# A GENERALIZED NONCOMMUTATIVE KOROVKIN THEOREM AND *-CLOSEDNESS OF CERTAIN SETS OF CONVERGENCE 

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## Introduction

Let $A$ be a complex $C^{*}$-algebra with identity $1_{A}$, and for $n=1,2, \ldots$, let $\phi_{n}: A \rightarrow A$ be a Schwarz map, i.e., a *linear map such that

$$
\phi_{n}(a)^{*} \phi_{n}(a) \leq \phi_{n}\left(a^{*} a\right)
$$

for all $a \in A$. Robertson [4] has proved that the set

$$
C=\left\{a \in A:\left\|\phi_{n}(a)-a\right\| \rightarrow 0,\left\|\phi_{n}\left(a^{*} a\right)-a^{*} a\right\| \rightarrow 0,\left\|\phi_{n}\left(a a^{*}\right)-a a^{*}\right\| \rightarrow 0\right\}
$$

is a $C^{*}$-subalgebra of $A$. This is a noncommutative analogue of a classical theorem of Korovkin which states that if $A=C([a, b])$, the set of all continuous functions on $[a, b]$, and $\phi_{n}: A \rightarrow A$ is a positive map for $n=1,2, \ldots$, then

$$
C=\left\{f \in A: \phi_{n}(f) \rightarrow f, \phi_{n}\left(|f|^{2}\right) \rightarrow|f|^{2} \text { uniformly on }[a, b]\right\}
$$

is a norm-closed and conjugate closed subalgebra of $A$; in particular, if, $1, t$ and $t^{2}$ belong to $C$, then by the Stone-Weierstrass theorem, $C=C([a, b])$.

Let $B$ be another $C^{*}$-algebra with identity $1_{B}, \phi: A \rightarrow B$ a homomorphism, and, for $n=1,2, \ldots, \phi_{n}: A \rightarrow B$ a Schwarz map. Note that each $\phi_{n}$ is a positive map with $\phi_{n}\left(1_{A}\right) \leq 1_{B}$. Consider the set

$$
D=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right)\right\}
$$

where the convergence is in the norm topology or in the weak topology. In Section 1 , we show that the set $D$ is a norm-closed (but not necessarily $*$-closed) subalgebra of $A$ (Theorem 1.2). By considering $D \cap D^{*}$, we obtain a straightforward generalization of Robertson's result (Corollary 1.4).

In case $A$ is commutative, the set $D$ is clearly $*$-closed. The purpose of this paper is to investigate the *-closedness of the set $D$ in case $A$ is noncommutative. Let $B=A$ and $\phi$ be the identity map. Robertson has asked whether the *-closedness of the set $D$ for all choices of Schwarz maps $\phi_{n}$ characterizes the commutativity of $A$. We answer this question in the negative by using the theorem proved in Section 1. We show that if $A=M_{2}$, the noncommutative $C^{*}$-algebra consisting of all $2 \times 2$ complex matrices, then the set $D$ is $*$-closed

[^0]for all choices of Schwarz maps $\phi_{n}$ (Theorem 2.3). Further, we show that $M_{2}$ is the only finite dimensional noncommutative $C^{*}$-algebra for which this result holds (Theorem 2.6).

As for infinite dimensional algebras, let $H$ be a Hilbert space of infinite dimension and let $\beta(H)$ (respectively, $\chi(H)$ ) denote the $C^{*}$-algebra of all bounded (respectively, compact) operators on $H$. We show that if $A=\beta(H)$ or if $A$ is an infinite dimensional noncommutative $C^{*}$-subalgebra of $x(H)$, then there is a Schwarz map $\phi: A \rightarrow A$ and there exists $T \in A$ such that

$$
\phi(T)=T, \quad \phi\left(T^{*} T\right)=T^{*} T, \quad \text { but } \quad \phi\left(T T^{*}\right) \neq T T^{*}
$$

so that the set

$$
D_{\phi}=\left\{T \in A: \phi(T)=T, \phi\left(T^{*} T\right)=T^{*} T\right\}
$$

is not $*$-closed. The question whether this can be done for any infinite dimensional noncommutative $C^{*}$-subalgebra of $\beta(H)$ remains open.

## 1. A generalization of Robertson's theorem

We begin with a convergence result for the $C^{*}$-algebra $\beta(H)$ of all bounded operators on a complex Hilbert space $H$.

Lemma 1.1. Let $\left(R_{n}\right),\left(S_{n}\right)$ and $\left(U_{n}\right)$ be sequences in $\beta(H)$ and $R, U \in \beta(H)$.
(a) For all $n$ and for all real numbers $t$, let

$$
t^{2} R_{n}+t S_{n}+U_{n} \geq 0
$$

and for all $n$ and for some positive real number $\alpha$, let $R_{n} \leq \alpha I$, where $I$ denotes the identity operator in $\beta(H)$. Let " $\rightarrow$ " denote either norm or weak convergence in $\beta(H)$. Then $U_{n} \rightarrow 0$ implies $S_{n} \rightarrow 0$.
(b) For all n, let

$$
R_{n}^{*} R_{n} \leq S_{n}
$$

and let $\left(R_{n}\right)$ and $\left(S_{n}\right)$ converge weakly to $R$ and $R * R$ respectively. Then, in fact, ( $R_{n}$ ) converges strongly to $R$. If, in addition, $\left(U_{n}\right)$ converges to $U$ weakly, then ( $U_{n} R_{n}$ ) converges weakly to UR.

Proof. (a) For all natural numbers $n$ and all real numbers $t$, we have

$$
-t S_{n} \leq t^{2} R_{n}+U_{n} \leq \alpha t^{2} I+U_{n}
$$

Let $\left\|U_{n}\right\| \rightarrow 0$. For a fixed $t \neq 0$ and all large enough $n, U_{n} \leq \alpha t^{2} I$. Hence

$$
-t S_{n} \leq 2 \alpha t^{2} I
$$

Changing $t$ to $-t$, we have $t S_{n} \leq 2 \alpha t^{2} I$. Thus, for any given $t>0$, and all large enough $n$,

$$
-2 \alpha t I \leq S_{n} \leq 2 \alpha t I
$$

Hence $\left\|R_{n}\right\| \rightarrow 0$.

Next, let $U_{n} \rightarrow 0$ weakly. Fix $x \in H$. Then, the above procedure shows that for any given $t>0$ and all large enough $n$,

$$
\left.\left.-2 \alpha t<x, x\rangle \leq<S_{n}(x), x\right\rangle \leq 2 \alpha t<x, x\right\rangle
$$

Hence $S_{n} \rightarrow 0$ weakly.
(b) Let $x \in H$. Then

$$
\begin{aligned}
\left\|R_{n}(x)-R(x)\right\|^{2} & =<R_{n}^{*} R_{n}(x), x>+<R^{*} R(x), x>-2 R e<R_{n}(x), R(x)> \\
\leq & <S_{n}(x), x>+<R^{*} R(x), x>-2 R e<R_{n}(x), R(x)>
\end{aligned}
$$

Since $R_{n} \rightarrow R$ and $S_{n} \rightarrow R * R$ weakly, we see that the right side of the above inequality tends to zero. Since $x \in H$ is arbitrary, $R_{n} \rightarrow R$ strongly.

Let $U_{n} \rightarrow U$ weakly. For $x, y \in H$,

$$
<\left(U_{n} R_{n}-U R\right)(x), y>\leq\left|<\left(U_{n} R_{n}-U_{n} R\right)(x), y>\left|+\left|<\left(U_{n} R-U R\right)(x), y\right|\right.\right.
$$

Now, $\left(U_{n}^{*}(x)\right)$ converges weakly in $H$ and hence it is bounded. Also,

$$
\left\|R_{n}(x)-R(x)\right\| \rightarrow 0
$$

by the above. Hence $\left\langle U_{n} R_{n}(x), y\right\rangle$ tends to $\langle U R(x), y\rangle$. Since $x, y \in H$ are arbitrary, $U_{n} R_{n} \rightarrow U R$ weakly, Q.E.D.

We now prove a generalization of Robertson's result [4]. Our proof, like Robertson's, uses an idea of Palmer [3].

Theorem 1.2. Let $A$ and $B$ be complex $C^{*}$-algebras with identities $1_{A}$ and $1_{B}$, respectively. Let $\left(\phi_{n}\right)$ be a sequence of Schwarz maps and $\phi$ a *homomorphism from $A$ to $B$. Then the set

$$
D=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right)\right\}
$$

is a norm-closed subalgebra of $A$, where " $\rightarrow$ " denotes either norm or weak convergence.

Proof. It is easy to see that $D$ is norm-closed. To see that $D$ is a subalgebra, it is enough to prove that if $a \in D$ and $\phi_{n}(b) \rightarrow \phi(b)$, then $\phi_{n}(b a) \rightarrow \phi(b a)$. Now, for any real number $t$,

$$
\begin{aligned}
& t\left\{\phi_{n}(b) \phi_{n}(a)+\phi_{n}(a)^{*} \phi_{n}(b)^{*}\right\} \\
& =\phi_{n}\left(t b^{*}+a\right)^{*} \phi_{n}\left(t b^{*}+a\right)-t^{2} \phi_{n}(b) \phi_{n}(b)^{*}-\phi_{n}(a)^{*} \phi_{n}(a) \\
& \leq \phi_{n}\left(\left(t b^{*}+a\right)^{*}\left(t b^{*}+a\right)\right)-t^{2} \phi_{n}\left(b b^{*}\right)-\phi_{n}(a)^{*} \phi_{n}(a)+t^{2}\left(\phi_{n}\left(b b^{*}\right)-\phi_{n}(b) \phi_{n}(b)^{*}\right) \\
& =t \phi_{n}\left(b a+a^{*} b^{*}\right)+\phi_{n}\left(a^{*} a\right)-\phi_{n}(a)^{*} \phi_{n}(a)+t^{2}\left(\phi_{n}\left(b b^{*}\right)-\phi_{n}(b) \phi_{n}(b)^{*}\right) .
\end{aligned}
$$

Hence, if we let

$$
\begin{aligned}
R_{n} & =\phi_{n}\left(b b^{*}\right)-\phi_{n}(b) \phi_{n}\left(b^{*}\right) \\
S_{n} & =\phi_{n}\left(b a+a^{*} b^{*}\right)-\phi_{n}(b) \phi_{n}(a)-\phi_{n}(a)^{*} \phi_{n}(b)^{*} \\
U_{n} & =\phi_{n}\left(a^{*} a\right)-\phi_{n}(a)^{*} \phi_{n}(a)
\end{aligned}
$$

we see that for all real numbers $t$,

$$
t^{2} R_{n}+t S_{n}+U_{n} \geq 0
$$

Since $\phi_{n}(a) \rightarrow \phi(a)$, where $\rightarrow$ denotes either norm or weak convergence, we have $\phi_{n}(a)^{*} \rightarrow \phi(a)^{*}$. If " $\rightarrow$ " denotes norm convergence, then clearly $\phi_{n}(a)^{*} \phi_{n}(a) \rightarrow$ $\phi(a)^{*} \phi(a)$, and if " $\rightarrow$ " denotes weak convergence, then Lemma 1.1(b) shows that

$$
\phi_{n}(a)^{*} \phi_{n}(a) \rightarrow \phi(a)^{*} \phi(a)
$$

Since $\phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right)=\phi(a)^{*} \phi(a)$, it follows that in both the cases,

$$
U_{n}=\phi_{n}\left(a^{*} a\right)-\phi_{n}(a)^{*} \phi_{n}(a) \rightarrow 0
$$

Hence, by Lemma 1.1(a),

$$
\begin{equation*}
S_{n}=\phi_{n}\left(b a+a^{*} b^{*}\right)-\phi_{n}(b) \phi_{n}(a)-\phi_{n}(a)^{*} \phi_{n}(b)^{*} \rightarrow 0 \tag{1}
\end{equation*}
$$

Since $\phi_{n}(b) \rightarrow \phi(b)$, we see, by using Lemma $1.1(b)$ in the case of weak convergence, that
(2)

$$
\phi_{n}(b) \phi_{n}(a) \rightarrow \phi(b) \phi(a)=\phi(b a)
$$

Taking adjoints in (2), we have

$$
\begin{equation*}
\phi_{n}(a)^{*} \phi_{n}(b)^{*} \rightarrow \phi\left(a^{*} b^{*}\right) \tag{3}
\end{equation*}
$$

From (1), (2) and (3), we obtain

$$
\begin{equation*}
\phi_{n}\left(b a+a^{*} b^{*}\right) \rightarrow \phi\left(b a+a^{*} b^{*}\right) \tag{4}
\end{equation*}
$$

Replacing $b$ by $i b$ in (4), we have

$$
\begin{equation*}
\phi_{n}\left(b a-a^{*} b^{*}\right) \rightarrow \phi\left(b a-a^{*} b^{*}\right) \tag{5}
\end{equation*}
$$

Adding (4) and (5), we obtain $\phi_{n}(b a) \rightarrow \phi(b a)$, as desired, Q.E.D.
Remark 1.3. We have not been able to settle the question of whether $D$ is a subalgebra when " $\rightarrow$ " denotes strong convergence. The difficulty lies in the fact that the adjoint operation is not continuous in the strong topology. Although the multiplication operation is not continuous in the weak topology, this problem is taken care of by Lemma 1.1(b). We cannot apply the same procedure for strong convergence, as the following example shows. Let $H$ be a separable infinite dimensional Hilbert space and $T$ denote a unilateral left shift operator on $H$. Let $R_{n}=T^{n}$ for $n=1,2, \ldots$ Then $\left(R_{n}\right)$ and $\left(R_{n}^{*} R_{n}\right)$ converge to the zero operator strongly, but $\left(R_{n}^{*}\right)$ does not converge to the zero operator strongly.

However, it is interesting to note that the following corollary is just as valid for strong convergence as it is for norm or weak convergence.

Corollary 1.4. Under the assumptions of Theorem 1.2, the set

$$
C=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right), \phi_{n}\left(a a^{*}\right) \rightarrow \phi\left(a a^{*}\right)\right\}
$$

is a $C^{*}$-subalgebra of $A$, where " $\rightarrow$ " denotes norm, weak or strong convergence.

Proof. For norm or weak convergence, the result follows immediately from Theorem 1.2 since $C=D \cap D^{*}$, where $D^{*}=\left\{d^{*}: d \in D\right\}$. Now, consider

$$
C_{w}=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right), \phi_{n}\left(a a^{*}\right) \rightarrow\left(a a^{*}\right) \text { weakly }\right\}
$$

and

$$
C_{s t}=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right), \phi_{n}\left(a a^{*}\right) \rightarrow\left(a a^{*}\right) \text { strongly }\right\} .
$$

Then $C_{s t} \subset C_{w}$. We show that, in fact, $C_{s t}=C_{w}$, which establishes the desired result.

Let $a \in C_{w}$. Let

$$
R_{n}=\phi_{n}(a), \quad R=\phi(a), \quad S_{n}=\phi_{n}\left(a^{*} a\right) .
$$

Since $\phi_{n}$ is a Schwarz map, we have $R_{n}^{*} R_{n} \leq S_{n}$. Now, $R_{n} \rightarrow R$ and $S_{n} \rightarrow R^{*} R$ weakly. Hence by Lemma 1.1(b), $\phi_{n}(a) \rightarrow \phi(a)$ strongly.

Next, by Theorem 1.2, the set

$$
D_{w}=\left\{a \in A: \phi_{n}(a) \rightarrow \phi(a), \phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right) \text { weakly }\right\}
$$

is an algebra. Since $a, a^{*} \in D_{w}$, we see that $\left(a^{*} a\right)^{2} \in D_{w}$. Again, letting

$$
R_{n}=\phi_{n}\left(a^{*} a\right), R=\phi\left(a^{*} a\right), S_{n}=\phi_{n}\left(\left(a^{*} a\right)^{2}\right)
$$

in Lemma 1.1(b), we see that $\phi_{n}\left(a^{*} a\right) \rightarrow \phi\left(a^{*} a\right)$ strongly. Similarly, it can be shown that $\phi_{n}\left(a a^{*}\right) \rightarrow \phi\left(a a^{*}\right)$ strongly, so that $a \in C_{s t}, \quad$ Q.E.D.

## 2. *-closedness of the set $D$

Let $A$ be a $C^{*}$-algebra with identity $1_{A}$, and, for $n=1,2, \ldots$, let $\phi_{n}: A \rightarrow A$ be a Schwarz map. If $A$ is commutative, the set

$$
D=\left\{a \in A:\left\|\phi_{n}(a)-a\right\| \rightarrow 0,\left\|\phi_{n}\left(a^{*} a\right)-a^{*} a\right\| \rightarrow 0\right\}
$$

is clearly $*$-closed. In general, this need not be the case as the following example shows.

Example 2.1. Let $H$ be a separable infinite dimensional Hilbert space and $S$ a unilateral right shift operator on $H$. For $T \in \beta(H)$, the $C^{*}$-algebra of all bounded operators on $H$, let

$$
\phi(T)=\phi_{n}(T)=S^{*} T S, \quad n=1,2, \ldots
$$

Then

$$
D=\left\{T \in \beta(H): S^{*} T S=T, S^{*} T^{*} T S=T^{*} T\right\}
$$

Now, since $S^{*} S$ is the identity operator on $H$, we see that $S \in D$; but $S^{*} \notin D$,
since $S S^{*}$ is not the identity operator on $H$. In fact, $T \in \beta(H)$ belongs to $D$ if and only if $T$ commutes with $S$, since for all $x \in H$,

$$
\|T S(x)-S T(x)\|^{2}=<\left(S^{*} T^{*}-T^{*} S^{*}\right)(T S-S T)(x), x>
$$

It is proved in [5] that the commutant of $S$ is the strong closure of the set of polynomials in $S$. Thus, $T \in D$ if and only if $T$ can be strongly approximated by a sequence of polynomials in $S$.

Next, we show that $A$ need not be commutative for the set $D$ to be *-closed for every choice of Schwarz maps $\phi_{n}$. Let $M_{k}$ denote the $C^{*}$-algebra of all $k \times k$ matrices with complex entries, and $I \in M_{k}$ denote the identity matrix.

Lemma 1.2. Let $\phi: M_{2} \rightarrow M_{2}$ be a positive linear map with $\phi(I) \leq I$. Let $T \in M_{2}$ be such that $T^{*} T$ is a one-dimensional projection and $\phi(T)=T$, $\phi\left(T^{*} T\right)=T^{*} T$. Then either $\phi(I)=I$ or $T$ is a normal matrix.

Proof. Let

$$
T=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

so that

$$
T^{*} T=\left[\begin{array}{lr}
|a|^{2}+|c|^{2} & \bar{a} b+\bar{c} d \\
a \bar{b}+c \bar{d} & |b|^{2}+|d|^{2}
\end{array}\right]
$$

Suppose, first, that

$$
T^{*} T=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]
$$

Then, $b=d=0$, so that

$$
T=\left[\begin{array}{ll}
a & 0 \\
c & 0
\end{array}\right]=a\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+c\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]
$$

If $c=0$, then $T$ is a normal matrix. Now, let $c \neq 0$. Since $T=\phi(T)$ and

$$
\phi\left(\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\right)=\phi\left(T^{*} T\right)=T^{*} T=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]
$$

we have

$$
\begin{aligned}
{\left[\begin{array}{ll}
a & 0 \\
c & 0
\end{array}\right] } & =a \phi\left(\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\right)+c \phi\left(\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]\right) \\
& =a\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+c \phi\left(\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]\right)
\end{aligned}
$$

i.e.,

$$
c\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]=c \phi\left(\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]\right)
$$

Since $c \neq 0$, we have

$$
\phi\left(\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]\right)=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] .
$$

Now, $\phi$ is a positive linear map on the $C^{*}$-algebra $M_{2}$ and $\phi(I) \leq I$. Hence, by Kadison's Schwarz inequality [2],

$$
\phi\left(S^{*}\right) \phi(S)+\phi(S) \phi\left(S^{*}\right) \leq \phi\left(S^{*} S+S S^{*}\right)
$$

for all $S \in M_{2}$. Putting

$$
S=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]
$$

we have

$$
\begin{aligned}
I & =S^{*} S+S S^{*} \\
& =\phi\left(S^{*}\right) \phi(S)+\phi(S) \phi\left(S^{*}\right) \\
& \leq \phi\left(S^{*} S+S S^{*}\right) \\
& =\phi(I) \\
& \leq I .
\end{aligned}
$$

This shows that $\phi(I)=I$.
Thus, we have proved the lemma in the case where

$$
T^{*} T=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]
$$

In the general case, since $T^{*} T$ is a one-dimensional projection, let $x=\left(x_{1}, x_{2}\right)$ and $y=\left(y_{1}, y_{2}\right)$ be pairs of complex numbers such that

$$
\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}=1=\left|y_{1}\right|^{2}+\left|y_{2}\right|^{2}, \quad x_{1} \overline{y_{1}}+x_{2} \overline{y_{2}}=0
$$

and

$$
T^{*} T(x)=x, T^{*} T(y)=0
$$

Let

$$
U=\left[\begin{array}{ll}
x_{1} & y_{1} \\
x_{2} & y_{2}
\end{array}\right]
$$

Then $U$ is a unitary matrix, and

$$
T^{*} T=U\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] U^{*}
$$

Define $\psi: M_{2} \rightarrow M_{2}$ by $\psi(R)=U^{*} \phi\left(U R U^{*}\right) U, R \in M_{2}$. Then it is easy to see that $\psi$ is a positive linear map on $M_{2}$ and $\psi(I) \leq I$. If we let $S=U^{*} T U$, then

$$
S^{*} S=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]
$$

and $\psi(S)=S, \psi\left(S^{*} S\right)=S^{*} S$. Hence, by the particular case considered above, we see that either $\psi(I)=I$, or $S$ is normal. This, in turn, shows that either $\phi(I)=I$ or $T$ is normal.

Theorem 2.3. For $n=1,2, \ldots$, let $\phi_{n}: M_{2} \rightarrow M_{2}$ be a Schwarz map. Then the set

$$
D=\left\{T \in M_{2}:\left\|\phi_{n}(T)-T\right\| \rightarrow 0,\left\|\phi_{n}\left(T^{*} T\right)-T^{*} T\right\| \rightarrow 0\right\}
$$

is $*$-closed, and hence is a $C^{*}$-subalgebra of $M_{2}$.
We prove the following:
(1) Let $\phi: M_{2} \rightarrow M_{2}$ be a Schwarz map. Then the set

$$
D_{\phi}=\left\{T \in M_{2}: \phi(T)=T, \phi\left(T^{*} T\right)=T^{*} T\right\}
$$

is a $C^{*}$-subalgebra of $M_{2}$.
(2) $D=\cap D_{\phi}$, where the intersection is taken over all cluster points $\phi$ of the sequence $\left(\phi_{n}\right)$ in

$$
P\left(M_{2}, M_{2}\right)=\left\{\psi: M_{2} \rightarrow M_{2}: \psi \text { positive linear, }\|\psi\| \leq 1\right\}
$$

Proof of (1). It follows from the Cayley-Hamilton theorem that for any $T \in M_{2}$,

$$
\begin{equation*}
T^{2}=\operatorname{tr}(T) T-\operatorname{det}(T) I \tag{*}
\end{equation*}
$$

where $\operatorname{tr}(T)$ is the $\operatorname{trace}$ of $T$ and $\operatorname{det}(T)$ is the determinant of $T$.
Let $T \in D_{\phi}$, i.e., $\phi(T)=T$ and $\phi\left(T^{*} T\right)=T^{*} T$. Assume that $\|T\|=1$.
Case (i). $I \notin D_{\phi}$, i.e., $\phi(I) \neq I$. Since $D_{\phi}$ is a subalgebra of $M_{2}$ by Theorem 1.2, we see that $T^{2} \in D_{\phi}$. Now by (*),

$$
T^{2}=\operatorname{tr}(T) T-\operatorname{det}(T) I
$$

so that

$$
\phi\left(T^{2}\right)=\operatorname{tr}(T) \phi(T)-\operatorname{det}(T) \phi(I)
$$

But $\phi(T)=T$ and $\phi\left(T^{2}\right)=T^{2}$. Hence

$$
\operatorname{det}(T) I=\operatorname{det}(T) \phi(I)
$$

Since $I \neq \phi(I)$, it follows that $\operatorname{det}(T)=0$. Consequently,

$$
\operatorname{det}\left(T^{*} T\right)=|\operatorname{det}(T)|^{2}=0
$$

Then, by (*),

$$
\left(T^{*} T\right)^{2}=\operatorname{tr}\left(T^{*} T\right) T^{*} T
$$

Since $\|T\|=1$, we have $\left\|T^{*} T\right\|=1=\left\|\left(T^{*} T\right)^{2}\right\|$. This shows that $\operatorname{tr}\left(T^{*} T\right)=1$. Hence $\left(T^{*} T\right)^{2}=T^{*} T$. Moreover, $T \neq 0$ and $\operatorname{det}\left(T^{*} T\right)=0$. Thus, $T^{*} T$ is a one-dimensional projection. Now, Lemma 2.2 shows that $T$ is normal, i.e., $T^{*} T=T T^{*}$. Hence $\phi\left(T T^{*}\right)=\phi\left(T^{*} T\right)=T^{*} T=T T^{*}$. Since $\phi\left(T^{*}\right)=T^{*}$ always, we see that $T^{*} \in D_{\phi}$.

Case (ii). $I \in D_{\phi}$, i.e., $\phi(I)=I$. Let $T=R+i S$ with $R^{*}=R$ and $S^{*}=S$. Then, by (*),

$$
2\left(T^{*} T+T T^{*}\right)=R^{2}+S^{2}=\operatorname{tr}(R) R+\operatorname{tr}(S) S-(\operatorname{det}(R)+\operatorname{det}(S)) I
$$

Since $\phi(T)=T$, we see that $\phi(R)=R$ and $\phi(S)=S$. Also, $\phi(I)=I$. Hence

$$
2 \phi\left(T^{*} T+T T^{*}\right)=\operatorname{tr}(R) R+\operatorname{tr}(S) S-(\operatorname{det}(R)+\operatorname{det}(S)) I
$$

This shows that

$$
T^{*} T+T T^{*}=\phi\left(T^{*} T+T T^{*}\right)
$$

But since $T \in D_{\phi}$, we have $T^{*} T=\phi\left(T^{*} T\right)$. Hence $T T^{*}=\phi\left(T T^{*}\right)$ and again, $T \in D_{\phi}$.

Thus $D_{\phi}$ is *-closed. By Theorem 1.2, it is a norm-closed subalgebra of $M_{2}$. Hence $D_{\phi}$ is a $C^{*}$-subalgebra of $M_{2}$.

Proof of (2). Let $\Phi$ denote the set of all cluster points of the sequence ( $\phi_{n}$ ) in $P\left(M_{2}, M_{2}\right)$. For $\phi \in \Phi$, clearly $D \subset D_{\phi}$. Let

$$
E=\cap\left\{D_{\phi}: \phi \in \Phi\right\}, \quad \text { and } \quad \psi_{n}=\phi_{n \mid E}, n=1,2, \ldots
$$

Then, by (1), $E$ is $C^{*}$-subalgebra of $M_{\mathbf{2}}$; it contains $D$ and

$$
\psi_{n} \in P\left(E, M_{2}\right)=\left\{\psi: E \rightarrow M_{2}, \psi \text { positive linear, }\|\psi\| \leq 1\right\}
$$

To show that $E$ is contained in $D$, we argue as follows. We claim that $\psi_{n}(T) \rightarrow T$ for all $T \in E$. Suppose this is not the case. Then, by the compactness of $P\left(E, M_{2}\right)$, there is $\psi \in P\left(E, M_{2}\right)$ and a subnet $\left(\psi_{\alpha}\right)$ of $\left(\psi_{n}\right)$ such that

$$
\psi_{\alpha}(T) \rightarrow \psi(T) \text { for all } T \in E, \text { and } \psi\left(T_{0}\right) \neq T_{0} \text { for some } T_{0} \in E .
$$

Let $\left(\phi_{\alpha}\right)$ be the corresponding subnet of $\left(\phi_{n}\right)$, so that $\phi_{\alpha_{\mid E}}=\psi_{\alpha}$. Now,

$$
\phi_{\alpha}(T)=\psi_{\alpha}(T) \rightarrow \psi(T) \quad \text { for all } T \in E
$$

Let $\phi$ be a cluster point of $\left(\phi_{\alpha}\right)$ in $P\left(M_{2}, M_{2}\right)$. Then $\phi(T)=\psi(T)$ for all $T \in E$. But $\phi$ is also a cluster point of $\left(\phi_{n}\right)$, i.e., $\phi \in \Phi$, while

$$
\phi\left(T_{0}\right)=\psi\left(T_{0}\right) \neq T_{0}
$$

This contradicts the fact that $T_{0} \in E$. Hence $\psi_{n}(T) \rightarrow T$ for all $T \in E$. Now, let $T \in E$. Since $E$ is a *subalgebra of $M_{2}$, we have $T^{*} T \in E$, so that

$$
\phi_{n}(T)=\psi_{n}(T) \rightarrow T \quad \text { and } \quad \phi_{n}\left(T^{*} T\right)=\psi_{n}\left(T^{*} T\right) \rightarrow T^{*} T
$$

i.e., $T \in D$. Thus, $D=E$ as desired, Q.E.D.

Remark 2.4. If $\phi: M_{2} \rightarrow M_{2}$ is merely a positive linear map with $\phi(I) \leq I$, then the set $D_{\phi}$ may not be closed even under addition, as the following example shows. Let

$$
\phi\left(\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\right)=\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]
$$

and

$$
R=\left[\begin{array}{ll}
i & 0 \\
0 & 0
\end{array}\right], S=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

Then $R$ and $S$ belong to $D_{\phi}$, but

$$
R+S=\left[\begin{array}{ll}
i & 1 \\
1 & 0
\end{array}\right]
$$

does not. Also $D_{\phi}$ is not closed under the Jordan product:
Let

$$
T=\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right], U=\left[\begin{array}{ll}
i & 1 \\
1 & i
\end{array}\right]
$$

Then $T$ and $U$ belong to $D_{\phi}$, but

$$
\frac{1}{2}(T U+U T)=\left[\begin{array}{cc}
i+1 & i+(1 / 2) \\
i+(1 / 2) & 1
\end{array}\right]
$$

does not. For this particular map $\phi, D_{\phi}$ is closed under the squares. Examples of positive linear maps $\phi: M_{2} \rightarrow M_{2}$ with $\phi(I) \leq I$ for which $D_{\phi}$ is not closed under the squares and/or $D_{\phi}$ is not $*$-closed are lacking.

The following example shows that $M_{2}$ cannot be replaced by any $M_{k}, k \geq 3$, in Theorem 2.3.

Example 2.5. Let $k \geq 3$ be an integer. Define $\phi^{(k)}: M_{k} \rightarrow M_{k}$ by

$$
\phi^{(k)}\left(\left[\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & \ldots & a_{1 k} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2 k} \\
a_{31} & a_{32} & a_{33} & \ldots & a_{3 k} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{k 1} & a_{k 2} & a_{k 3} & \ldots & a_{k k}
\end{array}\right]\right)=\left[\begin{array}{ccccc}
a_{11} & a_{12} & 0 & \ldots & 0 \\
a_{21} & a_{22} & 0 & \ldots & 0 \\
0 & 0 & a_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right] .
$$

Then $\phi^{(k)}$ is a Schwarz map on $M_{k}$. This can be proved as follows. Let $T=\left(a_{i j}\right), i, j=1, \ldots, n$. Then

$$
\phi^{(k)}(T)^{*} \phi^{(k)}(T)=\left[\begin{array}{cccc}
\left|a_{11}\right|^{2}+\left|a_{21}\right|^{2} & a_{11} \overline{a_{12}}+\overline{a_{21}} a_{22} & 0 & \ldots 0 \\
\overline{a_{11}} a_{12}+a_{21} \overline{a_{22}} & \left|a_{12}\right|^{2}+\left|a_{22}\right|^{2} & 0 & \ldots 0 \\
0 & 0 & \left|a_{22}\right|^{2} & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots 0
\end{array}\right]
$$

and

$$
\phi^{(k)}\left(T^{*} T\right)=\left[\begin{array}{cccc}
\sum_{j=1}^{k}\left|a_{j 1}\right|^{2} & \sum_{j=1}^{k} \bar{a}_{j 1} a_{j 2} & 0 & \ldots 0 \\
\sum_{j=1}^{k} a_{j 1} \overline{a_{j 2}} & \sum_{j=1}^{k}\left|a_{j 2}\right|^{2} & 0 & \ldots 0 \\
0 & 0 & \sum_{j=1}^{k}\left|a_{j 2}\right|^{2} & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots 0
\end{array}\right]
$$

Hence
$\phi^{(k)}\left(T^{*} T\right)-\phi^{(k)}\left(T^{*}\right) \phi^{(k)}(T)\left[\begin{array}{cccc}\sum_{j=3}^{k}\left|a_{j 1}\right|^{2} & \sum_{j=3}^{k} \bar{a}_{j 1} a_{j 2} & 0 & \ldots 0 \\ \sum_{j=3}^{k} a_{j 1} \overline{a_{j 2}} & \sum_{j=3}^{k}\left|a_{j 2}\right|^{2} & 0 & \ldots 0 \\ 0 & 0 & \sum_{j=3}^{k}\left|a_{j 2}\right|^{2} & \ldots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 0\end{array}\right]$
which is clearly a positive matrix.
Let

$$
D=\left\{T \in M_{k}: \phi^{(k)}(T)=T, \phi^{(k)}\left(T^{*} T\right)=T^{*} T\right\}
$$

and

$$
T^{(k)}=\left[\begin{array}{ccccc}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right]
$$

Then it can be easily seen that $T^{(k)}$ belongs to $D$, but its conjugate transpose does not. Thus, $D$ is not $*$-closed.

We now prove a kind of converse of Theorem 2.3.
Theorem 2.6. Let $A$ be a finite dimensional noncommutative $C^{*}$-algebra. Assume that for every Schwarz map $\phi: A \rightarrow A$, the set

$$
D_{\phi}=\left\{a \in A: \phi(a)=a, \phi\left(a^{*} a\right)=a^{*} a\right\}
$$

is *-closed. Then $A$ is isometrically *-isomorphic to $M_{2}$.
Proof. By Theorem 11.2, p. 50 of [6], $A$ is isometrically *-isomorphic to a direct sum of matrix algebras. Hence we can assume that

$$
A=M_{n_{1}} \oplus \ldots \oplus M_{n_{m}}
$$

for some non-negative integers $n_{1}, \ldots, n_{m}$. Let $d$ denote the vector space dimension of $A$. Then $d \geq 4$, since $A$ is noncommutative. If $d=4$, then $A=M_{2}$ and we are done. Let, now, $d \geq 5$. Again, since $A$ is noncommutative, we consider the following mutually exclusive and exhaustive cases. For $T \in A$, let $T=T^{\boldsymbol{T}_{1}} \oplus \ldots \oplus T_{n_{m}}$.

Case (i). At least one $n_{j}$, say, $n_{k}$, is at least 3. For $T \in A$, let

$$
\phi(T)=T_{n_{1}} \oplus \ldots \oplus \phi^{\left(n_{k}\right]}\left(T_{n_{k}}\right) \oplus \ldots \oplus T_{n_{m}},
$$

where $\phi^{\left(n_{k}\right)}: M_{n_{k}} \rightarrow M_{n_{k}}$ is the map considered in Example 2.5. Then it follows that $\phi: A \rightarrow A$ is a Schwarz map and

$$
T=0 \oplus \ldots \oplus T^{\left(n_{k}\right)} \oplus \ldots \oplus 0
$$

belongs to $D_{\phi}$, but $T^{*}$ does not.
Case (ii). $m \geq 2$ and all $n$ 's equal 2. An element of $M_{2} \oplus M_{2}$ can be regarded as a $4 \times 4$ matrix of the following form:

$$
T_{2} \oplus T_{2}=\left[\begin{array}{cccc}
a_{1} & a_{2} & 0 & 0 \\
a_{3} & a_{4} & 0 & 0 \\
0 & 0 & b_{1} & b_{2} \\
0 & 0 & b_{3} & b_{4}
\end{array}\right]
$$

Define

$$
\psi\left(T_{2} \oplus T_{2}\right)=\left[\begin{array}{cccc}
a_{1} & a_{2} & 0 & 0 \\
a_{3} & a_{4} & 0 & 0 \\
0 & 0 & a_{4} & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Then

$$
\psi: M_{2} \oplus M_{2} \rightarrow M_{2} \oplus M_{2}
$$

is the restriction of the Schwarz map $\phi^{(4)}: M_{4} \rightarrow M_{4}$ of Example 2.5 to the
$C^{*}$-subalgebra formed of all elements of the type $T_{2} \oplus T_{2}$. Define $\phi: A \rightarrow A$ by

$$
\phi\left(T_{2} \oplus T_{2} \oplus \ldots \oplus T_{2}\right)=\psi\left(T_{2} \oplus T_{2}\right) \oplus T_{2} \oplus \ldots \oplus T_{2}
$$

Then $\phi$ is a Schwarz map and

$$
T=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] \oplus\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] \oplus \ldots \oplus\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]
$$

belongs to $D_{\phi}$, but $T^{*}$ does not.
Case (iii). $A=M_{2} \oplus M_{1} \oplus M_{n_{3}} \oplus \ldots \oplus M_{n_{m}}$, where $n_{3}, \ldots, n_{m}$ are either 2 or 1. An element of $M_{2} \oplus M_{1}$ can be regarded as a $3 \times 3$ matrix of the following form:

$$
T_{2} \oplus T_{1}=\left[\begin{array}{ccc}
a_{1} & a_{2} & 0 \\
a_{3} & a_{4} & 0 \\
0 & 0 & a_{5}
\end{array}\right]
$$

Define

$$
\psi\left(T_{2} \oplus T_{1}\right)=\left[\begin{array}{ccc}
a_{1} & a_{2} & 0 \\
a_{3} & a_{4} & 0 \\
0 & 0 & a_{4}
\end{array}\right]
$$

Then

$$
\psi: M_{2} \oplus M_{1} \rightarrow M_{2} \oplus M_{1}
$$

is the restriction of the Schwarz map $\phi^{(3)}: M_{3} \rightarrow M_{3}$ of Example 2.5 to the $C^{*}$-algebra formed of all elements of the type $T_{2} \oplus T_{1}$. Define $\psi: A \rightarrow A$ by

$$
\psi\left(T_{2} \oplus T_{1} \oplus T_{n_{3}} \oplus \ldots \oplus T_{n_{m}}\right)=\psi\left(T_{2} \oplus T_{1}\right) \oplus T_{n_{3}} \oplus \ldots \oplus T_{n_{m}}
$$

Then $\psi$ is a Schwarz map and

$$
T=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] \oplus[0] \oplus 0 \oplus \ldots \oplus 0
$$

belongs to $D_{\phi}$, but $T^{*}$ does not, Q.E.D.
Remark 2.7. In Example 2.1, we have considered the infinite dimensional noncommutative $C^{*}$-algebra $\beta(H)$, where $H$ is a separable Hilbert space, and found a Schwarz map $\phi$ on $\beta(H)$ for which $D_{\phi}$ is not $*$-closed.

We can use Theorem 2.6 to show that if $H$ is a not necessarily separable Hilbert space, then there is a Schwarz map $\phi$ on $\beta(H)$ for which $D_{\phi}$ is not *-closed. For this purpose, let $G$ be a subspace of $H$ of dimension 3, and let $P$ denote the orthogonal projection of $H$ onto $G$. Let

$$
A=\{S P: S \in \beta(G)\}
$$

Then it is easy to see that $A$ is a finite dimensional noncommutative $C^{*}$-subalgebra of $\beta(H)$ and that $A$ is not isometrically *-isomorphic to $M_{2}$. Hence by Theorem 2.6, there is a Schwarz map $\psi: A \rightarrow A$ and some $S_{0} \in \beta(G)$ such that $S_{0} P \in D_{\psi}$ but $\left(S_{0} P\right) * \notin D_{\psi}$.

Define $\phi: \beta(H) \rightarrow \beta(H)$ by

$$
\phi(T)=\psi\left((P T)_{\mid G} P\right), \quad T \in \beta(H)
$$

Then $\phi$ is *linear. In fact, $\phi$ is a Schwarz map: Let $T \in \beta(H)$. Then since $\psi$ is a Schwarz map,

$$
\begin{aligned}
\phi(T)^{*} \phi(T) & =\psi\left((P T)_{\mid G} P\right)^{*} \psi\left((P T)_{\mid G} P\right) \\
& \leq \psi\left(\left((P T)_{\mid G} P\right)^{*}(P T)_{\mid G} P\right)
\end{aligned}
$$

Now, it can be easily seen that

$$
\left.\left((P T)_{\mid G} P\right)^{*}(P T)_{\mid G} P \leq\left(P T^{*} T\right)_{\mid G} P\right)
$$

Since $\psi$ is positive, we see that

$$
\phi(T)^{*} \phi(T) \leq \psi\left(\left(P T^{*} T\right)_{\mid G} P\right)=\phi\left(T^{*} T\right)
$$

Also, $\phi_{\mid A}=\psi$, since for $S \in \beta(G)$, we have

$$
\phi(S P)=\psi\left((P S P)_{\mid G} P\right)=\psi(S P)
$$

Hence $S_{0} P \in D_{\phi}$, but $\left(S_{0} P\right)^{*} \notin D_{\phi}$.
Again, since the range of $\phi$ is contained in the $C^{*}$-algebra $\varkappa(H)$ of all compact operators on $H$, we can consider the restriction of $\phi$ to $x(H)$ and obtain a Schwarz map on $x(H)$ for which the set $D_{\phi}$ is not $*$-closed.

Finally, if $A$ is any infinite dimensional noncommutative $C^{*}$-subalgebra of $\varkappa(H)$, then by Theorem 1.4.5 of [1],

$$
A=\underset{\alpha}{\oplus} x\left(H_{\alpha}\right)
$$

where each $H_{\alpha}$ is a Hilbert space. Hence we can find a Schwarz map $\phi: A \rightarrow A$ for which $D_{\phi}$ is not $*$-closed. We have not been able to answer the question whether this can be done for any infinite dimensional noncommutative $C^{*}$-algebra $A$.

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