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A GENERALIZED PÓLYA URN MODEL AND RELATED MULTIVARIATE DISTRIBUTIONS*

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Abstract. In this paper, we study a Pólya urn model containing balls of $(m+1)$ different labels under a general replacement scheme, which is characterized by an $(m+1) \times (m+1)$ addition matrix of integers without constraints on the values of these $(m+1)^2$ integers other than non-negativity. Let $X_1, X_2, \dots, X_n$ be trials obtained by the Pólya urn scheme (with possible outcomes: "0", "1", ..., "m"). We consider the multivariate distributions of the numbers of occurrences of runs of different types arising from the various enumeration schemes and give a recursive formula of the probability generating function. Some closed form expressions are derived as special cases, which have potential applications to various areas. Our methods for the derivation of the multivariate run-related distribution are very simple and suitable for numerical and symbolic calculations by means of computer algebra systems. The results presented here develop a general workable framework for the study of Pólya urn models. Our attempts are very useful for understanding non-classic urn models. Finally, numerical examples are also given in order to illustrate the feasibility of our results.

Key words and phrases: Pólya urn, replacement scheme, addition matrix, run, enumeration schemes, recursive scheme, probability generating function, double generating function, random structures.

1. Introduction

The theory and applications of urn models have been a popular subject of study for researchers in a wide range of areas such as statistics, probability theory and physics (see Kotz and Balakrishnan (1997), Feller (1968)). We can mention Pólya urn models as the interesting class of urn models, which was introduced by Eggenberger and Pólya (1923) as a model for the spread of contagious diseases (see Johnson and Kotz (1977), Inoue and Aki (2001), Inoue (2003)).

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We describe the Pólya urn scheme briefly. From an urn containing α_0 balls labeled 0 and α_1 balls labeled 1, a ball is drawn, its label is noted and the ball is returned to the urn along with additional balls depending on the label of the drawn ball; if a ball labeled i ($i = 0, 1$) is drawn, a_{ij} balls labeled j ($j = 0, 1$) are added. This scheme is characterized by the following 2×2 addition matrix of integers, $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$, whose rows are indexed by the label of the drawn ball and whose columns are indexed by the label of balls added.

One is interested in the exact distribution of the number of occurrences of "1" (or the exact distribution of the number of occurrences of "1"-runs of length k) within n draws from the urn. A number of Pólya urn models have been considered by many authors in various addition matrices, so that the distributions have been studied in a variety of different areas. The classical Pólya urn model ($a_{00} = a_{11}$, $a_{01} = a_{10} = 0$) was studied earlier and a detailed discussion can be found in Johnson and Kotz (1977). In the case where $a_{00} = a_{11}$, $a_{01} = a_{10} = 0$, Aki and Hirano (1988) obtained the Pólya distribution of order k. Friedman (1949) considered the generalization $a_{00} = a_{11}$, $a_{01} = a_{10}$ (see Shur (1984)). Furthermore, a multivariate Pólya urn model, which is characterized by an $m \times m$ ($m \geq 3$) addition matrix, is studied by many authors (see Tripsiannis *et al.* (2002), Johnson *et al.* (1997)).

For a long time, most investigations have been made under the special structure of the addition matrix with constant row sums; in the 2×2 case, $a_{00} + a_{01} = a_{10} + a_{11}$, which implies a steady linear growth of the urn size. It has been pointed out that the derivation of the exact distribution of the number of occurrences of "1" (or the number of occurrences of "1"-runs of length k) becomes very complicated in the case when $a_{00} + a_{01} \neq a_{10} + a_{11}$ (see Ling (1993)). Recently, Kotz *et al.* (2000) revealed why the case $a_{00} + a_{01} \neq a_{10} + a_{11}$ is considerably more challenging. They also described that the exact distribution is rather unwieldy for numerical purposes, even with the aid of a modern computer. The nature of urn schemes not satisfying the condition $a_{00} + a_{01} = a_{10} + a_{11}$ essentially differs from those that do. Even in the 2×2 case, urn schemes where the constraint is not imposed are generally more difficult to analyze than those where it was imposed.

Traditionally, enumerative combinatorial methods were used to obtain the exact distributions. Their derivation involves counting paths representing a realization of the urn development (see Sen and Jain (1997)). However, it is difficult to study the generalized Pólya urn models (not fixed row sum) by using the traditional combinatorial approach.

Our purpose in the present paper is to develop a general workable framework for the exact distribution theory for Pólya urn models. We give the method for the derivation of the exact distribution regardless of whether or not the constraint is imposed. In this paper, we consider a generalized Pólya urn model containing balls of $(m + 1)$ different labels, whose replacement is controlled by an $(m+1) \times (m+1)$ addition matrix without the constraint. We study the multivariate distributions of the numbers of occurrences of runs of different types by engaging the various enumeration schemes. The approach departs from the traditional combinatorial approach and provides a very efficient computational tool. Our results offer the key to the understanding of a class of non-classic Pólya urn models.

The rest of this paper is organized as follows. In Section 2, a Pólya urn model containing balls of $(m + 1)$ different labels is introduced, which is characterized by the general replacement scheme. Also we introduce the necessary notation that will be used in the remaining sections. In Section 3, we propose a method for the derivation of the multivariate run-related distribution. The present approach provides a computationally more efficient scheme. Section 4 gives some closed form expressions as special cases. All the results presented in the section are, to the best of our knowledge, new and have potential applications to other problems such as statistical tests based on multiple run and a class of multiple failure mode reliability system (see Boutsikas and Koutras (2002)). In Section 5, numerical examples are given in order to illustrate the feasibility of our main results, which nowadays can be easily achieved by computer algebra systems.

2. The Model

In this section, to begin with, we will introduce a Pólya urn model containing balls of $(m+1)$ different labels, which is characterized by the general replacement scheme. Next, we introduce the different ways of counting the numbers of runs that will be used in the remaining sections.

2.1 *Generalized Pólya urn models*

From an urn containing α_i balls labeled i $(i = 0, 1, \dots, m)$, a ball is drawn, its label is noted and the ball is returned to the urn along with additional balls depending on the label of the drawn ball; if a ball labeled i $(i = 0, 1, \dots, m)$ is drawn, a_{ij} balls labeled j $(j = 0, 1, \dots, m)$ are added. This scheme is characterized by the $(m+1) \times (m+1)$ addition matrix of non-negative integers $A = (a_{ij})$ $(i, j = 0, 1, \dots, m)$ whose rows are indexed by the label of the ball drawn and whose columns are indexed by the label of the balls added. We denote the urn composition and $(i+1)$-st row of the matrix A by $\boldsymbol{b}(= (\alpha_0, \alpha_1, \dots, \alpha_m))$ and $\boldsymbol{a}_i = (a_{i0}, a_{i1}, \dots, a_{im})$, respectively. If we draw a ball labeled i $(i = 0, 1, \dots, m)$ from the urn $\boldsymbol{b}$, then the urn composition changes from $\boldsymbol{b}$ to $\boldsymbol{b} + \boldsymbol{a}_i$. Also the total numbers of balls in the urn changes from $|\boldsymbol{b}|$ to $|\boldsymbol{b}| + |\boldsymbol{a}_i|$, where, $|\boldsymbol{b}| = \alpha_0 + \alpha_1 + \cdots + \alpha_m$ and $|\boldsymbol{a}_i| = a_{i0} + a_{i1} + \cdots + a_{im}$. Always starting with the newly constituted urn, this experiment is continued.

2.2 *Enumeration schemes*

In a sequence of Bernoulli trials (with two possible outcomes: "1" or "0"), there are various ways of counting the number of "1"-runs of length k in the literature (see Balakrishnan and Koutras (2002)). The four best-known types of the ways of counting the number of "1"-runs of length k are as follows.

(i) Type I enumeration scheme: the way of counting the number of non-overlapping and recurrent "1"-runs of length k, in the sense of Feller's (1968) counting,

(ii) Type II enumeration scheme: the way of counting the number of "1"-runs of length at least k, in the sense of Goldstein's (1990) counting (see Gibbons (1971)),

(iii) Type III enumeration scheme: the way of counting the number of overlapping "1"-runs of length k, in the sense of Ling's (1988) counting,

(iv) Type IV enumeration scheme: the way of counting the number of "1"-runs of size exactly k, in the sense of Mood's (1940) counting.

Let $X_1, X_2, \dots, X_n$ be a sequence of $\{0, 1, \dots, m\}$-valued random variables obtained by the Pólya urn scheme described in Subsection 2.1. Then we study the multivariate distributions of the numbers of occurrences of runs of different types by engaging the various enumeration schemes. The counting of "i"-runs of length $k_j^{(i)}$ $(i = 1, 2, \dots, m,$

$j = 1, 2, \ldots, r_i$) is performed by engaging Type $\beta^{(i)}(= I, II, III, IV)$ enumeration scheme. Let $\boldsymbol{N}(n, \boldsymbol{k}^{(i)}; \beta^{(i)}) = (N(n, k_1^{(i)}; \beta^{(i)}), \ldots, N(n, k_{r_i}^{(i)}; \beta^{(i)}))$ denote the random variables $(i = 1, 2, \ldots, m)$, where, $\boldsymbol{k}^{(i)} = (k_1^{(i)}, k_2^{(i)}, \ldots, k_{r_i}^{(i)})$ and $N(n, s; \beta^{(i)})$ represents the number of occurrences of "i"-runs of length s $(i = 1, 2, \ldots, m)$ by engaging Type $\beta^{(i)}(= I, II, III, IV)$ enumeration scheme.

We define

$$\text{(2.1)} \quad \boldsymbol{\mu}(j; \beta^{(i)}) = (\mu_1(j; \beta^{(i)}), \ldots, \mu_{r_i}(j; \beta^{(i)})) = \begin{cases} \left(\left[\frac{j}{k_1^{(i)}}\right], \ldots, \left[\frac{j}{k_{r_i}^{(i)}}\right]\right) & \beta^{(i)} = I, \\ (I(j \geq k_1^{(i)}), \ldots, I(j \geq k_{r_i}^{(i)})) & \beta^{(i)} = II, \\ ((j - (k_1^{(i)} - 1))^+, \ldots, (j - (k_{r_i}^{(i)} - 1))^+) & \beta^{(i)} = III, \\ (I(j = k_1^{(i)}), \ldots, I(j = k_{r_i}^{(i)})) & \beta^{(i)} = IV, \end{cases}$$

where, $(j - (k_s^{(i)} - 1))^+ = \max\{0, (j - (k_s^{(i)} - 1))\}$ and

$$I(u) = \begin{cases} 1 & u \text{ is true}, \\ 0 & \text{otherwise}. \end{cases}$$

3. Recursive formulae

In this section, we establish a recursive scheme for the evaluation of the probability generating function (p.g.f.) of $(\boldsymbol{N}(n, \boldsymbol{k}^{(1)}; \beta^{(1)}), \boldsymbol{N}(n, \boldsymbol{k}^{(2)}; \beta^{(2)}), \ldots, \boldsymbol{N}(n, \boldsymbol{k}^{(m)}; \beta^{(m)}))$.

Suppose that we have an initial urn composition $\boldsymbol{b}_0$ $(= (\alpha_{00}, \alpha_{01}, \ldots, \alpha_{0m}))$. Then, the p.g.f. of $(\boldsymbol{N}(n, \boldsymbol{k}^{(1)}; \beta^{(1)}), \boldsymbol{N}(n, \boldsymbol{k}^{(2)}; \beta^{(2)}), \ldots, \boldsymbol{N}(n, \boldsymbol{k}^{(m)}; \beta^{(m)}))$ will be denoted by $\phi_n(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta})$; i.e.,

$$\phi_n(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta}) = E[\boldsymbol{t}^{(1)\boldsymbol{N}(n, \boldsymbol{k}^{(1)}; \beta^{(1)})} \boldsymbol{t}^{(2)\boldsymbol{N}(n, \boldsymbol{k}^{(2)}; \beta^{(2)})} \cdots \boldsymbol{t}^{(m)\boldsymbol{N}(n, \boldsymbol{k}^{(m)}; \beta^{(m)})} \mid \boldsymbol{b}_0],$$

where, $\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{r_1}^{(1)}, \ldots, t_1^{(i)}, \ldots, t_{r_i}^{(i)}, \ldots, t_1^{(m)}, \ldots, t_{r_m}^{(m)})$, $\boldsymbol{\beta} = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(m)})$ and

$$\boldsymbol{t}^{(i)\boldsymbol{N}(n, \boldsymbol{k}^{(i)}; \beta^{(i)})} = t_1^{(i)N(n, k_1^{(i)}; \beta^{(i)})} \cdots t_{r_i}^{(i)N(n, k_{r_i}^{(i)}; \beta^{(i)})}.$$

Suppose that we have $X_1 = s$ $(s = 0, 1, \ldots, m)$ and the urn composition before the next trial is $\boldsymbol{b}$ $(= (\alpha_0, \alpha_1, \ldots, \alpha_m))$. Then we denote $\phi_n^{(s)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta})$ by the conditional p.g.f. of the numbers of occurrences of "i"-runs of length $k_j^{(i)}$ $(j = 1, 2, \ldots, r_i,\ i = 1, 2, \ldots, m)$ by engaging Type $\beta^{(i)}(= I, II, III, IV)$ enumeration scheme in the sequence $X_1, X_2, X_3, \ldots, X_{n+1}$ given $X_1 = s$. From the definition, we see that $\phi_n^{(0)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = \phi_n(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta})$. The conditional p.g.f. $\phi_n^{(s)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta})$ $s = 0, 1, \ldots, m$ will be defined by

$$\phi_n^{(s)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = E[\boldsymbol{t}^{(1)\boldsymbol{N}(n+1, \boldsymbol{k}^{(1)}; \beta^{(1)})} \boldsymbol{t}^{(2)\boldsymbol{N}(n+1, \boldsymbol{k}^{(2)}; \beta^{(2)})} \cdots \boldsymbol{t}^{(m)\boldsymbol{N}(n+1, \boldsymbol{k}^{(m)}; \beta^{(m)})} \mid \boldsymbol{b}, X_1 = s].$$

THEOREM 3.1. *For an initial urn composition* $\boldsymbol{b}_0$ *and every possible urn composition* $\boldsymbol{b}$, *the p.g.f.* $\phi_n(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta})$ *and the conditional p.g.f.'s* $\phi_n^{(i)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta})$ $i = 0, 1, \ldots, m$

satisfy the recurrence relations.

$$\phi_n(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta}) = \sum_{i=0}^{m} \frac{\alpha_{0i}}{|\boldsymbol{b}_0|} \phi_{n-1}^{(i)}(\boldsymbol{b}_0 + \boldsymbol{a}_i, \boldsymbol{t}; \boldsymbol{\beta}) \qquad n \geq 1, \tag{3.1}$$

$$\phi_0(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta}) = 1, \tag{3.2}$$

$$\phi_n^{(0)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = \sum_{i=0}^{m} \frac{\alpha_i}{|\boldsymbol{b}|} \phi_{n-1}^{(i)}(\boldsymbol{b} + \boldsymbol{a}_i, \boldsymbol{t}; \boldsymbol{\beta}) \qquad n \geq 1, \tag{3.3}$$

$$\phi_0^{(0)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = 1, \tag{3.4}$$

$$\begin{aligned} \phi_n^{(i)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = & \sum_{i' \neq i} \sum_{j=0}^{n-1} \frac{\alpha_i^{[j, a_{ii}]}}{|\boldsymbol{b}|^{[j, |\boldsymbol{a}_i|]}} \frac{\alpha_{i'} + j a_{ii'}}{|\boldsymbol{b}| + j|\boldsymbol{a}_i|} \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(j+1;\beta^{(i)})}} \\ & \times \phi_{n-j-1}^{(i')}(\boldsymbol{b} + j\boldsymbol{a}_i + \boldsymbol{a}_{i'}, \boldsymbol{t}; \boldsymbol{\beta}) \\ & + \frac{\alpha_i^{[n, a_{ii}]}}{|\boldsymbol{b}|^{[n, |\boldsymbol{a}_i|]}} \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(n+1;\beta^{(i)})}} \\ & \qquad n \geq 1, \ \beta^{(i)} = I, II, III, IV, \ i = 1, 2, \ldots, m, \end{aligned} \tag{3.5}$$

$$\phi_0^{(i)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(1;\beta^{(i)})}} \qquad i = 1, 2, \ldots, m, \tag{3.6}$$

where, $\boldsymbol{\mu}(j; \beta^{(i)})$ *is given by* (2.1), $a^{[x,c]} = a(a+c)\cdots(a+(x-1)c)$ *with* $a^{[0,c]} = 1$, $|\boldsymbol{b}_0| = \sum_{i=0}^{m} \alpha_{0i}$, $|\boldsymbol{b}| = \sum_{i=0}^{m} \alpha_i$, $|\boldsymbol{a}_i| = \sum_{j=0}^{m} a_{ij}$ *and* $\boldsymbol{t}^{(i)^{\boldsymbol{\mu}(j+1;\beta^{(i)})}} = t_1^{(i)^{\mu_1(j+1;\beta^{(i)})}} \cdots t_{r_i}^{(i)^{\mu_{r_i}(j+1;\beta^{(i)})}}$.

PROOF. Given the initial urn composition $\boldsymbol{b}_0$ $(= (\alpha_{00}, \alpha_{01}, \ldots, \alpha_{0m}))$, we observe the first trial. Then we obtain

$$\begin{aligned} & E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}_0] \\ & \quad = \sum_{i=0}^{m} \frac{\alpha_{0i}}{|\boldsymbol{b}_0|} E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}_0, X_1 = i]. \end{aligned}$$

Since we see that

$$\begin{aligned} & E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}_0, X_1 = i] \\ & \quad = \phi_{n-1}^{(i)}(\boldsymbol{b}_0 + \boldsymbol{a}_i, \boldsymbol{t}; \boldsymbol{\beta}) \qquad \text{for} \quad i = 0, 1, \ldots, m, \end{aligned}$$

therefore we have the equation (3.1). Since $\phi_n^{(0)}(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta}) = \phi_n(\boldsymbol{b}, \boldsymbol{t}; \boldsymbol{\beta})$ from the definition, we have the equation (3.3).

Suppose that we have $X_1 = i$ $(i = 1, 2, \ldots, m)$ and the urn composition before the next trial is $\boldsymbol{b}$ $(= (\alpha_0, \alpha_1, \ldots, \alpha_m))$. Considering the event $B_{j+2}^{i'}$ that the first i' $(i' \neq i)$ occurs at the $(j+2)$-th trials $(j = 0, 1, \ldots, n-1)$ and the event C that the first i' $(i' \neq i)$ does not occur in $X_2, X_3, \ldots, X_{n+1}$.

Then we obtain

$$E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}, X_1 = i]$$

$$= \sum_{i' \neq i} \sum_{j=0}^{n-1} \frac{\alpha_i^{[j,a_{ii}]}}{|\boldsymbol{b}|^{[j,|\boldsymbol{a}_i|]}} \frac{\alpha_{i'} + j a_{ii'}}{|\boldsymbol{b}| + j|\boldsymbol{a}_i|}$$

$$\times E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}, X_1 = i, B_{j+2}^{i'}]$$

$$+ \frac{\alpha_i^{[n,a_{ii}]}}{|\boldsymbol{b}|^{[n,|\boldsymbol{a}_i|]}} E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots$$

$$\boldsymbol{t}^{(m)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}, X_1 = i, C], \qquad \text{for} \quad i = 1, 2, \ldots, m.$$

Since we see that

$$E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}, X_1 = i, B_{j+2}^{i'}]$$

$$= \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(j+1;\beta^{(i)})}} \phi_{n-j-1}^{(i')}(\boldsymbol{b} + j\boldsymbol{a}_i + \boldsymbol{a}_{i'}, \boldsymbol{t}; \boldsymbol{\beta}) \qquad \text{for} \quad i' \neq i, \ i = 0, 1, \ldots, m,$$

$$E[\boldsymbol{t}^{(1)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(1)};\beta^{(1)})}} \boldsymbol{t}^{(2)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(2)};\beta^{(2)})}} \cdots \boldsymbol{t}^{(m)^{\boldsymbol{N}(n+1,\boldsymbol{k}^{(m)};\beta^{(m)})}} \mid \boldsymbol{b}, X_1 = i, C]$$

$$= \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(n+1;\beta^{(i)})}}, \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

therefore we have the equation (3.5). It is easy to check the equations (3.2), (3.4) and (3.6). The proof is completed. □

The method established in Theorem 3.1 is a very efficient recursive scheme. The advantage of this method is that it survives in a much broader framework than the ones used so far. When the sequence $X_1, X_2, \ldots, X_n$ is constructed from other random trials (for example, Markov chain), the exact distribution of $(N(n, k_1^{(i)}; \beta^{(i)}), \ldots, N(n, k_{r_i}^{(i)}; \beta^{(i)}))$ could be easily obtained through Theorem 3.1 after trivial modifications.

Remark 3.1. Recently, Aki and Hirano (2000) introduced a generalized enumeration scheme which is called ℓ-overlapping counting (see Inoue and Aki (2003)). In Theorem 3.1, setting

$$\begin{aligned} \boldsymbol{\mu}(j; \beta^{(i)}) &= (\mu_1(j; \beta^{(i)}), \ldots, \mu_{r_i}(j; \beta^{(i)})) \\ &= \left(\left[\frac{j - \ell_1^{(i)}}{k_1^{(i)} - \ell_1^{(i)}} \right]^+, \ldots, \left[\frac{j - \ell_{r_i}^{(i)}}{k_{r_i}^{(i)} - \ell_{r_i}^{(i)}} \right]^+ \right), \end{aligned}$$

where, $[\frac{j-\ell_s^{(i)}}{k_s^{(i)}-\ell_s^{(i)}}]^+ = \max\{0, [\frac{j-\ell_s^{(i)}}{k_s^{(i)}-\ell_s^{(i)}}]\}$ and $0 \leq \ell_s^{(i)} \leq k_s^{(i)} - 1$, the methods in Theorem 3.1 can be extended to cover this case easily.

Remark 3.2. It is possible to study the multivariate run-related distributions by engaging various enumeration schemes through some modifications on $\boldsymbol{\mu}(j; \beta^{(i)})$. If we count the number of occurrences of "i"-runs of length $k_j^{(i)}$ by engaging Type $\beta_j^{(i)}$ enumeration scheme ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, r_i$, $\beta_j^{(i)} = I, II, III, IV$), we should replace $\boldsymbol{\mu}(j; \beta^{(i)})$ in Theorem 3.1 by

$$\boldsymbol{\mu}(j; \boldsymbol{\beta}^{(i)}) = (\mu(j; \beta_1^{(i)}), \mu(j; \beta_2^{(i)}), \ldots, \mu(j; \beta_{r_i}^{(i)})),$$

where, $\boldsymbol{\beta}^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_{r_i}^{(i)})$ and for $s = 1, 2, \dots, r_i$,

$$\mu(j; \beta_s^{(i)}) = \begin{cases} \left[\frac{j}{k_s^{(i)}}\right] & \beta_s^{(i)} = I, \\ I(j \geq k_s^{(i)}) & \beta_s^{(i)} = II, \\ (j - (k_s^{(i)} - 1))^+ & \beta_s^{(i)} = III, \\ I(j = k_s^{(i)}) & \beta_s^{(i)} = IV. \end{cases}$$

In closing, we would like to mention that Theorem 3.1 provides a tool for the evaluation of the r-th descending factorial moment and r-th order moment of $N(n, k_j^{(i)}; \beta^{(i)})$. The recursive scheme for the descending factorial moment of $N(n, k_j^{(i)}; \beta^{(i)})$ can be derived from the r-th order derivatives of the p.g.f.'s directly. The recursive scheme for the r-th order moment $N(n, k_j^{(i)}; \beta^{(i)})$ can be derived from the r-th order derivatives of the moment generating functions obtained by replacing $t_j^{(i)}$ by $e^{t_j^{(i)}}$. We can also derive a recursive scheme for the covariance between $N(n, s_1; u_1)$ and $N(n, s_2; u_2)$. The details can be worked out easily and are thus omitted here.

4. Multivariate distributions as special cases

In this section, we assume that the addition matrix A is equal to the $(m+1) \times (m+1)$ zero matrix. Then the initial urn composition $\boldsymbol{b}_0$ $(= (\alpha_{00}, \alpha_{01}, \dots, \alpha_{0m}))$ does not change with each draw. In this section, we will denote the p.g.f.'s simply by $\phi_n(\boldsymbol{t}; \boldsymbol{\beta})$ and $\phi_n^{(i)}(\boldsymbol{t}; \boldsymbol{\beta})$ (instead of $\phi_n(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta})$ and $\phi_n^{(i)}(\boldsymbol{b}, \mathbf{t}; \boldsymbol{\beta})$). Easily we see that $\phi_n^{(0)}(\boldsymbol{t}; \boldsymbol{\beta}) = \phi_n(\boldsymbol{t}; \boldsymbol{\beta})$ from the definition.

Using Theorem 3.1, we obtain the next corollary.

COROLLARY 4.1. *The p.g.f. $\phi_n(\boldsymbol{t}; \boldsymbol{\beta})$ and the conditional p.g.f.'s $\phi_n^{(i)}(\boldsymbol{t}; \boldsymbol{\beta})$ $i = 1, 2, \dots, m$ satisfy the recurrence relations.*

$$\phi_n(\boldsymbol{t}; \boldsymbol{\beta}) = p_0 \phi_{n-1}(\boldsymbol{t}; \boldsymbol{\beta}) + \sum_{i=1}^{m} p_i \phi_{n-1}^{(i)}(\boldsymbol{t}; \boldsymbol{\beta}) \qquad n \geq 1,$$

$$\phi_0(\boldsymbol{t}; \boldsymbol{\beta}) = 1,$$

$$\begin{aligned} \phi_n^{(i)}(\boldsymbol{t}; \boldsymbol{\beta}) &= \sum_{j=0}^{n-1} p_i^j p_0 \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(j+1; \boldsymbol{\beta}^{(i)})}} \phi_{n-j-1}(\boldsymbol{t}; \boldsymbol{\beta}) \\ &\quad + \sum_{i' \neq 0, i} \sum_{j=0}^{n-1} p_i^j p_{i'} \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(j+1; \boldsymbol{\beta}^{(i)})}} \phi_{n-j-1}^{(i')}(\boldsymbol{t}; \boldsymbol{\beta}) + p_i^n \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(n+1; \boldsymbol{\beta}^{(i)})}}, \\ &\qquad\qquad n \geq 1, \ \beta^{(i)} = I, II, III, IV, \ i = 1, 2, \dots, m, \end{aligned}$$

$$\phi_0^{(i)}(\boldsymbol{t}; \boldsymbol{\beta}) = \boldsymbol{t}^{(i)^{\boldsymbol{\mu}(1; \boldsymbol{\beta}^{(i)})}} \qquad i = 1, 2, \dots, m,$$

where, $\boldsymbol{\mu}(j; \beta^{(i)})$ is given by (2.1) and $p_i = \alpha_{0i}/|\boldsymbol{b}_0|$ $(i = 0, 1, \dots, m)$.

We will introduce the following double generating functions

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) = \sum_{n=0}^{\infty} \phi_n(\boldsymbol{t}; \boldsymbol{\beta}) z^n,$$

$$\Phi^{(i)}(\boldsymbol{t}, z; \boldsymbol{\beta}) = \sum_{n=0}^{\infty} \phi_n^{(i)}(\boldsymbol{t}; \boldsymbol{\beta}) z^n \qquad i = 1, 2, \ldots, m.$$

Using Corollary 4.1, we can easily obtain the system of equations of the double generating functions. The next corollary provides the details.

COROLLARY 4.2. *The double generating functions* $\Phi(\boldsymbol{t}, z; \boldsymbol{\beta})$ *and* $\Phi^{(i)}(\boldsymbol{t}, z; \boldsymbol{\beta})$ $i = 1, 2, \ldots, m$ *satisfy the system of equations*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) = 1 + p_0 z \Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) + \sum_{i=1}^{m} p_i z \Phi^{(i)}(\boldsymbol{t}, z; \boldsymbol{\beta}),$$
$$\Phi^{(i)}(\boldsymbol{t}, z; \boldsymbol{\beta}) = P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)}) + p_0 z P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)}) \Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) + P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)}) \sum_{i' \neq 0, i} p_{i'} z \Phi^{(i')}(\boldsymbol{t}, z; \boldsymbol{\beta})$$
$$\beta^{(i)} = I, II, III, IV, \ i = 1, 2, \ldots, m,$$

where, $\boldsymbol{\mu}(j; \beta^{(i)})$ *is given by* (2.1), $\boldsymbol{t}^{(i)} = (t_1^{(i)}, \ldots, t_{r_i}^{(i)})$ *and*

$$P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)}) = \sum_{j=0}^{\infty} (p_i z)^j \boldsymbol{t}^{(i)\boldsymbol{\mu}(j+1; \beta^{(i)})}. \tag{4.1}$$

Corollary 4.2 reveals the following compact formula for the double generating function $\Phi(\boldsymbol{t}, z; \boldsymbol{\beta})$.

PROPOSITION 4.1. *The double generating function of* $(\boldsymbol{N}(n, \boldsymbol{k}^{(1)}; \beta^{(1)}), \ldots, \boldsymbol{N}(n, \boldsymbol{k}^{(m)}; \beta^{(m)}))$ *is given by*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) = \frac{1}{1 - p_0 z - \sum_{i=1}^{m} \dfrac{p_i z P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)})}{1 + p_i z P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)})}},$$

where, $P(\boldsymbol{t}^{(i)}, p_i z; \beta^{(i)})$ *is as in* (4.1).

Example 4.1. *Joint distribution of* $(N(n, k_1^{(1)}; \beta^{(1)}), \ldots, N(n, k_1^{(m)}; \beta^{(m)}))$. We consider the joint distribution of $(N(n, k_1^{(1)}; \beta^{(1)}), \ldots, N(n, k_1^{(m)}; \beta^{(m)}))$. The double generating function $\Phi(\boldsymbol{t}, z; \boldsymbol{\beta})$ is given by

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\beta}) = \frac{1}{1 - p_0 z - \sum_{i=1}^{m} \dfrac{p_i z P(t_1^{(i)}, p_i z; \beta^{(i)})}{1 + p_i z P(t_1^{(i)}, p_i z; \beta^{(i)})}},$$

where, for $i = 1, 2, \ldots, m$,

$$P(t_1^{(i)}, p_i z; \beta^{(i)}) = \sum_{j=0}^{\infty} (p_i z)^j t_1^{(i)\mu_1(j+1; \beta^{(i)})}$$

$$= \begin{cases} \dfrac{1-(p_iz)^{k_1^{(i)}-1}+(p_iz)^{k_1^{(i)}-1}t_1^{(i)}(1-p_iz)}{(1-p_iz)(1-(p_iz)^{k_1^{(i)}}t_1^{(i)})} & \beta^{(i)} = I, \\ \dfrac{1-(p_iz)^{k_1^{(i)}-1}(1-t_1^{(i)})}{1-p_iz} & \beta^{(i)} = II, \\ \dfrac{1-(p_iz)^{k_1^{(i)}-1}(1-t_1^{(i)})-p_izt_1^{(i)}}{(1-p_iz)(1-p_izt_1^{(i)})} & \beta^{(i)} = III, \\ \dfrac{1-(p_iz)^{k_1^{(i)}-1}(1-t_1^{(i)})(1-p_iz)}{1-p_iz} & \beta^{(i)} = IV. \end{cases}$$

Remark 4.1. In the case where $m = 2$ $r_1 = 1$ and $r_2 = 1$, Balakrishnan and Koutras (2002) studied the joint distributions of $(N(n,k_1^{(1)};I), N(n,k_1^{(2)};I))$, $(N(n,k_1^{(1)};II), N(n,k_1^{(2)};II))$ and $(N(n,k_1^{(1)};III), N(n,k_1^{(2)};III))$, which are called Type I, Type II and Type III trinomial distributions of order $(k_1^{(1)}, k_1^{(2)})$. The results presented in this section are generalizations of these trinomial distributions.

5. Numerical examples

The results presented in Theorem 3.1 are suitable for computation. In this section, we illustrative how to obtain the multivariate run-related distribution by making use of Theorem 3.1 and a computer algebra system.

Example 5.1. *The joint distribution of* $(N(n,k_1^{(1)};I), N(n,k_1^{(2)};III))$. Under the following initial urn composition and the addition matrix

$$\boldsymbol{b}_0 = (1,6,4), \qquad A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 1 & 5 & 2 \end{pmatrix},$$

we consider the joint distribution of $(N(n,k_1^{(1)};I), N(n,k_1^{(2)};III))$. For $n = 10$, $k_1^{(1)} = 4$ and $k_1^{(2)} = 6$, the values of probabilities are given in Table 1.

Example 5.2. *The joint distribution of* $(N(n,k_1^{(1)};I), N(n,k_1^{(2)};I))$. Under the following initial urn composition and the addition matrix

$$\boldsymbol{b}_0 = (1,2,3), \qquad A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix},$$

Table 1. Values of probabilities in Example 5.1.

	$N(n,k_1^{(1)};I)=0$	$N(n,k_1^{(1)};I)=1$	$N(n,k_1^{(1)};I)=2$	Marginal
$N(n,k_1^{(2)};III)=0$	0.712578	0.267807	0.017249	0.997636
$N(n,k_1^{(2)};III)=1$	0.001710	0.000107	0	0.001817
$N(n,k_1^{(2)};III)=2$	0.000427	0	0	0.000427
$N(n,k_1^{(2)};III)=3$	0.000095	0	0	0.000095
$N(n,k_1^{(2)};III)=4$	0.000019	0	0	0.000019
$N(n,k_1^{(2)};III)=5$	0.383×10^{-6}	0	0	0.383×10^{-6}
Marginal	0.714834	0.267915	0.017249	1

we consider the joint distribution of $(N(n, k_1^{(1)}; I), N(n, k_1^{(2)}; I))$. For $n = 10$, $k_1^{(1)} = 2$ and $k_1^{(2)} = 3$, the p.g.f. is

$$\begin{aligned}\phi_{10}(\boldsymbol{b}_0, \boldsymbol{t}; \boldsymbol{\beta}) = {} & 0.06023023367 t_1^{(1)^2} t_1^{(2)} + 0.0005015316096 t_1^{(2)^3} + 0.02423774105 t_1^{(1)^4} \\ & + 0.01038354171 t_1^{(1)} t_1^{(2)^2} + 0.01632694797 t_1^{(2)^2} + 0.1103976042 \\ & + 0.2170400500 t_1^{(1)^2} + 0.2339697290 t_1^{(1)} + 0.08736207037 t_1^{(2)} \\ & + 0.1221110338 t_1^{(1)} t_1^{(2)} + 0.1051554984 t_1^{(1)^3} + 0.009732021762 t_1^{(1)^3} t_1^{(2)} \\ & + 0.0009754937684 t_1^{(1)^2} t_1^{(2)^2} + 0.001576502720 t_1^{(1)^5}.\end{aligned}$$

For $n = 50$, $k_1^{(1)} = 2$ and $k_1^{(2)} = 3$, we give Fig. 1, which is the three-dimensional plot of the exact joint probability function of $(N(n, k_1^{(1)}; I), N(n, k_1^{(2)}; I))$.

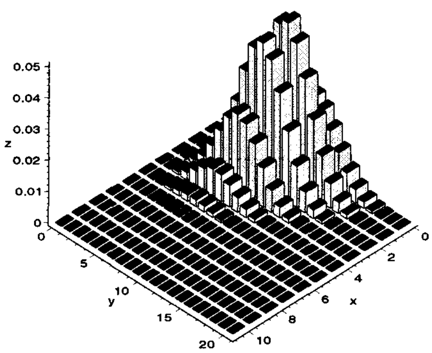

Fig. 1. Probability function for $k_1^{(1)} = 2$, $k_1^{(2)} = 3$ and $n = 50$.

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