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A Generalized Subspace Approach for Mobile Positioning With Time-of-Arrival Measurements

H. C. So, *Member, IEEE*, and Frankie K. W. Chan

***Abstract*—The problem of locating mobile terminals has received considerable attention particularly in the field of wireless communications. In this correspondence, a simple subspace-based algorithm for mobile positioning with the use of time-of-arrival measurements deduced from signals received at three or more reference base stations is derived and analyzed. It is shown that the proposed approach is a generalization of the mobile localization method based on multidimensional similarity analysis. Computer simulations are included to contrast the estimator performance with Cramér–Rao lower bound.**

***Index Terms*—Fast algorithm, mobile terminal, position estimation, range measurements.**

I. Introduction

Mobile terminal (MT) positioning has been receiving considerable interest, especially after the Federal Communications Commission in the United States has adopted rules to improve the Emergency 911 (E-911) services by mandating the accuracy of locating an E-911 caller to be within a specified range, even for a wireless phone user [1]. Apart from emergency assistance, mobile position information is also the key enabler for a large number of innovative applications such as personal localization and monitoring, fleet management, asset tracking, travel services, location-based advertising, and billing [2].

Common positioning approaches [3] are based on time-of-arrival (TOA), received signal strength, time-difference-of-arrival, and/or angle-of-arrival measurements determined from the MT signal received at several reference base stations (BSs) with known locations. In this correspondence, we focus on two-dimensional (2-D) MT localization given the TOA information. In the TOA method, the one-way propagation time of the signal travelling between the MT and each of the BSs is measured. Each TOA measurement then provides a circle centered at the BS on which the MT must lie. With three or more BSs, the measurements are converted into a set of circular equations, from which the MT position can be determined with the knowledge of the BS geometry.

The optimum TOA-based localization approach involves solving the nonlinear circular equations in an iterative manner, and commonly used techniques [4] include linearization via Taylor-series expansion, the steepest descent method, and Newton-type iteration. However, this approach is computationally intensive and sufficiently precise initial estimates are required to obtain the global solution. On the other hand, computationally efficient but suboptimum position estimators, which allow real-time realization as well as ensure global convergence, have also been proposed in the literature [5]–[9]. In the least-squares (LS) calibration method [5], the nonlinear equations are reorganized into a set of linear equations via introduction of an extra variable, which is a function of the source position, and these linear equations are then

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The authors are with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: hcso@ee.cityu.edu.hk; k.w.chan@student.cityu.edu.hk).

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solved straightforwardly by using LS. Alternatively, the common variable in the linear equations can be eliminated via subtraction of each equation from all others, and this technique is referred to as the *linear least-squares estimator* [6]. Based on a new geometrical formulation, Caffery has proposed the straight-lines-of-position (SLOP) method [7] where a generalized set of linear equations is constructed. Instead of forming linear equations, computationally simple subspace based positioning algorithms [8], [9] have also been derived using the squared TOA measurements or, equivalently, the squared distance measurements. In [8], classical multidimensional scaling (MDS) is modified, while Wan *et al.* [9] have derived a noise subspace-based algorithm with a linear constraint for the three-BS case. Inspired by the multidimensional similarity matrix utilized in [9], we develop a new subspace based localization approach which allows any number of BSs.

The rest of the correspondence is organized as follows. The development of the subspace based mobile positioning algorithm is presented in Section II. When there are three receiving BSs, it is proved that the proposed method is identical to [9]. The bias and variance for the position estimate are also derived. Simulation results are included in Section III to validate our theoretical calculation and to evaluate the accuracy of the subspace positioning technique. Finally, concluding remarks are provided in Section IV.

II. Algorithm Development

Let $\mathbf{z} = [x\ y]$ be the MT position to be determined and the known coordinates of the ith BS be $[x_i\ y_i], i = 1, 2, \ldots, M$, where $M \geq 3$ is the total number of receiving BSs. The distances between the MT and BSs are determined from the corresponding TOA measurements, which are modeled as

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} + q_i, \quad i = 1, 2, \ldots, M \tag{1}$$

where q_i is the noise in r_i or range error at the ith BS. In our study, we assume line-of-sight propagation between the MT and all BSs such that each q_i is a zero-mean random process.

Define an $M \times 2$ matrix $\mathbf{X}$ of the form

$$\mathbf{X} = \begin{bmatrix} x_1 - x & y_1 - y \\ x_2 - x & y_2 - y \\ \vdots & \vdots \\ x_M - x & y_M - y \end{bmatrix} \tag{2}$$

which is parameterized by $[x \quad y]$. Following [9], we define the multidimensional similarity matrix, namely, $\mathbf{D} = \mathbf{X}\mathbf{X}^T$, where T denotes transpose operation, which is a rank-2 symmetric matrix and its (m, n) entry is given by

$$[\mathbf{D}]_{m,n} = 0.5\left(d_m^2 + d_n^2 - d_{mn}^2\right) \tag{3}$$

where d_m denotes the noise-free version of r_m, $m = 1, 2, \ldots, M$, and $d_{mn} = d_{nm} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}$ is of known value because it represents the distance between the mth and nth BSs. Although the exact form of $\mathbf{D}$ is unavailable, we are able to construct its approximate version at sufficient small noise conditions, denoted by $\hat{\mathbf{D}}$, with the use of the noisy $\{r_m\}$ and noise-free $\{d_{mn}\}$. As a generalization of [9], the (m, n) entry of $\hat{\mathbf{D}}$ is

$$[\hat{\mathbf{D}}]_{m,n} = 0.5\left(r_m^2 + r_n^2 - d_{mn}^2\right). \tag{4}$$

Decomposing the symmetric $\hat{\mathbf{D}}$ by eigenvalue factorization yields

$$\hat{\mathbf{D}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \tag{5}$$

where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M)$ is the diagonal matrix of eigenvalues of $\hat{\mathbf{D}}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$, and $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_M]$ is an orthonormal matrix whose columns are the corresponding eigenvectors. Since the rank of the ideal $\mathbf{D}$ is 2, an LS estimate of $\mathbf{X}$ up to a rotation, denoted by $\hat{\mathbf{X}}^{\mathbf{r}}$, can be computed as [8], [10]

$$\hat{\mathbf{X}}^{\mathbf{r}} = \arg\min_{\tilde{\mathbf{X}}} \|\hat{\mathbf{D}} - \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\|_F^2 = \mathbf{U}_s\mathbf{\Lambda}_s^{\frac{1}{2}} \tag{6}$$

where $\tilde{\mathbf{X}}$ is the variable matrix for $\mathbf{X}$, $\|\|_F$ represents the Frobenius norm, $\mathbf{U}_s = [\mathbf{u}_1\ \mathbf{u}_2]$ corresponds to the signal subspace, and $\mathbf{\Lambda}_s^{(1/2)} = \mathrm{diag}(\lambda_1^{(1/2)}, \lambda_2^{(1/2)})$. It is noteworthy that (6) is an important result in classical MDS, and interested readers can refer to [10] for its derivation. The relationship between $\hat{\mathbf{X}}^{\mathbf{r}}$ and $\mathbf{X}$ is then

$$\mathbf{X} \approx \hat{\mathbf{X}}^{\mathbf{r}}\Omega \tag{7}$$

where $\mathbf{\Omega}$ is an unknown rotation matrix to be determined. In the absence of noise, we have $\mathbf{X} = \hat{\mathbf{X}}^{\mathbf{r}}\mathbf{\Omega}$. From (7), an optimal estimate of $\mathbf{\Omega}$ in the LS sense is easily shown to be

$$\hat{\mathbf{\Omega}} = (\hat{\mathbf{X}}^{\mathbf{r}T}\hat{\mathbf{X}}^{\mathbf{r}})^{-1}\hat{\mathbf{X}}^{\mathbf{r}T}\mathbf{X} = \mathbf{\Lambda}_s^{-\frac{1}{2}}\mathbf{U}_s^T\mathbf{X}. \tag{8}$$

Substituting (8) into (7), we get

$$\mathbf{X} \approx \mathbf{U}_s\mathbf{U}_s^T\mathbf{X}. \tag{9}$$

It is observed that there are M linear equations in terms of x and another M similar linear equations as a function of y, and thus we can apply the LS technique to solve for the MT position from the overdetermined system of (9). Alternatively, we can perform the position estimation based on the noise subspace as follows. Since $\mathbf{I}_M - \mathbf{U}_s\mathbf{U}_s^T = \mathbf{U}_n\mathbf{U}_n^T$, where $\mathbf{I}_M$ is the $M \times M$ identity matrix and $\mathbf{U}_n = [\mathbf{u}_3\ \mathbf{u}_4 \cdots \mathbf{u}_M]$ denotes the noise subspace, (9) is rearranged as

$$\mathbf{U}_n\mathbf{U}_n^T\mathbf{1}_M[x \quad y] \approx \mathbf{U}_n\mathbf{U}_n^T \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix} \tag{10}$$

where $\mathbf{1}_M$ denotes an $M \times 1$ vector with all elements equal to unity. Solving (10) in the LS sense, the position estimate, denoted by $\hat{\mathbf{z}} = [\hat{x} \quad \hat{y}]$, is

$$\hat{\mathbf{z}} = \left(\mathbf{U}_n\mathbf{U}_n^T\mathbf{1}_M\right)^{\dagger}\mathbf{U}_n\mathbf{U}_n^T \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix} = \frac{\mathbf{1}_M^T\mathbf{U}_n\mathbf{U}_n^T}{\mathbf{1}_M^T\mathbf{U}_n\mathbf{U}_n^T\mathbf{1}_M} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix} \tag{11}$$

where † represents the pseudoinverse. In particular, when there are only three BSs, we have $\mathbf{U}_n = \mathbf{u}_3$, and (11) is then simplified to

$$\hat{\mathbf{z}} = \frac{\mathbf{u}_3^T}{\mathbf{u}_3^T \mathbf{1}_M} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix} \tag{12}$$

which is exactly the solution given by [9]. It is noteworthy that, unlike [9], which utilizes a linear constraint, we start from the signal subspace, and we can allow any number of BSs as long as $M \geq 3$, while the former only operates for $M = 3$.

When $\hat{\mathbf{\Omega}} \rightarrow \mathbf{\Omega}$, which is reasonably true for a sufficiently small noise condition, our proposed subspace solution can be approximated as

$$[\hat{x} \quad \hat{y}] = \arg\min_{\tilde{x}, \tilde{y}} J \tag{13}$$

where $J = \|\hat{\mathbf{D}} - \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\|_F^2$ such that $\tilde{\mathbf{X}}$ is now restricted as

$$\tilde{\mathbf{X}} = \begin{bmatrix} x_1 - \tilde{x} & y_1 - \tilde{y} \\ x_2 - \tilde{x} & y_2 - \tilde{y} \\ \vdots & \vdots \\ x_M - \tilde{x} & y_M - \tilde{y} \end{bmatrix}. \tag{14}$$

The biases of $\hat{x}$ and $\hat{y}$ in (13) are given by [8]

$$\begin{aligned} \mathbf{E}\hat{\mathbf{z}} - \mathbf{z} &\approx -\mathbf{E}\left\{\left(\frac{\partial^2 J}{\partial \tilde{\mathbf{z}} \tilde{\mathbf{z}}^T}\right)^{-1}\right\} \mathbf{E}\left(\frac{\partial J}{\partial \tilde{\mathbf{z}}}\right)\Bigg|_{\tilde{\mathbf{z}}=\mathbf{z}} \\ &\approx -\left(\mathbf{E}\left(\frac{\partial^2 J}{\partial \tilde{\mathbf{z}} \tilde{\mathbf{z}}^T}\right)\right)^{-1} \mathbf{E}\left(\frac{\partial J}{\partial \tilde{\mathbf{z}}}\right)\Bigg|_{\tilde{\mathbf{z}}=\mathbf{z}} \end{aligned} \tag{15}$$

where $\tilde{\mathbf{z}} = [\tilde{x} \; \tilde{y}]$ and $\mathbf{E}$ is the expectation operator. Note that the interchange of the expectation operator and matrix inverse is valid for sufficiently high signal-to-noise ratio (SNR) conditions.

On the other hand, the variances of $\hat{x}$ and $\hat{y}$ in (13) are given by the first and second diagonal elements of the following 2×2 matrix, respectively [8]:

$$\mathbf{E}\left(\frac{\partial^2 J}{\partial \tilde{\mathbf{z}} \tilde{\mathbf{z}}^T}\right)\Bigg|_{\tilde{\mathbf{z}}=\mathbf{z}}^{-1} \mathbf{E}\left\{\left(\frac{\partial J}{\partial \tilde{\mathbf{z}}}\right)\left(\frac{\partial J}{\partial \tilde{\mathbf{z}}}\right)^T\right\}\Bigg|_{\tilde{\mathbf{z}}=\mathbf{z}} \mathbf{E}\left(\frac{\partial^2 J}{\partial \tilde{\mathbf{z}} \tilde{\mathbf{z}}^T}\right)\Bigg|_{\tilde{\mathbf{z}}=\mathbf{z}}^{-1}. \tag{16}$$

For simplicity but without loss of generality, we consider uncorrelated $\{q_i\}$ with $\mathbf{E}\{q_i^2\} = \sigma_i^2, i = 1, 2, \ldots, M$, and the required terms to compute (15) and (16) have been calculated in the Appendix.

In the following, the computational complexity of the modified MDS, SLOP, as well as proposed methods is analyzed in terms of number of floating-point operations (FLOPS). We only investigate the dominant operations involved, namely, the singular value decomposition (SVD) and LS computations. The calculation of FLOPS is briefly described as follows. A dot product of length n involves $2n$ FLOPS because there are n multiplications and n additions. Furthermore, performing SVD of a matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$ needs $4m^2n + 22n^2$ FLOPS. Here, QR factorization is utilized to obtain the LS solution of $\mathbf{Ax} \approx \mathbf{b}$, although other approaches can be considered. In summary, the LS computation requires $2mn^2$ FLOPS for the QR factorization of $\mathbf{A} = \mathbf{QR}, 2mn$ FLOPS for vector construction of $\mathbf{b} = \mathbf{Q}^T\mathbf{b}$, as well as n^2 FLOPS for the backward substitution of $\mathbf{Rx} = \mathbf{b}$. As a result, the numbers of FLOPS in the SLOP, modified MDS, and proposed methods are $6M^2 - 6M + 4, 4M^3 + 34M^2 + 64M + 162$, and $4M^3 + 27M^2 - 4M - 13$, respectively. It is observed that the SLOP approach is the most computationally efficient with complexity of $O(M^2)$ while the complexity of the remaining two methods is of $O(M^3)$, but their difference will not be significant when the BS number is small.

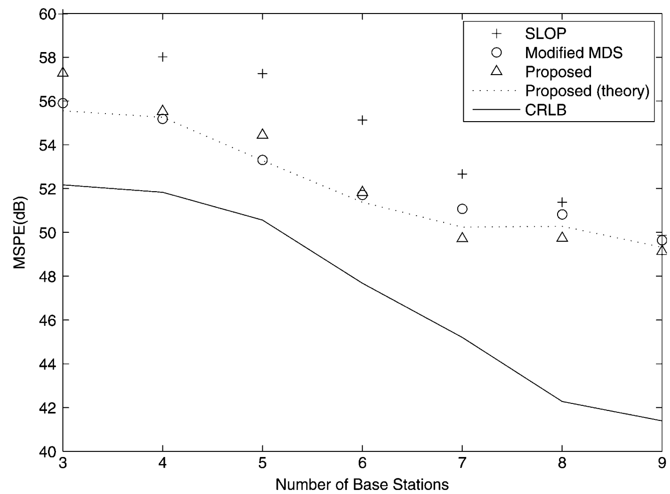

Fig. 1. Mean-square position error versus number of BSs when $\mathbf{z} = [-5000 \; -500]$ m.

III. Numerical Examples

Computer simulation had been conducted to evaluate the performance of the proposed TOA-based positioning approach. We compared the mean square position errors (MSPEs) of the subspace estimator with the modified MDS [8], the SLOP method [7], as well as Cramér–Rao lower bound (CRLB) in MT localization. For presentation simplicity, the SNR in each range measurement was assigned identical with SNR $= d_i^2/\sigma_i^2$ where σ_i^2 was the variance of the zero-mean white Gaussian range error q_i. All results were averages of 10 000 independent runs.

In the first scenario, the MT was fixed at $[-5000 \quad -500]$ m, and we started with three BSs with coordinates $[0 \; 0]$ m, $[0 \; 6000]$ m and $[6000 \quad 6000]$ m. The BSs with coordinates $[6000 \; 0]$ m, $[6000 \quad -6000]$ m, $[-6000 \; 0]$ m, $[-6000 \quad -6000]$ m, $[-6000 \; 0]$ m, and $[-6000 \quad 6000]$ m were then added successively. Fig. 1 shows the MSPEs versus number of BSs when the SNR was kept at 30 dB. It is seen that the proposed algorithm had similar MSPEs with the modified MDS but it outperformed the SLOP method for $3 \leq M \leq 7$, and all of them were suboptimal estimators since their performance could not attain the CRLB. We also observe that the theoretical mean-square error, which was computed as the sum of variance and squared bias based on (15) and (16), agreed with the simulation results. Fig. 2 shows the MSPEs versus SNR when $M = 9$ with the BS geometry equalled to the previous test. We see that for SNR $\geq$ 15 dB, the MSPEs of the subspace method were close to those of the modified MDS and SLOP methods and decreased with the CRLB. However, for smaller SNR conditions, the proposed method was inferior to the SLOP technique, which is a common phenomenon for subspace-based approaches. The theoretical mean-square error development was also confirmed for sufficiently small noise conditions, namely, SNR > 10 dB. It is worthy to point out that the variance dominated the squared bias for sufficiently high-SNR conditions, and thus we may ignore the bias of (15) in the computation for simplicity.

The above two tests were repeated when the position of the MT was uniformly distributed within the square bounded by $[-3000 \quad -3000]$ m, $[-3000 \; 3000]$ m, $[3000 \quad -3000]$ m, and $[3000 \; 3000]$ m in each trial. In Fig. 3, it is seen that the proposed algorithm outperformed the SLOP method for $3 \leq M \leq 6$, and all had comparable performance for $7 \leq M \leq 9$. While in Fig. 4, we observe that for SNR $\geq$ 15 dB,

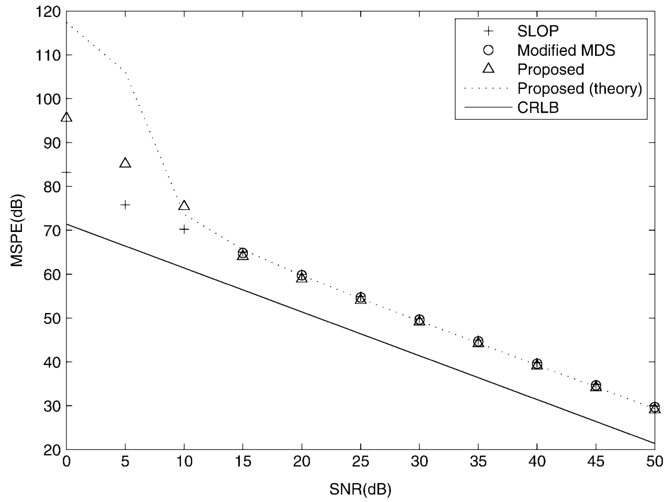

Fig. 2. Mean-square position error versus SNR when $\mathbf{z} = [-5000 \ -500]$ m.

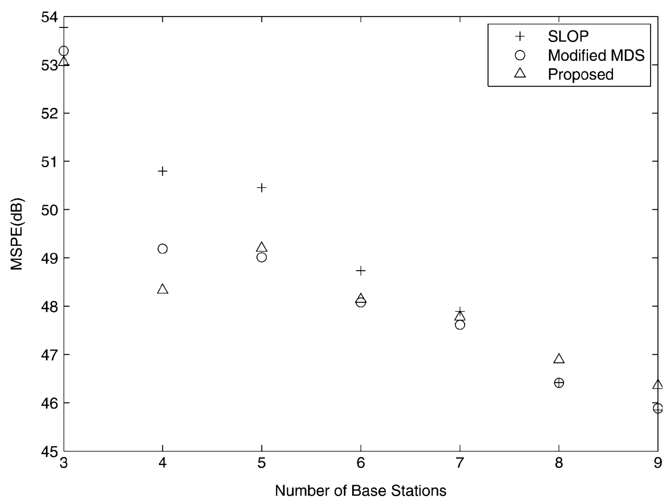

Fig. 3. Mean-square position error versus number of BSs for randomly distributed MT.

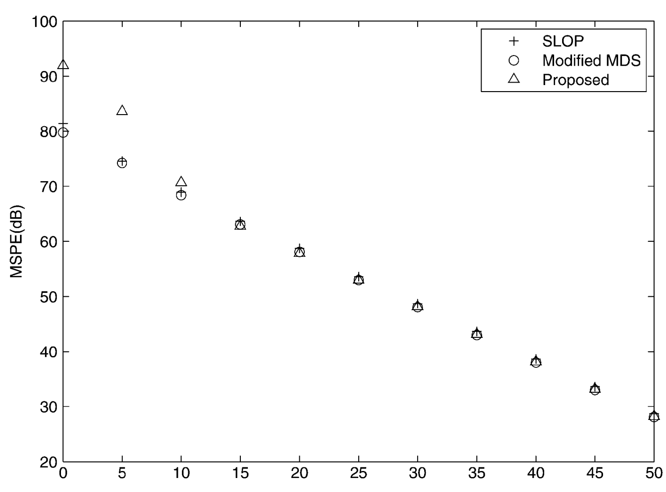

Fig. 4. Mean-square position error versus SNR for randomly distributed MT.

the MSPEs of the proposed estimator were more or less the same as the modified MDS algorithm but were a bit smaller than those of the SLOP method.

IV. CONCLUDING REMARKS

A novel subspace-based approach has been devised for mobile terminal localization using distance measurements. It is shown that the proposed approach, which can be used with at least three receiving base stations, is a generalized version of the mobile localization method [9] based on multidimensional similarity analysis. Theoretical positioning accuracy of the subspace estimator is produced and verified by computer simulations.

APPENDIX

The required terms for computing (15) and (16) are determined as follows. First, we notice that

$$J = \left\| \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T - \hat{\mathbf{D}} \right\|_F^2 = \mathrm{tr}\{\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T - 2\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\hat{\mathbf{D}} + \hat{\mathbf{D}}^T\hat{\mathbf{D}}\} \tag{A1}$$

where tr denotes the trace operator. Then, we have

$$\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}} = \mathbf{K}_x\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\mathbf{K}_x^T \tag{A2}$$

$$\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}} = \mathbf{K}_y\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\mathbf{K}_y^T \tag{A3}$$

$$\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}} = \frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}}\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}} \tag{A4}$$

$$\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}} = \frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}}\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}} \tag{A5}$$

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}^2} = 2\mathbf{K}_x\mathbf{K}_x^T = 2\mathbf{1}_M\mathbf{1}_M^T \tag{A6}$$

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}\partial \tilde{y}} = \mathbf{K}_x\mathbf{K}_y^T + \mathbf{K}_y\mathbf{K}_x^T = \mathbf{0}_{M\times M} \tag{A7}$$

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}^2} = 2\mathbf{K}_y\mathbf{K}_y^T = 2\mathbf{1}_M\mathbf{1}_M^T \tag{A8}$$

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}^2} = 2\mathbf{1}_M\mathbf{1}_M^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T + 2\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{1}_M\mathbf{1}_M^T + 2(\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}})^2 \tag{A9}$$

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}\tilde{y}} = \frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}}\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}} + \frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}}\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}} \tag{A10}$$

and

$$\frac{\partial^2 \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}^2} = 2\mathbf{1}_M\mathbf{1}_M^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T + 2\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{1}_M\mathbf{1}_M^T + 2(\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{y}})^2 \tag{A11}$$

where $\mathbf{K}_x = [-\mathbf{1}_M \quad \mathbf{0}_{M\times 1}], \mathbf{K}_y = [\mathbf{0}_{M\times 1} \quad -\mathbf{1}_M]$ and $\mathbf{0}_{i\times j}$ stands for the $i \times j$ zero matrix. By using (A2) to (A11), we get $(\partial J/\partial \tilde{x}), (\partial J/\partial \tilde{y}), \ (\partial^2 J/\partial \tilde{x}^2), (\partial^2 J)/(\partial \tilde{x}\partial \tilde{y}), \ (\partial^2 J/\partial \tilde{y}^2)$, and their corresponding expected values, which are shown as follows:

$$\frac{\partial J}{\partial \tilde{x}} = 2\mathrm{tr}\left\{(\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T - \hat{\mathbf{D}})\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}}\right\}$$

$$\Rightarrow \mathbf{E}\left(\frac{\partial J}{\partial \tilde{x}}\right) = \mathrm{tr}\left\{\left(\sum \mathbf{1}_M^T + \mathbf{1}_M \sum^T\right)\left(\mathbf{K}_x\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\mathbf{K}_x^T\right)\right\}$$

$$\Rightarrow \mathbf{E}\left(\frac{\partial J}{\partial \tilde{y}}\right) = \mathrm{tr}\left\{\left(\sum \mathbf{1}_M^T + \mathbf{1}_M \sum^T\right)\left(\mathbf{K}_y\tilde{\mathbf{X}}^T + \tilde{\mathbf{X}}\mathbf{K}_y^T\right)\right\}.$$

On the other hand

$$\mathbf{E}\left(\frac{\partial J}{\partial \tilde{x}}\right)^2\Bigg|_{\tilde{x}=x} = 4\mathbf{E}\left\{\mathrm{tr}^2\left(\mathbf{Q}\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\right)\right\} = 4\mathbf{E}\left\{\mathrm{tr}^2\left[0.5\mathbf{q}\mathbf{1}_M^T + 0.5\mathbf{1}_M\mathbf{q}^T\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\right]\right\} = 4\mathbf{E}\left\{\left(\mathbf{1}_M^T\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{q}\right)^2\right\} = 4\left(\mathbf{1}_M^T\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{\Sigma}\right)^2 + 4\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{1}_M\right)\odot\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{1}_M\right)\right\}^T \times(2\mathbf{\Sigma}\odot\mathbf{\Sigma}+4\mathbf{r}\odot\mathbf{\Sigma})$$

where $\odot$ denotes the Schur product, $\mathbf{Q} = \hat{\mathbf{D}} - \mathbf{X}\mathbf{X}^T$, $\mathbf{q} = [2r_1q_1+q_1^2 \quad 2r_2q_2+q_2^2 \quad \cdots \quad 2r_Mq_N+q_N^2]^T$, $\mathbf{\Sigma} = [\sigma_1^2 \quad \sigma_2^2 \quad \cdots \quad \sigma_M^2]^T$, and $\mathbf{r} = [r_1^2 \quad r_2^2 \quad \cdots \quad r_N^2]^T$. Similarly, we have

$$\mathbf{E}\left(\frac{\partial J}{\partial \tilde{y}}\right)^2\Bigg|_{\tilde{y}=y} = 4\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\mathbf{1}_M\right)^T\mathbf{\Sigma}\right\}^2 + 4\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\mathbf{1}_M\right)\odot\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\mathbf{1}_M\right)\right\}^T \times(2\mathbf{\Sigma}\odot\mathbf{\Sigma}+4\mathbf{r}\odot\mathbf{\Sigma})$$

and

$$\mathbf{E}\left(\frac{\partial J}{\partial \tilde{x}}\frac{\partial J}{\partial \tilde{y}}\right)\Bigg|_{\tilde{x}=x,\tilde{y}=y} = 4\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{1}_M\right)^T\mathbf{\Sigma}\right\}\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\mathbf{1}_M\right)^T\mathbf{\Sigma}\right\} + 4\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\mathbf{1}_M\right)\odot\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\mathbf{1}_M\right)\right\}^T \times(2\mathbf{\Sigma}\odot\mathbf{\Sigma}+4\mathbf{r}\odot\mathbf{\Sigma}).$$

For the second-order derivatives, we have

$$\frac{\partial^2 J}{\partial \tilde{x}^2} = 2\mathrm{tr}\left\{2\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{1}_M\mathbf{1}_M{}^T + \left(\frac{\partial \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T}{\partial \tilde{x}}\right)^2 - 2\hat{\mathbf{D}}\mathbf{1}_M\mathbf{1}_M{}^T\right\}$$
$$\Rightarrow \frac{\partial^2 J}{\partial \tilde{x}^2}\Bigg|_{\tilde{x}=x} = 2\mathrm{tr}\left\{\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\right)^2 - 2\mathbf{Q}\mathbf{1}_M\mathbf{1}_M{}^T\right\}$$
$$\Rightarrow \mathbf{E}\ \frac{\partial^2 J}{\partial \tilde{x}^2}\Bigg|_{\tilde{x}=x} = 2\mathrm{tr}\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\right)^2 - 4M\mathbf{1}_M^T\mathbf{\Sigma}.$$

By similar derivation, we obtain

$$\mathbf{E}\left(\frac{\partial^2 J}{\partial \tilde{y}^2}\right)\Bigg|_{\tilde{y}=y} = 2\mathrm{tr}\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\right)^2 - 4M\mathbf{1}_M^T\mathbf{\Sigma}$$

and

$$\mathbf{E}\left(\frac{\partial^2 J}{\partial \tilde{x}\partial \tilde{y}}\right)\Bigg|_{\tilde{x}=x,\tilde{y}=y} = 2\mathrm{tr}\left(\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial x}\frac{\partial \mathbf{X}\mathbf{X}^T}{\partial y}\right).$$

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