}_\varphi$.

Henceforth in this section we assume that $\varphi(u) \to \infty$ as $u \to \infty$.

The basic properties of the topology $\mathcal{J}_I^\varphi$ are included in the following theorems.

Theorem 2.1. *The space $(L^\varphi, \mathcal{J}_I^\varphi)$ is complete.*

Proof. It is known that the balls $B_\varphi(2^n)$ are closed subsets of $(L^{\varphi_0}, \mathcal{J}_\mu)$ (see [22, 0.3.6]), so the spaces $(B_\varphi(2^n), \mathcal{J}_{\mu|B_\varphi(2^n)})$ $(n \geq 0)$ are complete. Hence, by [22, Theorem 1.1.10] the space $(L^\varphi, \mathcal{J}_I^\varphi)$ is complete.

Theorem 2.2. *For a subset $Z \subset L^\varphi$ the following statements are equivalent:*

(i) $\sup\{||x||_\varphi : x \in Z\} < \infty$.

(ii) *Z is bounded for* $\mathcal{J}_I^\varphi$.

Proof. By Lemma 1.2 the balls $B_\varphi(2^n)$ are bounded subsets of $(L^\varphi, \mathcal{J}_{\mu|_{L^\varphi}})$. Moreover, the balls $B_\varphi(2^n)$ are also closed in $(L^\varphi, \mathcal{J}_{\mu|_{L^\varphi}})$ (see [22, 0.3.6]). In view of (2.2) and (2.3) $\mathcal{J}_I^\varphi$ is the finest of all linear topologies ξ on L^φ such that $\xi|_{F_n} = \mathcal{J}_{\mu|_{F_n}}$. Hence by [22, Corollary 1.1.12] the equivalence $(i) \Leftrightarrow (ii)$ holds.

Theorem 2.3. *For a subset* $Z \subset L^\varphi$ *the following statements are equivalent:*

(i) *Z is relatively compact for* $\mathcal{J}_I^\varphi$.

(ii) *Z is relatively compact for* $\mathcal{J}_{\mu|_{L^\varphi}}$ *and*

$$\sup\{||x||_\varphi : x \in Z\} < \infty .$$

Proof. It follows from Theorem 2.2 and (2.3).

Let us recall that a sequence (x_n) in L^φ is said to be γ_φ-convergent to $x \in L^\varphi$, in symbols $x_n \xrightarrow{\gamma_\varphi} x$, whenever

$$x_n \to x \ (\mu - \Omega) \qquad \text{and} \qquad \sup_n ||x_n||_\varphi < \infty.$$

Theorem 2.4. *For a sequence* (x_n) *in* L^φ *the following statements are equivalent:*

(i) $x_n \to 0$ *for* $\mathcal{J}_I^\varphi$.

(ii) $x_n \xrightarrow{\gamma_\varphi} 0$.

Moreover, $\mathcal{J}_I^\varphi$ *is the finest of all linear topologies* ξ *on* L^φ *which satisfy the condition:*

$$x_n \xrightarrow{\gamma_\varphi} 0 \qquad \textit{implies} \qquad x_n \to 0 \ \textit{for} \ \xi. \tag{+}$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from Theorem 2.2 and (2.3).

Now let ξ be a linear topology on L^φ for which the condition (+) holds. Then $\xi|_{B_\varphi(r)} \subset \mathcal{J}_{\mu|_{B_\varphi(r)}}$ for $r > 0$, because $\mathcal{J}_\mu$ is a metrizable linear topology. Hence by (2.2) we get that $\xi \subset \mathcal{J}_I^\varphi$.

Definition 2.2. *Let (X, η) be a linear topological space. A linear mapping $A: L^\varphi \to X$ is said to be γ_φ-linear, if*

$$x_n \xrightarrow{\gamma_\varphi} 0 \qquad \textit{implies} \qquad A(x_n) \to 0 \textit{ for } \eta.$$

The next theorem gives a characterization of γ_φ-linear functionals on L^φ.

Theorem 2.5. *For a linear topological space (X, η) and a linear mapping $A: L^\varphi \to X$ the following statements are equivalent:*

(i) A is $(\mathcal{J}_I^\varphi, \eta)$-continuous.

(ii) A is γ_φ-linear.

(iii) For every $r > 0$, the restriction $A_{|B_\varphi(r)}$ is $(\mathcal{J}_{\mu|B_\varphi(r)}, \eta)$-continuous.

Proof. $(i) \Rightarrow (ii)$ It follows from Theorem 2.4.

$(ii) \Rightarrow (iii)$ it is obvious.

$(iii) \Rightarrow (i)$ Let W be a neighbourhood of zero in X for η. Then there exists a sequence $(W_n : n \geq 0)$ of neighbourhoods of zero for η such that $\sum_{n=0}^{N} W_n \subset W$ for every $N \geq 0$. Thus by our assumption there exists a sequence $(\varepsilon_n : n \geq 0)$ of positive numbers such that $A(B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \subset W_n$. Thus for $N \geq 0$

$$A\Big(\sum_{n=0}^{N}(B_\varphi(2^n) \cap B_\mu(\varepsilon_n))\Big) \subset \sum_{n=0}^{N} W_n \subset W,$$

so

$$A\Big(\bigcup_{N=0}^{\infty}\big(\sum_{n=0}^{N}(B_{\varphi}(2^n)\cap B_{\mu}(\varepsilon_n))\big)\Big)\subset$$

$$\subset\bigcup_{N=0}^{\infty}A\Big(\sum_{n=0}^{N}(B_{\varphi}(2^n)\cap B_{\mu}(\varepsilon_n))\Big)\subset W.$$

This means that A is $(\mathcal{J}_I^{\varphi},\eta)$-continuous.

Now we are going to compare the topology $\mathcal{J}_I^{\varphi}$ with the mixed topology $\gamma[\mathcal{J}_{\varphi},\mathcal{J}_{\mu|L^{\varphi}}]$ in the sense of Wiweger (see [24]). For this purpose we shall need the following

Theorem 2.6. *Assume that (Ω,Σ,μ) is an atomless measure space or that μ is the counting measure on $\mathcal{N}$. If $(L^{\varphi},\mathcal{J}_{\varphi})$ is a locally bounded space then for a subset Z of L^{φ} the following statements are equivalent:*

(i) Z is bounded for $\mathcal{J}_I^{\varphi}$.

(ii) $\sup\{\|x\|_{\varphi}:\ x\in Z\}<\infty$.

(iii) Z is bounded for $\mathcal{J}_{\varphi}$.

Proof. $(i)\Leftrightarrow(ii)$ See Theorem 2.2.

$(ii)\Rightarrow(iii)$ In view of [22, 0.3.10.2] $\sup\{\|x\|_{\varphi}:\ x\in Z\}<\infty$ if and only if Z is additively bounded (see [22, 0.3.10.1]), so arguing as in the proof of [15, Lemma 2.5] we obtain that Z is bounded for $\mathcal{J}_{\varphi}$.

$(iii)\Rightarrow(i)$ Obvious.

Theorem 2.7. *Assume that (Ω,Σ,μ) is an atomless measure space or that μ is the counting measure on $\mathcal{N}$. If $(L^{\varphi},\mathcal{J}_{\varphi})$ is a locally bounded space, then the generalized mixed topology $\mathcal{J}_I^{\varphi}$ coincides with the mixed topology $\gamma[\mathcal{J}_{\varphi},\mathcal{J}_{\mu|L^{\varphi}}]$.*

Proof. In view of Theorem 2.6 it follows from [24, 2.2.1 and 2.2.2].

3. SOME PROJECTIVE TOPOLOGY ON ORLICZ SPACES

In [5], [6] H.W. Davis, F.J.Muray and J. Weber studied the spaces

$$L^P(S) = \bigcap_{p\in S} L^p \ (S \subset [1,\infty))$$

endowed with the appropriate projective topology.

There are some results concerning a representation of an Orlicz space L^φ as the intersection of some family of other Orlicz spaces (see [10], [17], [18]). In this section we examine the appropriate projective topology on L^φ. In section 4 we shall show that this projective topology coincides with the generalized mixed topology $\mathcal{J}_I^\varphi$.

We start with some equalities among Orlicz spaces, proved in [17] and [18], which are of key importance in this section. At the very beginning we distinguish some classes of Orlicz functions.

An Orlicz function φ continuous for all $u \geq 0$, taking only finite values, vanishing only at zero, and not bounded is usually called a φ-function. By Φ we will denote the collection of all φ-functions.

A Young function φ vanishing only at zero and taking only finite values is called an N-function whenever $\varphi(u)/u \to 0$ as $u \to 0$ and $\varphi(u)/u \to \infty$ as $u \to \infty$. By Φ_N we will denote the collection of all N-functions.

Let Φ_1 be the set of all Orlicz functions φ vanishing only at zero and such that $\varphi(u) \to \infty$ as $u \to \infty$. Denote by

$$\Phi_{11} = \{\varphi \in \Phi_1 : \ \varphi(u) < \infty \text{ for } u > 0\},$$

$$\Phi_{12} = \{\varphi \in \Phi_1 : \ \varphi \text{ jumps to } \infty\}.$$

Then $\Phi_1 = \Phi_{11} \cup \Phi_{12}$. In view of [17, Theorem 3.1, 3.2, 3.7 and 3.8] we get

Theorem 3.1. *Let $\varphi \in \Phi_{1i}$ $(i = 1,2)$. Then the following equalities hold:*

$$L^\varphi = \bigcap\{L^\psi : \ \psi \in \Psi_{1i}^\varphi\} = \bigcap\{E^\psi : \ \psi \in \Psi_{1i}^\varphi\}$$

where

$$\Psi_{11}^{\varphi} = \{\psi \in \Phi :\ \psi \overset{a}{\prec\!\!\prec} \varphi\},\ \Psi_{12}^{\varphi} = \{\psi \in \Phi :\ \psi \overset{s}{\prec\!\!\prec} \varphi\}.$$

Moreover, if μ is an atomless measure or the counting measure on $\mathcal{N}$, then the strict inclusion $L^{\varphi} \subsetneq E^{\psi}$ holds for each $\psi \in \Psi_{1i}^{\varphi}$.

Next let Φ_1^c be the set of all Young functions φ vanishing only at zero and such that $\varphi(u)/u \to \infty$ as $u \to \infty$.

Denote by

$$\Phi_{11}^c = \{\varphi \in \Phi_1^c :\ \varphi(u) < \infty \text{ for } u > 0 \text{ and } \varphi(u)/u \to 0 \text{ as } u \to 0\},$$

$$\Phi_{12}^c = \{\varphi \in \Phi_1^c :\ \varphi \text{ jumps to } \infty \text{ and } \varphi(u)/u \to 0 \text{ as } u \to 0\},$$

$$\Phi_{13}^c = \{\varphi \in \Phi_1^c :\ \varphi(u) < \infty \text{ for } u > 0 \text{ and } \varphi(u)/u \to a \text{ as } u \to 0,\ a > 0\},$$

$$\Phi_{14}^c = \{\varphi \in \Phi_1^c :\ \varphi \text{ jumps to } \infty \text{ and } \varphi(u)/u \to a \text{ as } u \to 0,\ a > 0\}.$$

Then $\Phi_1^c = \bigcup_{i=1}^{4} \Phi_{1i}^c$, where the sets Φ_{1i}^c are pairwise disjoint. It is seen that $\Phi_{11}^c = \Phi_N$. According to [18, Theorems 2.1-2.4] we get

Theorem 3.2. *Let $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$. Then the following equalities hold:*

$$L^{\varphi} = \bigcap\{L^{\psi} :\ \psi \in \Psi_{1i}^{\varphi}(N)\} = \bigcap\{E^{\psi} :\ \psi \in \Psi_{1i}^{\varphi}(N)\}$$

where

$$\Psi_{11}^{\varphi}(N) = \{\psi \in \Phi_N :\ \psi \overset{a}{\prec\!\!\prec} \varphi\},\ \Psi_{12}^{\varphi}(N) = \{\psi \in \Phi_N :\ \psi \overset{s}{\prec\!\!\prec} \varphi\},$$

$$\Psi_{13}^{\varphi}(N) = \{\psi \in \Phi_N :\ \psi \overset{l}{\prec\!\!\prec} \varphi\},\ \Psi_{14}^{\varphi}(N) = \Phi_N.$$

Next, let Φ_2^c be the set of all Young functions φ taking only finite values and such that $\varphi(u)/u \to 0$ as $u \to 0$.

Denote by

$$\Phi_{21}^c = \{\varphi \in \Phi_2^c :\ \varphi(u) > 0 \text{ for } u > 0 \text{ and } \varphi(u)/u \to \infty \text{ as} u \to \infty\}\ ,$$

$\Phi_{22}^c = \{\varphi \in \Phi_2^c :\ \varphi(u) > 0 \text{ for } u > 0 \text{ and } \varphi(u)/u \to a \text{ as } u \to \infty,\ a > 0\}$,

$\Phi_{23}^c = \{\varphi \in \Phi_2^c :\ \varphi(u) = 0 \text{ near zero and } \varphi(u)/u \to \infty \text{ as } u \to \infty\}$,

$\Phi_{24}^c = \{\varphi \in \Phi_2^c :\ \varphi(u) = 0 \text{ near zero and } \varphi(u)/u \to a \text{ as } u \to \infty,\ a > 0\}$.

Then $\Phi_2^c = \bigcup_{i=1}^{4} \Phi_{2i}^c$, where the sets Φ_{2i}^c are pairwise disjoint. It is seen that $\Phi_{21}^c = \Phi_N$. According to [18, Theorems 1.1-1.4] we have

Theorem 3.3. *Let $\varphi \in \Phi_{2i}^c$ $(i = 1,2,3,4)$. Then the following equalities hold*

$$E^\varphi = \bigcup\{E^\psi :\ \psi \in \Psi_{2i}^\varphi(N)\} = \bigcup\{L^\psi :\ \psi \in \Psi_{2i}^\varphi(N)\}$$

where

$\Psi_{21}^\varphi(N) = \{\psi \in \Phi_N :\ \varphi \overset{a}{\prec\!\!\prec} \psi\}$, $\Psi_{22}^\varphi(N) = \{\psi \in \Psi_N :\ \varphi \overset{s}{\prec\!\!\prec} \psi\}$,
$\Psi_{23}^\varphi(N) = \{\psi \in \Phi_N :\ \varphi \overset{l}{\prec\!\!\prec} \psi\}$, $\Psi_{24}^\varphi(N) = \Phi_N$.

At last, in view of [18, Lemma 3.1 and Theorem 3.3] we get

Theorem 3.4 *Let φ_1 and φ_2 be a pair of complementary Young functions. Then $\varphi_1 \in \Phi_{1i}^c$ if and only if $\varphi_2 \in \Phi_{2i}^c$ $(i = 1,2,3,4)$, and moreover, the sets $\Psi_{1i}^\varphi(N)$ and $\Psi_{2i}^\varphi(N)$ are mutually related in such a way that*

$$\left(\Psi_{1i}^{\varphi_1}(N)\right)^* = \Psi_{2i}^{\varphi_2}(N) \text{ and } \left(\Psi_{2i}^{\varphi_2}(N)\right)^* = \Psi_{1i}^{\varphi_1}(N).$$

We shall need the following

Corollary 3.5. *Let $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$. Then*

$$E^{\varphi^*} = \bigcup\{L^{\psi^*} :\ \psi \in \Psi_{1i}^\varphi(N)\}.$$

Proof. Since $\varphi \in \Phi_{2i}^c$ and $(\Psi_{1i}^{\varphi}(N))^* = \Psi_{2i}^{\varphi^*}(N)$ (see Theorem 3.4) by Theorem 3.3 we get

$$\begin{aligned}\bigcup\{L^{\psi^*} : \psi \in \Psi_{1i}^{\varphi}(N)\} &= \bigcup\{L^{\psi} : \psi \in (\Psi_{1i}^{\varphi}(N))^*\} \\ &= \bigcup\{L^{\psi} : \psi \in \Psi_{2i}^{\varphi^*}(N)\} = E^{\varphi^*}.\end{aligned}$$

We are now ready to define our projective topology on L^{φ}.

Definition 3.1. *Let $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. By $\mathcal{J}_P^{\varphi}$ we will denote the projective topology on L^{φ} with respect to the family $\{(E^{\psi}, \mathcal{J}_{\psi|E^{\psi}}) : \psi \in \Psi_{1i}^{\varphi}\}$, i.e., $\mathcal{J}_P^{\varphi}$ is defined to be the coarsest of all linear topologies ξ on L^{φ} for which $\mathcal{J}_{\psi|L^{\varphi}} \subset \xi$ holds for every $\psi \in \Psi_{1i}^{\varphi}$. Thus*

$$\mathcal{J}_P^{\varphi} = \sup \{\mathcal{J}_{\psi|L^{\varphi}} : \psi \in \Psi_{1i}^{\varphi}\}.$$

For φ being a φ-function the topology $\mathcal{J}_P^{\varphi}$ has been examined in [14], [15], [16]. It is easy to verify that all properties of $\mathcal{J}_P^{\varphi}$ which are obtained in [14], [15], [16] for φ being a φ-function remain valid for $\varphi \in \Phi_{11}$. In this section we extend results from [14], [15], [16] to the case of φ belonging to Φ_1.

From the definition of $\mathcal{J}_P^{\varphi}$ we have

Theorem 3.6. *Let $\varphi \in \Phi_1$. Then $\mathcal{J}_{\mu|L^{\varphi}} \subset \mathcal{J}_P^{\varphi} \subset \mathcal{J}_{\varphi}$.*

Theorem 3.7. *Let $\varphi \in \Phi_1$ and let μ be an infinite atomless measure. Then there exists a sequence (x_n) in L^{φ} such that $x_n \to 0$ for $\mathcal{J}_P^{\varphi}$ and $m_{\varphi}(x_n) = 1$ for $n \in \mathcal{N}$. Hence the strict inclusion $\mathcal{J}_P \subsetneq \mathcal{J}_{\varphi}$ holds.*

Proof. For $\varphi \in \Phi_{11}$ this fact is proved in [13, Theorem 2.5]. Now let $\varphi \in \Phi_{12}$, i.e., $\varphi(u) < \infty$ for $u \le a$ and $\varphi(u) = \infty$ for $u > a$. Let (u_n) be a sequence of positive numbers such that $u_n \downarrow 0$ and $u_1 < a$. Let (Ω_n) be a sequence of measurable subsets of Ω such that $\mu(\Omega_n) = 1/\varphi(u_n)$. Define

$$x_n(t) = \begin{cases} u_n & \text{for} \quad t \in \Omega_n \,, \\ 0 & \text{for} \quad t \notin \Omega_n \,. \end{cases}$$

We shall show that $x_n \to 0$ for $\mathcal{J}_P^{\varphi}$, i.e., $\|x_n\|_{\psi} \to 0$ for each $\psi \in \Psi_{12}^{\varphi}$. Indeed, let $\psi \overset{s}{\prec\!\!\prec} \varphi$ and let $\varepsilon > 0$ be given. Then there exists $u_0 > 0$ such that $\psi(u/\varepsilon) \leq \varepsilon\varphi(u)$ for $u \leq u_0$. Let $n_0 \in \mathcal{N}$ be such that $u_n \leq u_0$ for $n \geq n_0$. Then for $n \geq n_0$ we have $m_{\psi}(x_n/\varepsilon) = \psi(u_n/\varepsilon)/\varphi(u_n) \leq \varepsilon$ i.e., $\|x_n\|_{\psi} \leq \varepsilon$. On the other hand, $m_{\varphi}(x_n) = \varphi(u_n)/\varphi(u_n) = 1$.

Arguing as in the proof of [13, Theorem 1.2] we get

Theorem 3.8. *Let $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. Then the topology $\mathcal{J}_P^{\varphi}$ has a base of neighbourhoods of zero consisting of all sets of the form:*

$$B_{\psi}(r) \cap L^{\varphi}$$

where $\psi \in \Psi_{1i}^{\varphi}$ and $r > 0$.

Repeating the arguments of the proof of [13, Theorem 5.1] and using the equalities from Theorem 3.1 we get

Theorem 3.9. *Let $\phi \in \Phi_1$. Then the space $(L^{\varphi}, \mathcal{J}_P^{\varphi})$ is complete.*

Since the space $(L^{\varphi}, \mathcal{J}_{\varphi})$ is complete, from Theorems 3.6 and 3.7, in view of the Open Mapping Theorem it follows

Theorem 3.10. *Let $\varphi \in \Phi_1$ and let μ be an infinite atomless measure. Then te space $(L^{\varphi}, \mathcal{J}_P^{\varphi})$ is not metrizable.*

To the end of this section we will assume that $\varphi \in \Phi_1^c$. We start with the following lemma.

Lemma 3.11. *Let $\varphi \in \Phi_{1i}^c$ $(i = 1, 2, 3, 4)$ and let ψ be a φ-function such that $\psi \overset{a}{\prec\!\!\prec} \varphi$ if $i = 1$ (resp. $\psi \overset{s}{\prec\!\!\prec} \varphi$ if $i = 2$, $\psi \overset{a}{\prec\!\!\prec} \varphi$ if $i = 3$, $\psi \overset{s}{\prec\!\!\prec} \varphi$ if $i = 4$). Then there exists an N-function ψ_0 such that $\psi(u) \leq \psi_0(2u)$ for $u \geq 0$ and $\psi_0 \overset{a}{\prec\!\!\prec} \varphi$ if $i = 1$ (resp. $\psi_0 \overset{s}{\prec\!\!\prec} \varphi$ if $i = 2$, $\psi_0 \overset{l}{\prec\!\!\prec} \varphi$ if $i = 3$).*

Proof. Let ψ_1 be an arbitrary N-function such that $\psi_1 \overset{a}{\prec\!\!\prec} \varphi$ if $i = 1$ (resp. $\psi_1 \overset{s}{\prec\!\!\prec} \varphi$ if $i = 2$, $\psi_1 \overset{l}{\prec\!\!\prec} \varphi$ if $i = 3$, $\psi_1 \overset{s}{\prec\!\!\prec} \varphi$ if $i = 4$). Let $\psi_2 = \psi \vee \psi_1$. Next, let us put $p(0) = 0$ and $p(s) = \sup_{0<t\leq s} (\psi_2(t)/t)$ for $s > 0$. Let

$$\psi_0(u) = \int_0^u p(s)\, ds \text{ for } u \geq 0.$$

It is seen that ψ_0 is an N-function. Arguing as in he proof of [13, Lemma 1.4] we can verify that ψ_0 satisfies the desired properties.

Theorem 3.12. *Let $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$. Then the topology $\mathcal{J}_P^\varphi$ is generated by the family of B-norms $\{|||\cdot|||_{\psi|L^\varphi} : \psi \in \Psi_{1i}^\varphi(N)\}$.*

Proof. For example, let $\varphi \in \Phi_{13}^c$. Then $\varphi \in \Phi_{11}$. Given $\psi \in \Psi_{11}^\varphi$ and $r > 0$, in view of Lemma 3.11 there exists $\psi_0 \in \Psi_{13}^\varphi(N)$ and such that $\psi(u) \leq \psi_0(2u)$ for $u \geq 0$. Hence

$$||x||_\psi \leq ||2x||_{\psi_0} \text{ for all } x \in L^{\psi_0}. \tag{1}$$

On the other hand, since the F-norms $||\cdot||_{\psi_0}$ and $|||\cdot|||_{\psi_0}$ are equivalent on L^{ψ_0}, there exists $r_1 > 0$ such that

$$B_{(\psi_0)}(r_1) \subset B_{\psi_0}(r). \tag{2}$$

We shall show that $B_{(\psi_0)}(r_1/2) \cap L^\varphi \subset B_\psi(r)$. Indeed, let $x \in B_{(\psi_0)}(r_1/2) \cap L^\varphi$. Then $|||2x|||_{\psi_0} \leq r_1$, hence by (2) we get $||2x||_{\psi_0} \leq r$ and next, by (1) we see that $||x||_\psi \leq r$.

For i=1,2,4 the proof is similar.

Now we are ready to establish the general form of $\mathcal{J}_P^\varphi$-continuous linear functionals on L^φ.

Theorem 3.13. *Let $\varphi \in \Phi_1^c$ and let μ be a σ-finite measure. Then for a linear functional f on L^φ the following statements are equivalent:*

(i) f is continuous for $\mathcal{J}_P^\varphi$.

(ii) There exists a unique $y \in E^{\varphi^*}$ such that

$$f(x) = f_y(x) = \int_\Omega x(t)y(t)\,d\mu \qquad \textit{for } x \in L^\varphi.$$

Proof. $(i) \Rightarrow (ii)$. Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. In view of Theorem 3.12 there exist $\psi \in \Psi^\varphi_{1i}(N)$ and $r > 0$ such that f is bounded on $B_{(\psi)}(r) \cap L^\varphi$. This means that f is continuous on the linear subspace $(L^\varphi, \mathcal{J}_{\psi|L^\varphi})$ of the normed space $(E^\psi, \mathcal{J}_{\psi|E^\psi})$. Hence, by the Hahn-Banach theorem there exists a $\mathcal{J}_{\psi|E^\psi}$-continuous linear functional $\bar{f}$ on E^ψ such that $\bar{f}(x) = f(x)$ for $x \in L^\varphi$. According to [11, p. 56] there exists $y \in L^{\psi^*} \subset E^{\varphi^*}$ such that

$$\bar{f}(x) = \int_\Omega x(t)y(t)\,d\mu \qquad \text{for } x \in E^\psi.$$

Hence

$$f(x) = f_y(x) = \int_\Omega x(t)y(t)\,d\mu \qquad \text{for } x \in L^\varphi. \tag{1}$$

Now assume that there exists another $y' \in E^{\varphi^*}$ such that

$$f(x) = f_{y'}(x) = \int_\Omega x(t)y'(t)\,d\mu \text{ for } x \in L^\varphi. \tag{2}$$

Then, for example, there exists a measurable set $A \subset \{t \in \Omega : y'(t) > y(t)\}$ such that $0 < \mu(A) < \infty$. Hence by (1) and (2) we get

$$\int_\Omega \chi_A(t)\,(y'(t) - y(t))\,d\mu = \int_A (y'(t) - y(t))\,d\mu = 0.$$

This contradiction establishes that there exists a unique $y \in E^{\varphi^*}$ such that (1) holds.

$(ii) \Rightarrow (i)$ Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. According to Corollary 3.5 there exists $\psi \in \Psi^\varphi_{1i}(N)$ such that $y \in L^{\psi^*}$. Then $L^\varphi \subset E^\psi \subset L^\psi$ and using the Holder's inequality we get that $|f_y(x)| \leq ||y||^0_{\psi^*} |||x|||_\psi$ for

$x \in L^\varphi$ (here $\|\cdot\|^0_{\psi^*}$ denotes the Orlicz norm on L^{ψ^*}). This means that f_y is $\mathcal{J}_{\psi|L^\varphi}$-continuous, so f_y is $\mathcal{J}^\varphi_P$-continuous, because $\mathcal{J}_{\psi|L^\varphi} \subset \mathcal{J}^\varphi_P$.

4. THE IDENTITY OF THE TOPOLOGIES $\mathcal{J}^\varphi_I$ AND $\mathcal{J}^\varphi_P$ ON L^φ

In this section we shall prove that the topologies $\mathcal{J}^\varphi_I$ and $\mathcal{J}^\varphi_P$ coincide on L^φ. We start with the following lemma.

Lemma 4.1. *Let $\varphi \in \Phi_{12}$ and ψ be a φ-function. Then the following statements hold:*

(i) For every $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup\{\|x_A\|_\psi : x \in B_\varphi(r)\} < \varepsilon \qquad \text{for } A \in \Sigma,\ \mu(A) < \delta.$$

(ii) If $\psi \overset{s}{\prec\!\prec} \varphi$, then for every $r > 0$

$$\mathcal{J}_{\mu|B_\varphi(r)} = \mathcal{J}_{\psi|B_\varphi(r)}.$$

Proof. (i) Assume that $\varphi(u) < \infty$ for $0 \le u \le a$ and $\varphi(u) = \infty$ for $u > a$, where $a > 0$. Given $x \in B_\varphi(r)$ we have $\int_\Omega \varphi(|x(t)|/r)\, d\mu \le r$, so $|x(t)|/r \le a$ a.e. on Ω. Given $\varepsilon > 0$ let $\delta = \varepsilon/\psi(ar/\varepsilon)$. Then for $A \in \Sigma$ with $\mu(A) < \delta$

$$\int_\Omega \psi(|x_A(t)|/\varepsilon)\, d\mu \le \psi(ar/\varepsilon)\mu(A) \le \varepsilon$$

i.e., $\|x_A\|_\psi \le \varepsilon$.

(ii) Since the inclusion $\mathcal{J}_{\mu|L^\varphi} \subset \mathcal{J}_{\psi|L^\varphi}$ holds it is enough to show that $\mathcal{J}_{\psi|B_\varphi(r)} \subset \mathcal{J}_{\mu|B_\varphi(r)}$ holds for every $r > 0$. To this end we shall show that for any $x \in B_\varphi(r)$ and $\xi > 0$ there exists $\eta_0 > 0$ such that

$$B_\mu(x, \eta_0) \cap B_\varphi(r) \subset B_\psi(x, \xi)$$

where for $\eta > 0$

$$B_\mu(x,\eta) = \{y \in L^{\varphi_0} : \ ||y-x||_\mu \le \eta\}$$
$$= \{y \in L^{\varphi_0} : \ \mu(\{t \in \Omega : \ |y(t)-x(t)| > \eta\}) \le \eta\}$$

and

$$B_\psi(x,\eta) = \{y \in L^\psi : \ ||y-x||_\psi \le \eta\}.$$

Indeed, let $x \in B_\varphi(r)$ and $\xi > 0$ be given. For $\eta > 0$ and $y \in B_\varphi(r)$ let put

$$E(\eta,y) = \{t \in \Omega : \ |y(t)-x(t)| > \eta\}, \ G(\eta,y) = \Omega \setminus E(\eta,y).$$

It is seen that

$$m_\varphi((y-x)/2r) \le 2r. \tag{1}$$

Since $\psi \overset{s}{\prec\!\prec} \varphi$, there exists $u_0 > 0$ such that

$$\psi(\frac{2u}{\xi}) \le \frac{\xi}{4r}\varphi(\frac{u}{2r}) \qquad \text{for } 0 \le u \le u_0. \tag{2}$$

Moreover, in view of (i) there exists $\delta > 0$ such that

$$||(y-x)_A||_\psi \le \frac{1}{2}\xi \qquad \text{for } A \in \Sigma \text{ with } \mu(A) < \delta. \tag{3}$$

Now let $\eta_0 = min(u_0,\delta)$ and let $y \in B_\mu(x,\eta_0) \cap B_\varphi(r)$.
Then $\mu(E(\eta_0,y)) \le \eta_0 \le \delta$, and hence from (3) we get

$$||(y-x)_{E(\eta_0,y)}||_\psi \le \frac{1}{2}\xi. \tag{4}$$

On the other hand, since $\eta_0 \le u_0$, from (2) and (1) we get

$$m_\psi\Big(\frac{2}{\xi}(y-x)_{G(\eta_0,y)}\Big) = \int_{G(\eta_0,y)} \psi\Big(\frac{2|y(t)-x(t)|}{\xi}\Big)\, d\mu$$
$$\le \frac{\xi}{4r}\int_\Omega \varphi\Big(\frac{|y(t)-x(t)|}{2r}\Big)\, d\mu = \frac{\xi}{2}.$$

Hence

$$||(y-x)_{G(\eta_0,y)}||_\psi \le \frac{\xi}{2}. \tag{5}$$

Thus from (1), (4) and (5) we get

$$||y-x||_\psi \le ||(y-x)_{E(\eta_0,y)}||_\psi + ||(y-x)_{G(\eta_0,y)}||_\psi \le \xi$$

and this means that $y \in B_\psi(x,\xi)$.

As an application of Lemma 4.1 we get

Corollary 4.2. *Let $\varphi \in \Phi_1$. Then for every $r > 0$*

$$\mathcal{J}^\varphi_{P|B_\varphi(r)} = \mathcal{J}_{\mu|B_\varphi(r)}.$$

Proof. This equality is proved in [14, Theorem 1.4] for φ being a φ-function, but the proof can be applied for $\varphi \in \Phi_{11}$. For $\varphi \in \Phi_{12}$ our equality follows Lemma 4.1, because

$$\mathcal{J}^\varphi_{P|B_\varphi(r)} = \sup\{\mathcal{J}_{\psi|B_\varphi(r)} :\ \psi \in \Psi^\varphi_{12}\} = \mathcal{J}_{\mu|B_\varphi(r)}.$$

In view of Corollary 4.2 and (2.2) we have: $\mathcal{J}^\varphi_P \subset \mathcal{J}^\varphi_I$. Repeating the arguments of the proof of [14, Theorem 2.2] we get

Theorem 4.3. *Let $\varphi \in \Phi_1$. If a sequence (x_n) in L^φ is modular convergent to $x \in L^\varphi$ (i.e., $m_\varphi(\lambda(x_n - x)) \to 0$ for some $\lambda > 0$) then $x_n \to 0$ for $\mathcal{J}^\varphi_I$*

It is well known that the set of all simple integrable functions $\Im$ is dense in L^φ in the sense of modular convergene. Therefore, in view of the previous theorem and the inclusion $\mathcal{J}_P^\varphi \subset \mathcal{J}_I^\varphi$we get:

Theorem 4.4. *Let $\varphi \in \Phi_1$. The set of all simple integrable functions $\Im$ is dense in L^φ with respect to $\mathcal{J}_P^\varphi$ and $\mathcal{J}_I^\varphi$.*

Now we are in position to prove our main theorem.

Theorem 4.5. *Let $\varphi \in \Phi_1$. Then the equality*

$$\mathcal{J}_I^\varphi = \mathcal{J}_P^\varphi$$

holds, i.e., for $\varphi \in \Phi_{1i}$ $(i = 1, 2)$ the generalized mixed topology $\mathcal{J}_I^\varphi$ is generated by the family $\{\|\cdot\|_{\psi|L^\varphi} : \psi \in \Psi_{1i}^\varphi\}$.

Proof. For $\varphi \in \Phi_{11}$ this equality is proved in [14, Theorem 2.4].

Next let $\varphi \in \Phi_{12}$. It is enough to show that the inclusion $\mathcal{J}_I^\varphi \subset \mathcal{J}_P^\varphi$ holds. Since the spaces $(L^\varphi, \mathcal{J}_P^\varphi)$ and $(L^\varphi, \mathcal{J}_I^\varphi)$ are complete (see Theorems 2.1 and 3.9), in view of Theorem 4.4 and [4, Corollary of Lemma 4, p. 34] if suffices to show that

$$\mathcal{J}_{I|\Im}^\varphi \subset \mathcal{J}_{P|\Im}^\varphi.$$

To his end, in view of Definition 2.1, given a sequence of positive numbers $(\varepsilon_n : n \geq 0)$ we shall find $\psi_0 \in \Psi_{12}^\varphi$ (i.e., $\psi_0 \stackrel{s}{\prec\!\!\prec} \varphi$) and $r_0 > 0$ such that

$$B_{\psi_0}(r_0) \cap \Im \subset \bigcup_{N=0}^{\infty} \Big(\sum_{n=0}^{N} (B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \Big). \tag{1}$$

Without loss of generality we can assume that $\varepsilon_n \downarrow 0$, $\varepsilon_0 < 1$ and $\varepsilon_0 \varphi(1) < 1$. Moreover, for the reasons of convenience we can assume that $\varphi(u) < \infty$ for $u \leq 1$ and $\varphi(u) = \infty$ for $u > 1$.

Let us choose subsequence (ε_{k_n}) of (ε_n) in such a way that:

(a) $k_0 = 0$.

(b) k_1 is the smallest natural number such that

$$\varphi(\varepsilon_{k_1}) < \frac{1}{\varepsilon_{k_1}}.$$

(c) Given k_n $(n \geq 1)$ we take $k_{n+1} > k_n$ such that

$$\varphi(\varepsilon_{k_n})/2 > \varphi(\varepsilon_{k_{n+1}}) \quad \text{and} \qquad \varepsilon_{k_n} < \varepsilon_{n+1}/2.$$

Let

$$N(t) = \sup\{n \in \mathcal{N} : \ \varepsilon_{k_n} \geq t\} \text{ for } t \in \varepsilon_{k_1}.$$

We shall now define a φ-function ψ_0 such that $\psi_0 \overset{s}{\prec\!\!\prec} \varphi$ and

$$\psi_0(u) \geq \frac{1}{N(u)}\varphi\left(\frac{u}{N(u)}\right) \ \text{ for } 0 \leq u \leq \varepsilon_{k_2} \ , \tag{2}$$

$$\psi_0(n) \geq \frac{1}{\varepsilon_{n+1}} \qquad \text{for } n \geq 1 \ . \tag{3}$$

Let us denote by:

$A'_n = \{t > 0 : \ \varepsilon_{k_{n+1}} < t \leq \varepsilon_{k_n}\}$, $n = 1, 2, \ldots$,
$A = \{t > 0 : \ \varepsilon_{k_1} < t < 1\}$,
$A''_n = \{t > 0 : \ n \leq t < n+1\}$, $n = 1, 2, \ldots$,

and

$B'_n = \{s > 0 : \ \varphi(\varepsilon_{k_{n+1}}) < s \leq \varphi(\varepsilon_{k_n})\}$, $n = 1, 2, \ldots$,
$B = \{s > 0 : \ \varphi(\varepsilon_{k_1}) < s < \varphi(1)/2\}$,
$B''_n = \{s > 0 : \ \varphi(1)\, n/2 \leq s < \varphi(1)(n+1)/2\}$, $n = 1, 2, \ldots$.

Let us put

$$p(t) = \begin{cases} 0 & \text{for} \quad t = 0\ , \\ \frac{2}{n-1} & \text{for} \quad t \in A'_n\ ,\ n \geq 2\ , \\ 2 & \text{for} \quad t \in A'_1 \cup A \cup A''_1\ , \\ n & \text{for} \quad t \in A''_n\ ,\ n \geq 2\ , \end{cases}$$

and

$$q(s) = \begin{cases} 0 & \text{for} \quad s = 0\ , \\ \frac{2}{n-1} & \text{for} \quad s \in B'_n\ ,\ n \geq 3 \\ 1 & \text{for} \quad s \in B'_2 \cup B'_1 \cup B\ , \\ \frac{2}{\varphi(1)\varepsilon_{n+1}} & \text{for} \quad s \in B''_n\ ,\ n \geq 1\ . \end{cases}$$

Next, define for $u \geq 0$ and $v \geq 0$

$$\xi(u) = \int_0^u p(t)\, dt \quad \text{and} \quad \eta(v) = \int_0^v q(s)\, ds.$$

Let us put

$$\varphi_0(u) = \begin{cases} \varphi(u) & \text{for} \quad u \leq 1 \\ \varphi(1)u & \text{for} \quad u > 1 \end{cases}$$

At last let

$$\psi_0(u) = \eta(\varphi_0(\xi(u))) = \int_0^{\varphi_0(\xi(u))} q(s)\, ds \quad \text{for } u \geq 0\ .$$

Similarly as in the proof of [14, Theorem 2.4] *we can show that* $\psi_0 \overset{s}{\prec\!\prec} \varphi$ and that the conditions (2) and (3) hold.

Now let us put

$$r_0 = min(\frac{1}{2}, d_0\ \varepsilon_0\ \varphi(\varepsilon_0)) \tag{4}$$

Where

$$\frac{1}{d_0} = \sup\left\{\frac{\varphi(u)}{\psi_0(u)}:\ \varepsilon_{k_2} < u \le 1\right\}.$$

We shall now show that the inclusion (1) holds.
Indeed, let

$$x = \sum_{i \in I} \lambda_i \chi_{H_i} \in B_{\psi_0}(r_0)$$

where I is a finite subset of $\mathcal{N}$, and $\mu(H_i) < \infty$. Denote by

$$K = \{i \in I:\ \varepsilon_{k_2} < |\lambda_i| \le 1\}\ ,$$

$$L = \{i \in I:\ |\lambda_i| \le \varepsilon_{k_2}\},\ J = \{i \in I:\ |\lambda_i| > 1\}.$$

Let

$$x_1 = \sum_{i \in K} \lambda_i \chi_{H_i}\ ,\ x_2 = \sum_{i \in L} \lambda_i \chi_{H_i}\ ,\ x_3 = \sum_{i \in J} \lambda_i \chi_{H_i}\ .$$

Since $x \in B_{\psi_0}(r_0)$ and $r_0 < 1$ we have

$$m_{\psi_0}(x) = \sum_{i \in I} \psi_0(|\lambda_i|)\mu(H_i) = c \le r_0.$$

Write

$$c_i = \psi_0(|\lambda_i|)\mu(H_i)\ .$$

Arguing as in the proof of [14, Theorem 2.4] we get

$$x_1 \in B_\varphi(1) \cap B_\mu(\varepsilon_0) \tag{5}$$

and moreover, using (2) we can find $N_1 \in \mathcal{N}$ such that

$$x_2 \in \sum_{n=1}^{N_1} \left(B_\varphi(\frac{1}{2}\ 2^n) \cap B_\mu(\frac{1}{2}\ \varepsilon_n)\right). \tag{6}$$

Now we shall find $N_2 \in \mathcal{N}$ such that

$$x_3 \in \sum_{n=1}^{N_2} \Big(B_\varphi(\frac{1}{2}\, 2^n) \cap B_\mu(\frac{1}{2}\, \varepsilon_n) \Big). \tag{7}$$

Let

$$n_i = \sup\{n \in \mathcal{N} : \ n \leq |\lambda_i|\} \text{ for } i \in J.$$

Then $n_i \geq 1$ and $n_i \leq |\lambda_i| < n_i + 1$. Let $j_1, \ldots, J_{m_0}$ be the different numbers in the set $\{n_i : \ i \in J\}$ and let us assume that $j_1 < \ldots < j_{m_0}$. Write

$$J_l = \{i \in J : \ n_i = j_l\} \qquad \text{for } 1 \leq l \leq m_0 \ .$$

Then

$$x_3 = \sum_{i \in J} \lambda_i \chi_{H_i} = \sum_{l=1}^{m_0} \big(\sum_{i \in J_l} \lambda_i \chi_{H_i} \big).$$

and let

$$y_l = \sum_{i \in J_l} \lambda_i \chi_{H_i} \text{ for } 1 \leq l \leq m_0.$$

Then $j_l \leq |\lambda_i| < j_l + 1$ for $i \in J_l$ and using (4) we get

$$m_\varphi(y_l/(j_l+1)) = \sum_{i \in J_l} \varphi(|\lambda_i|/(j_l+1))\mu(H_i)$$

$$\leq d_0^{-1} \sum_{i \in J_l} \psi_0(|\lambda_i|/(j_{l+1}))\mu(H_i) \leq d_0^{-1} r_0 < j_l + 1.$$

Thus

$$y_l \in B_\varphi(j_l+1) \subset B_\varphi(\frac{1}{2}\, 2^{j_l+1}). \tag{8}$$

Let

$$E_{y_l}(\varepsilon) = \{t \in \Omega : \ |y_l(t)| > \varepsilon\} \text{ for any } \varepsilon > 0.$$

Then

$$E_{y_l}(\frac{1}{2}\, \varepsilon_{j_l+1}) = \bigcup_{i \in J_l} H_i.$$

Hence, using (3) we get

$$\mu\big(E_{y_l}(\frac{1}{2}\,\varepsilon_{j_l+1})\big) \le \sum_{i\in J_l}\mu(H_i) \le \sum_{i\in J_l} c_i/\psi_0(|\lambda_i|)$$

$$\le \sum_{i\in J_l} c_i/\psi_0(j_l) \le \big(\sum_{i\in J_l} c_i\big)/\psi_0(j_l) \le \frac{1}{2}\,\varepsilon_{j_l+1}$$

and this means that

$$y_l \in B_\mu\big(\frac{1}{2}\,\varepsilon_{j_l+1}\big). \tag{9}$$

Thus from (8) and (9) we have

$$y_l \in B_\varphi\big(\frac{1}{2}\,2^{j_l+1}\big) \cap B_\mu\big(\frac{1}{2}\,\varepsilon_{j_l+1}\big).$$

Hence for $N_2 = j_{m_0} + 1$ we obtain

$$x_3 \in \sum_{l=1}^{m_0}\big(B_\varphi\big(\frac{1}{2}2^{j_l+1}\big) \cap B_\mu\big(\frac{1}{2}\varepsilon_{j_l+1}\big)\big) \subset \sum_{n=1}^{N_2}\big(B_\varphi\big(\frac{1}{2}2^n\big) \cap B_\mu\big(\frac{1}{2}\varepsilon_n\big)\big).$$

At last, using (5), (6) and (7), for $N_0 = max(N_1, N_2)$ we get

$$x = x_1 + x_2 + x_3 \in B_\varphi(1) \cap B_\mu(\varepsilon_0) + \sum_{n=1}^{N_0}\big(B_\varphi(2^n) \cap B_\mu(\varepsilon_n)\big)$$

$$\subset \bigcup_{N=0}^{\infty}\Big(\sum_{n=0}^{N}\big(B_\varphi(2^n) \cap B_\mu(\varepsilon_n)\big)\Big).$$

Thus the proof is completed.

5. A TOPOLOGICAL CHARACTERIZATION OF THE γ_φ-CONVERGENCE IN L^φ

In this section by applying of Theorem 4.5 we obtain a topological characterization of the γ_φ-convergence in L^φ.

Theorem 5.1. *Let $\varphi \in \Phi_{1i}$ $(i = 1,2)$. Then for a sequence (x_n) in L^φ and $x \in L^\varphi$ the following statements are equivalent:*

(i) $x_n \to x$ *for* $\mathcal{J}_I^\varphi$.

(ii) $\|x_n - x\|_\psi \to 0$ *for every* $\psi \in \Psi_{1i}^\varphi$.

(iii) $x_n \to x$ $(\mu - \Omega)$ *and* $\sup\limits_n \|x_n\|_\varphi < \infty$.

Moreover, for $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$ the above statements are equivalent to

(iv) $|||x_n - x|||_\psi \to 0$ *for every* $\psi \in \Psi_{1i}^\varphi(N)$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. It follows from Theorem 2.4. and Theorem 4.5.

$(i) \Leftrightarrow (iv)$. It follows from Theorem 4.5 and Theorem 3.12.

Now we apply the above theorem to some classes of Orlicz spaces. Let

$$\chi_p(u) = u^p \text{ for } u \geq 0 \text{ and } \chi_\infty(u) = \begin{cases} 0 & \text{for } \ 0 \leq u \leq 1, \\ \infty & \text{for } \ u > 1. \end{cases}$$

Let $\|\cdot\|_p$ and $\|\cdot\|_\infty$ denote the usual norms in L^p $(p \geq 1)$ and L^∞ respectively.

Examples

A. For $p \geq 1$ let

$$\varphi(u) = \begin{cases} u^p & \text{for } \ 0 \leq u \leq 1, \\ \infty & \text{for } \ u > 1. \end{cases}$$

Hence $\varphi(u) = \chi_p(u) \vee \chi_\infty(u)$ for $u \geq 0$, so $L^\varphi = L^p \cap L^\infty$ by Lemma 1.1. We see that $\varphi \in \Phi^c_{14}$ for $p = 1$ and $\varphi \in \Phi^c_{12}$ for $p > 1$. Hence by applying of Theorem 5.1 and Lemma 1.1 we get the following two theorems:

Theorem 5.2. *For a sequence (x_n) in $L^1 \cap L^\infty$ and $x \in L^1 \cap L^\infty$ the following statements are equivalent:*

(i) $x_n \to x\ (\mu - \Omega)$ and $\sup_n ||x_n||_1 < \infty$, $\sup_n ||x_n||_\infty < \infty$.

(ii) $|||x_n - x|||_\psi \to 0$ for every N − function ψ.

Theorem 5.3. *Let $p > 1$. For a sequence (x_n) in $L^p \cap L^\infty$ and $x \in L^p \cap L^\infty$ the following statements are equivalent:*

(i) $x_n \to x\ (\mu - \Omega)$ and $\sup_n ||x_n||_p < \infty$, $\sup_n ||x_n||_\infty < \infty$.

(ii) $|||x_n - x|||_\psi \to 0$ for every N-function ψ such that $\psi(u)/u^p \to 0$ as $u \to 0$.

B. For $p > 1$ let

$$\varphi(u) = \begin{cases} u & \text{for} \quad 0 \leq u \leq 1, \\ u^p & \text{for} \quad u > 1. \end{cases}$$

Then $\varphi(u) = \chi_1(u) \vee \chi_p(u)$ for $u \geq 0$, so $L^\varphi = L^1 \cap L^p$. Then $\varphi \in \Phi^c_{13}$ and using Theorem 5.1 and Lemma 1.1 we get:

Theorem 5.4. *Let $p > 1$. For a sequence (x_n) in $L^1 \cap L^p$ and $x \in L^1 \cap L^p$ the following statements are equivalent:*

(i) $x_n \to x\ (\mu - \Omega)$ and $\sup_n ||x_n||_1 < \infty$, $\sup_n ||x_n||_p < \infty$.

(ii) $|||x_n - x|||_\psi \to 0$ for every N-function ψ such that $\psi(u)/u^p \to 0$ as $u \to \infty$.

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