A generating function for the spherical harmonics in p dimensions

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Received 13 November 2012; accepted 25 January 2013

A generating function for the spherical harmonics in three or more dimensions is given. This function allows us to find in a simple manner the explicit expressions for the spherical harmonics in three dimensions. The generating functions for the spherical harmonics given here are analogous to certain well-known generating functions for solutions of the wave equation.

Keywords: Spherical harmonics; generating functions; Laplace equation; wave equation.

Se da una función generatriz para los armónicos esféricos en tres o más dimensiones. Esta función permite hallar en una forma simple las expresiones explícitas para los armónicos esféricos en tres dimensiones. Las funciones generatrices para los armónicos esféricos dadas aquí son análogas a ciertas funciones generatrices bien conocidas para soluciones de la ecuación de onda.

Descriptores: Armónicos esféricos; funciones generatrices; ecuación de Laplace; ecuación de onda.

PACS: 02.30.Gp; 02.30.Jr; 02.30.Uu

1. Introduction

The standard spherical harmonics, Y_{lm}, arise in the solution by separation of variables in spherical coordinates of various partial differential equations (PDEs), such as the Laplace equation, the wave equation, and the Schrödinger equation for a particle in a central potential (see, *e.g.*, Refs. 1, 2). These functions can also be defined as the common eigenfunctions of the square of the orbital angular momentum operator $\mathbf{L}^2$ and the z-component of the angular momentum operator L_z (see, *e.g.*, Refs. 2,3).

Apart from a normalization constant, $Y_{lm}(\theta, \phi)$ is the product of the associated Legendre function $P_l^m(\cos\theta)$ and $\exp(\mathrm{i}m\phi)$. As is well known, the explicit expression of the functions Y_{lm} can be obtained with the aid of the ladder operators $L_x \pm \mathrm{i}L_y$ (see, *e.g.*, Refs. 2, 4), and the associated Legendre functions can be obtained by means of differentiations, making use of a Rodrigues formula. Actually, there exist an assortment of expressions for the standard spherical harmonics, including differential and integral expressions, power series expressions, and relations to other functions, many of which are given, without derivation, in Ref. 5.

The spherical harmonics can be defined for any dimension $p \geqslant 2$ in the following manner. Let $(x^1, x^2, \ldots, x^p)$ be Cartesian coordinates in $\mathbb{R}^p$, and

$$r \equiv \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^p)^2};$$

if f is a homogeneous function of degree n of the x^i that satisfies the Laplace equation, then f/r^n is a spherical harmonic of degree n (see also Ref. 4). The spherical harmonics for $p > 3$ are also useful in mathematical physics; for instance, the spherical harmonics in four dimensions appear in the solution of the Schrödinger equation for the hydrogen atom (see, *e.g.*, Refs. 6-8) and the spherical top, and are related to the Wigner functions [5,6,8] and the spin-weighted spherical harmonics (see, *e.g.*, Refs. 9,10). (Alternatively, the spherical harmonics in p dimensions can be defined as the eigenfunctions of the Laplace–Beltrami operator of the sphere

$$\mathrm{S}^{p-1} \equiv \{(x^1, x^2, \ldots, x^p) \in \mathbb{R}^p | (x^1)^2 + (x^2)^2 + \cdots + (x^p)^2 = 1\}.$$

See Eqs. (16) and (18) below.)

The aim of this paper is to present a generating function for the spherical harmonics in three and four dimensions which enables us to readily find some properties of these functions and the explicit expression of the spherical harmonics in three dimensions. This generating function is closely related to expressions given in Refs. 5 and 11 for the spherical harmonics in three dimensions, and in Ref. 4 for the Legendre polynomials in p dimensions. It is also shown that similar results can be derived for the wave equation, which is the analog of the Laplace equation when the signature of the metric is Lorentzian. In Sec. 2, we establish the basic result of this paper and develop the generating function for the spherical harmonics in three dimensions. In Sec. 3 the spherical harmonics in four dimensions are considered and an integral representation for them is derived. In Sec. 4 we consider the wave equation and identify two well-known expansions of plane waves as generating functions of separable solutions of the wave equation in two and three dimensions, which are analogous to the generating functions of spherical harmonics presented here.

2. Solutions of the Laplace equation

Throughout this paper we shall consider real- or complex-valued functions defined in $\mathbb{R}^p$. If the components of the metric tensor, g_{ij}, with respect to a coordinate system $(x^1, x^2, \ldots, x^p)$ are constant, then the Laplace operator is given by

$$\Delta = g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j},$$

where (g^{ij}) is the inverse of the matrix (g_{ij}), and there is sum over repeated indices. The basic result to be employed in what follows is given by the following proposition.

Proposition 1. The homogeneous function $(k_1x^1 + k_2x^2 + \cdots + k_px^p)^n$ is a solution of the Laplace equation, $\Delta f = 0$, for $n = 0, 1, 2, \ldots$, if and only if the constants k_i are the components of a null vector

$$g^{ij}k_ik_j = 0. \tag{1}$$

Indeed, a straightforward computation yields

$$\begin{aligned} g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(k_mx^m)^n &= g^{ij}\frac{\partial}{\partial x^i}[n(k_mx^m)^{n-1}k_j] \\ &= n(n-1)(k_mx^m)^{n-2}g^{ij}k_ik_j. \end{aligned}$$

In the case of $\mathbb{R}^3$ (with its usual metric tensor), Eq. (1) has no nontrivial real solutions; however, for any $u \in \mathbb{R}$, the vector $(k_1, k_2, k_3) = (\mathrm{i}\cos u, \mathrm{i}\sin u, 1)$ is null and therefore, for $l = 0, 1, 2, \ldots$, $(z + \mathrm{i}x\cos u + \mathrm{i}y\sin u)^l$ is a (complex-valued) solution of the Laplace equation (which depends on the parameter u).

Since

$$\begin{aligned} z + \mathrm{i}x\cos u + \mathrm{i}y\sin u &= z \\ &+ \frac{\mathrm{i}}{2}(x - \mathrm{i}y)\mathrm{e}^{\mathrm{i}u} + \frac{\mathrm{i}}{2}(x + \mathrm{i}y)\mathrm{e}^{-\mathrm{i}u}, \end{aligned}$$

for x, y, z fixed and $l \in \mathbb{N}$, $(z + \mathrm{i}x\cos u + \mathrm{i}y\sin u)^l$ must be a linear combination of $\{\mathrm{e}^{-\mathrm{i}lu}, \mathrm{e}^{-\mathrm{i}(l-1)u}, \ldots, \mathrm{e}^{\mathrm{i}lu}\}$, with coefficients that depend on x, y, z,

$$(z + \mathrm{i}x\cos u + \mathrm{i}y\sin u)^l = \sum_{m=-l}^{l} F_{lm}(x, y, z)\,\mathrm{e}^{-\mathrm{i}mu}. \tag{2}$$

According to Proposition 1, the right-hand side of Eq. (2) satisfies the Laplace equation, hence

$$\sum_{m=-l}^{l}[\Delta F_{lm}(x, y, z)]\,\mathrm{e}^{-\mathrm{i}mu} = 0$$

and, by virtue of the linear independence of the set $\{\mathrm{e}^{-\mathrm{i}lu}, \mathrm{e}^{-\mathrm{i}(l-1)u}, \ldots, \mathrm{e}^{\mathrm{i}lu}\}$, it follows that $\Delta F_{lm} = 0$. Furthermore, expressing x, y, z in terms of the spherical coordinates r, θ, ϕ we see that

$$\begin{aligned} z + \mathrm{i}x\cos u + \mathrm{i}y\sin u = r\Bigg[\cos\theta \\ + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(\phi-u)} + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(\phi-u)}\Bigg], \end{aligned}$$

and therefore the right-hand side of Eq. (2) depends on ϕ and u only through their difference; thus

$$F_{lm}(r, \theta, \phi) = r^l f_{lm}(\theta)\,\mathrm{e}^{\mathrm{i}m\phi},$$

where f_{lm} is a function of θ only, which must be proportional to the associated Legendre function $P_l^m(\cos\theta)$. Hence, we conclude that

$$\begin{aligned} &\left[\cos\theta + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(\phi-u)} + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(\phi-u)}\right]^l \\ &\qquad = \sum_{m=-l}^{l} N_{lm}Y_{lm}(\theta, \phi)\,\mathrm{e}^{-\mathrm{i}mu}, \end{aligned} \tag{3}$$

where the N_{lm} are constants. (Equations (3) and (11) are equivalent to Eq. (16) of Sec. 5.1 of Ref. 5.)

Thus, for instance, the explicit expression of $Y_{31}(\theta, \phi)$ is obtained by considering those terms in the expansion of

$$\left[\cos\theta + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(\phi-u)} + \frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(\phi-u)}\right]^3$$

that contain the factor $\mathrm{e}^{-\mathrm{i}u}$; in this manner we find that

$$\begin{aligned} N_{31}Y_{31}(\theta, \phi) &= 3\cos\theta\cos\theta\frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ &+ 3\frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\frac{\mathrm{i}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ &= \mathrm{i}\frac{3}{8}(5\cos^2\theta - 1)\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}. \end{aligned}$$

The value of the factor N_{lm} is obtained below [see Eq. (11)].

Similarly, from Eq. (3) we see that, for $l = 0, 1, 2, \ldots$,

$$\left(\frac{\mathrm{i}}{2}\right)^l \sin^l\theta\,\mathrm{e}^{\mathrm{i}l\phi} = N_{ll}Y_{ll}(\theta, \phi).$$

The modulus of N_{ll} can be readily obtained from the normalization condition

$$1 = \int_{\mathrm{S}^2}|Y_{ll}|^2\,\mathrm{d}\Omega,$$

where $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is the solid angle element and the integral is over the unit sphere $r = 1$. Hence,

$$\begin{aligned} 1 &= \frac{1}{2^{2l}|N_{ll}|^2}\int_{\mathrm{S}^2}\sin^{2l}\theta\,\mathrm{d}\Omega = \frac{1}{2^{2l}|N_{ll}|^2} \\ &\times \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin^{2l+1}\theta\,\mathrm{d}\theta = \frac{4\pi(l!)^2}{|N_{ll}|^2(2l+1)!}. \end{aligned}$$

Thus, following the Condon-Shortley phase convention [3,5], N_{ll} is given by

$$N_{ll} = (-\mathrm{i})^l l!\sqrt{\frac{4\pi}{(2l+1)!}}. \tag{4}$$

Making use of the orthogonality of the set $\{e^{-ilu}, e^{-i(l-1)u}, \ldots, e^{ilu}\}$ on the interval $[0, 2\pi]$, from Eq. (3) we obtain the integral representation

$$Y_{lm}(\theta,\phi) = \frac{1}{2\pi N_{lm}} \int_0^{2\pi} \left[\cos\theta + \frac{i}{2}\sin\theta\, e^{-i(\phi-u)} + \frac{i}{2}\sin\theta\, e^{i(\phi-u)}\right]^l e^{imu}\, du \quad (5)$$

or, by means of the change of variable $w \equiv u - \phi$,

$$Y_{lm}(\theta,\phi) = \frac{1}{2\pi N_{lm}} e^{im\phi} \times \int_0^{2\pi} (\cos\theta + i\sin\theta\cos w)^l\, e^{imw}\, dw. \quad (6)$$

Expression (5) is equivalent to Eq. (7) of Sec. 5.3 of Ref. 5 (cf. also Refs. 11, 14). Hence, the integral in the last equation must be proportional to $P_l^m(\cos\theta)$, and, setting $m = 0$, we conclude that the Legendre polynomials have the integral representation

$$P_l(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta + i\sin\theta\cos w)^l\, dw. \quad (7)$$

The proportionality factor has been determined using the fact that $P_l(1) = 1$. (Equation (7) is known as Laplace's integral representation for the Legendre polynomials [11,12,14]. Cf. also Ref. 4.) Alternatively, from Eq. (7), introducing an auxiliary variable t, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x) t^n &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} (tx + it\sqrt{1-x^2}\,\cos w)^n\, dw \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - tx - it\sqrt{1-x^2}\,\cos w} dw \\ &= \frac{1}{\sqrt{1-2tx+t^2}}, \end{aligned} \quad (8)$$

which is the well-known generating function of the Legendre polynomials.

The value of the constants N_{lm} introduced in Eq. (3) can be readily obtained using the fact that, with the Condon-Shortley phase convention,

$$L_\pm Y_{lm} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, Y_{l,m\pm1}, \quad (9)$$

where $L_\pm = L_x \pm iL_y$ are the usual ladder operators which, in spherical coordinates, have the form [2]

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta \frac{\partial}{\partial\phi}\right).$$

Indeed, letting

$$K(\theta,\phi,u) \equiv \cos\theta + \frac{i}{2}\sin\theta\, e^{-i(\phi-u)} + \frac{i}{2}\sin\theta\, e^{i(\phi-u)}, \quad (10)$$

so that the integrand in Eq. (5) amounts to $[K(\theta,\phi,u)]^l\, e^{imu}$, we find that

$$e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) K(\theta,\phi,u) = ie^{iu}K(\theta,\phi,u) - e^{iu}\frac{\partial}{\partial u}K(\theta,\phi,u)$$

and from Eq. (5) it follows that

$$\begin{aligned}2\pi N_{lm} L_+ Y_{lm} &= \int_0^{2\pi} l[K(\theta,\phi,u)]^{l-1} e^{imu} L_+ K(\theta,\phi,u) du \\ &= \hbar\int_0^{2\pi}\left\{il[K(\theta,\phi,u)]^l e^{i(m+1)u} - e^{i(m+1)u}\frac{\partial}{\partial u}K(\theta,\phi,u)\right\} du.\end{aligned}$$

Integrating by parts the last term and using Eq. (5) again, we get

$$2\pi N_{lm} L_+ Y_{lm} = i\hbar(l+m+1) 2\pi N_{l,m+1} Y_{l,m+1}.$$

Comparing with Eq. (9) we obtain the recurrence relation

$$\sqrt{l-m}\, N_{lm} = i\sqrt{l+m+1}\, N_{l,m+1},$$

which together with Eq. (4) yield

$$N_{lm} = i^{-m}\sqrt{\frac{4\pi}{2l+1}\frac{l!}{(l+m)!}\frac{l!}{(l-m)!}}. \quad (11)$$

In terms of the polar coordinates (r,θ) of $\mathbb{R}^2$, the functions $(1/\sqrt{2\pi})\, e^{im\theta}$ and $(1/\sqrt{2\pi})\, e^{-im\theta}$ are (normalized) spherical harmonics of degree m in two dimensions; therefore, according to Eq. (5), the function $[K(\theta,\phi,u)]^l$ is the kernel of an integral transform that maps the spherical harmonics of degree m in two dimensions, with $m \leqslant l$, into the set $\{Y_{l,-l}(\theta,\phi), Y_{l,-l+1}(\theta,\phi), \ldots, Y_{l,l}(\theta,\phi)\}$ of spherical harmonics in three dimensions.

We close this section with two further examples of the applications of Eq. (3). When $\theta = 0$, Eq. (3) reduces to

$$1 = \sum_{m=-l}^{l} N_{lm} Y_{lm}(0,\phi)\, e^{-imu},$$

which, with the aid of Eq. (11), implies that

$$Y_{lm}(0,\phi) = \begin{cases} 0 & \text{for } m \neq 0, \\ \sqrt{\dfrac{2l+1}{4\pi}} & \text{for } m = 0. \end{cases}$$

Since $[K(\theta,\phi,u)]^l\,[K(\theta,\phi,u)]^p = [K(\theta,\phi,u)]^{l+p}$, making use repeatedly of Eq. (3) we obtain

$$Y_{l+p,r}(\theta,\phi) = \frac{1}{N_{l+p,r}} \times \sum_m N_{lm} N_{p,r-m}\, Y_{lm}(\theta,\phi)\, Y_{p,r-m}(\theta,\phi). \quad (12)$$

(A similar expression is given in Ref. 5 in terms of the Clebsch–Gordan coefficients.)

3. Spherical harmonics in four dimensions

For the case of the spherical harmonics in four dimensions we have to follow a procedure slightly different from that followed in the case of three dimensions.

The spherical coordinates in $\mathbb{R}^4$, (r,χ,θ,ϕ), are related to the Cartesian coordinates (x,y,z,w) by

$$\begin{aligned} x &= r\sin\chi\sin\theta\cos\phi, \\ y &= r\sin\chi\sin\theta\sin\phi, \\ z &= r\sin\chi\cos\theta, \\ w &= r\cos\chi, \end{aligned} \quad (13)$$

and, therefore, the Laplace operator of $\mathbb{R}^4$ has the expression

$$\Delta_{\mathbb{R}^4} = \frac{1}{r^3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathrm{S}^3} \quad (14)$$

with

$$\Delta_{\mathrm{S}^3} = \frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \quad (15)$$

being the Laplace-Beltrami operator of S^3, the unit sphere $r=1$. Hence, a spherical harmonic of degree n in four dimensions, $f(\chi,\theta,\phi)$, satisfies the eigenvalue equation

$$\Delta_{\mathrm{S}^3} f = -n(n+2)f, \quad (16)$$

which follows from Eq. (14) and the fact that $r^n f$ is a solution of the Laplace equation. Using Eq. (15) one finds that Eq. (16) admits separable solutions of the form

$$P^l_{n,4}(\cos\chi)\, Y_{lm}(\theta,\phi), \quad (17)$$

where $P^l_{n,4}$ is an associated Legendre function in four dimensions [4], which satisfies the equation

$$\frac{1}{\sin^2\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\sin^2\chi\frac{\mathrm{d}}{\mathrm{d}\chi}P^l_{n,4}(\cos\chi)\right] + \left[n(n+2) - \frac{l(l+1)}{\sin^2\chi}\right]P^l_{n,4}(\cos\chi) = 0,$$

and we have made use of the fact that the usual spherical harmonics are eigenfunctions of the Laplace-Beltrami operator of S^2

$$\Delta_{\mathrm{S}^2} Y_{lm} = -l(l+1) Y_{lm} \quad (18)$$

with

$$\Delta_{\mathrm{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

(note that $\mathbf{L}^2 = -\hbar^2\Delta_{\mathrm{S}^2}$).

The vector $(k_1,k_2,k_3,k_4) = (\mathrm{i}\sin u\cos v, \mathrm{i}\sin u\sin v, \mathrm{i}\cos u, 1)$ is null for all $u,v\in\mathbb{R}$ and, according to Proposition 1, the functions

$$F_{nlm}(x,y,z,w) \equiv \int_{\mathrm{S}^2} (w + \mathrm{i}z\cos u + \mathrm{i}x\sin u\cos v + \mathrm{i}y\sin u\sin v)^n\, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)}, \quad (19)$$

where $\mathrm{d}\Omega_{(u,v)} = \sin u\,\mathrm{d}u\,\mathrm{d}v$ is the solid angle element corresponding to the angles u,v, are solutions of the Laplace equation (cf. Ref. 11). Since F_{nlm} is a homogeneous function of degree n in x,y,z,w, it follows that F_{nlm}/r^n is a spherical harmonic of degree n and, as we shall show, this function is, up to a constant factor, the spherical harmonic (17).

Indeed, letting

$$K(\chi,\theta,\phi,u,v) \equiv \cos\chi + \mathrm{i}\sin\chi\,[\cos\theta\cos u + \sin\theta\sin u\cos(\phi - v)], \quad (20)$$

from Eqs. (19) and (13) we have

$$F_{nlm}(r,\chi,\theta,\phi) = r^n \int_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n\, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)}$$

and making use of the fact that

$$\partial K(\chi,\theta,\phi,u,v)/\partial\phi = -\partial K(\chi,\theta,\phi,u,v)/\partial v,$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial\phi}\int_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n Y_{lm}(u,v)\mathrm{d}\Omega_{(u,v)} &= -\int_{\mathrm{S}^2} Y_{lm}(u,v)\frac{\partial}{\partial v}[K(\chi,\theta,\phi,u,v)]^n\,\mathrm{d}\Omega_{(u,v)} \\ &= \int_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n \frac{\partial}{\partial v} Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)} \\ &= \mathrm{i}m\int_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n\, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)}, \end{aligned}$$

where we have integrated by parts, which implies that $F_{nlm}(r,\chi,\theta,\phi)$ depends on ϕ only through the factor $\mathrm{e}^{\mathrm{i}m\phi}$. In a similar manner one finds that the value of the function $\Delta_{\mathrm{S}^2}[K(\chi,\theta,\phi,u,v)]^n$ is invariant under the exchange of (θ,ϕ) with (u,v) (just like the function $K(\chi,\theta,\phi,u,v)$

itself) hence, denoting by $\Delta_{\mathrm{S}^2(u,v)}$ the Laplace–Beltrami operator of the sphere S^2 in the variables u, v, and making use of the self-adjointness of this operator,

$$\begin{aligned}\Delta_{\mathrm{S}^2} \int\limits_{\mathrm{S}^2} &[K(\chi,\theta,\phi,u,v)]^n \, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)} \\ &= \int\limits_{\mathrm{S}^2} \Delta_{\mathrm{S}^2(u,v)}[K(\chi,\theta,\phi,u,v)]^n \, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)} \\ &= \int\limits_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n \, \Delta_{\mathrm{S}^2(u,v)} Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)} \\ &= -l(l+1) \int\limits_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n \, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)}\end{aligned}$$

[see Eq. (18)]. We conclude that $F_{nlm}(r,\chi,\theta,\phi)$ is proportional to $Y_{lm}(\theta,\phi)$ and, therefore, to the spherical harmonic (17).

Thus, denoting by Y_{nlm} the normalized spherical harmonics (17), we conclude that

$$Y_{nlm}(\chi,\theta,\phi) = \frac{1}{N_{nlm}} \times \int\limits_{\mathrm{S}^2} [K(\chi,\theta,\phi,u,v)]^n \, Y_{lm}(u,v)\,\mathrm{d}\Omega_{(u,v)}, \qquad (21)$$

where N_{nlm} is a normalization constant [cf. Eq. (5)], which amounts to

$$\{\cos\chi + \mathrm{i}\sin\chi\,[\cos\theta\cos u + \sin\theta\sin u\cos(\phi - v)]\}^n = \sum_{l=0}^{n}\sum_{m=-l}^{l} N_{nlm} Y_{nlm}(\chi,\theta,\phi)\, Y^*_{lm}(u,v), \qquad (22)$$

where * denotes complex conjugation [cf. Eq. (3)]. Even though Eq. (22) is analogous to Eq. (3), Eq. (22) does not seem convenient to find the explicit expression for the spherical harmonics in four dimensions owing to the presence of the functions $Y^*_{lm}(u,v)$ on the right-hand side of Eq. (22). Note that each term in the right-hand side of Eq. (22), $N_{nlm} Y_{nlm}(\chi,\theta,\phi)\, Y^*_{lm}(u,v)$, is the product of five functions of one variable each.

The functions $Y_{n00}(\chi,\theta,\phi)$ are, up to a constant factor, the Legendre polynomials in four dimensions, $P_{n,4}(\cos\chi)$ [4]. Using the fact that Y_{00} is a constant, and that the angle γ between the directions (θ,ϕ) and (u,v) is given by $\cos\gamma = \cos\theta\cos u + \sin\theta\sin u\cos(\phi - v)$, from Eq. (21), and the condition $P_{n,4}(1) = 1$, one obtains the integral representation

$$\begin{aligned}P_{n,4}(\cos\chi) &= \frac{1}{2}\int\limits_{-\pi}^{\pi} (\cos\chi + \mathrm{i}\sin\chi\,\cos\gamma)^n \sin\gamma\,\mathrm{d}\gamma \\ &= \frac{1}{2}\int\limits_{-1}^{1} (\cos\chi + \mathrm{i}\mu\sin\chi)^n\,\mathrm{d}\mu \qquad (23)\end{aligned}$$

(cf. Ref. 4). Then, using an auxiliary variable t, we have

$$\begin{aligned}\sum_{n=0}^{\infty} &(n+1) P_{n,4}(x) t^n \\ &= \frac{1}{2}\int\limits_{-1}^{1} \sum_{n=0}^{\infty} (n+1)(tx + \mathrm{i}t\sqrt{1-x^2}\,\mu)^n\,\mathrm{d}\mu \\ &= \frac{1}{2\pi}\int\limits_{-1}^{1} \frac{\mathrm{d}\mu}{(1 - tx - \mathrm{i}t\sqrt{1-x^2}\,\mu)^2} \\ &= \frac{1}{1 - 2tx + t^2},\end{aligned}$$

which gives a generating function for the Legendre polynomials in four dimensions [4] [cf. Eq. (8)].

4. Other dimensions, other signatures

It should be clear that we can write down formulas analogous to Eqs. (3), (5), (21), and (22) for the spherical harmonics in p dimensions in terms of the spherical harmonics in $p-1$ dimensions, for $p > 4$. On the other hand, Proposition 1 also holds when the metric g_{ij} is not positive definite and, in the case where $(g_{ij}) = \mathrm{diag}\,(-1,1,1,1)$, the Laplace operator becomes the d'Alembert operator which appears in the wave equation. However, as pointed out in Ref. 11, instead of homogeneous polynomials it is preferable to have solutions with an oscillatory time dependence. As a consequence of Proposition 1, or by a direct computation, one finds that

$$\exp\mathrm{i}(k_0 ct + k_1 x + k_2 y + k_3 z)$$

is a solution of the wave equation

$$g^{ij}\frac{\partial}{\partial x^i}\frac{\partial f}{\partial x^j} = 0,$$

with $(g_{ij}) = \mathrm{diag}\,(-1,1,1,1)$, $i,j = 0,1,2,3$, and $(x^0,x^1,x^2,x^3) = (ct,x,y,z)$, provided that the constant vector (k_0,k_1,k_2,k_3) is null, that is $(k_0)^2 = (k_1)^2 + (k_2)^2 + (k_3)^2$.

For all $k,u,v \in \mathbb{R}$, the vector $k(1, \sin u\cos v, \sin u\sin v, \cos u)$ is null and, therefore, $\exp\mathrm{i}k(ct + z\cos u + x\sin u\cos v + y\sin u\sin v)$ must be a superposition of solutions of the wave equation, with coefficients that depend on u and v. In fact, as is well known,

$$\exp\mathrm{i}k(ct + z\cos u + x\sin u\cos v + y\sin u\sin v) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{i}^l \mathrm{e}^{\mathrm{i}kct} j_l(kr)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(u,v), \quad (24)$$

where the j_l are spherical Bessel functions and (r,θ,ϕ) are the spherical coordinates of the point (x,y,z) (see, *e.g.*, Refs. 1, 5). That is, Eq. (24) is an expression of a solution of the wave equation as a superposition of *separable* solutions of the wave equation, with coefficients that depend on u, v (which happen to be spherical harmonics!). Equation (24) is analogous to Eq. (22) in the sense that it can be regarded as a generating function (of solutions of the wave equation) and it shows that, for each $k > 0$, the function $\exp \mathrm{i}k(ct + z\cos u + x\sin u\cos v + y\sin u\sin v)$ is the kernel of an integral transform that maps the spherical harmonic $Y_{lm}(u, v)$ into a multiple of $\mathrm{e}^{\mathrm{i}kct} j_l(kr)\, Y_{lm}(\theta, \phi)$.

An analog of Eq. (24) in two spatial dimensions is

$$\exp \mathrm{i}k(ct + x\cos u + y\sin u) = \sum_{m=-\infty}^{\infty} \mathrm{i}^m \mathrm{e}^{\mathrm{i}kct} J_m(kr) \mathrm{e}^{\mathrm{i}m\theta}\, \mathrm{e}^{-\mathrm{i}mu}, \qquad (25)$$

where the J_m are Bessel functions (of integral order) and (r, θ) are the polar coordinates of the point (x, y). Here again the spherical harmonics in two dimensions, $\mathrm{e}^{-\mathrm{i}mu}$, appear [cf. Eq. (3)]. It may be remarked that even though Eq. (25) can be readily obtained from the usual generating function for the Bessel functions,

$$\exp \frac{x}{2}\left(t - \frac{1}{t}\right) = \sum_{m=-\infty}^{\infty} J_m(x) t^m,$$

the generating function (25) generates entire *solutions* of the wave equation in two spatial dimensions.

Finally, making use of the generating function (25), one can derive an addition formula for the Bessel functions, analogous to Eq. (12). Eliminating the common factor $\mathrm{e}^{\mathrm{i}kct}$ from Eq. (25) one obtains

$$\exp \mathrm{i}kr(\cos\theta\cos u + \sin\theta\sin u) = \sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m(kr)\, \mathrm{e}^{\mathrm{i}m\theta}\, \mathrm{e}^{-\mathrm{i}mu}, \qquad (26)$$

hence

$$\begin{aligned}
&\exp \mathrm{i}kr_1(\cos\theta_1\cos u + \sin\theta_1\sin u) \\
&\quad \times \exp[-\mathrm{i}kr_2(\cos\theta_2\cos u + \sin\theta_2\sin u)] \\
&\quad = \sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m(kr_1)\, \mathrm{e}^{\mathrm{i}m\theta_1}\, \mathrm{e}^{-\mathrm{i}mu} \\
&\quad \times \sum_{m'=-\infty}^{\infty} (-\mathrm{i})^{m'} J_{m'}(kr_2)\, \mathrm{e}^{-\mathrm{i}m'\theta_2}\, \mathrm{e}^{\mathrm{i}m'u}.
\end{aligned}$$

The left-hand side of the last equation is equal to

$$\begin{aligned}
&\exp \mathrm{i}k\big[(r_1\cos\theta_1 - r_2\cos\theta_2)\cos u \\
&\quad + (r_1\sin\theta_1 - r_2\sin\theta_2)\sin u\big] \\
&\quad = \exp \mathrm{i}kR(\cos\theta_3\cos u + \sin\theta_3\sin u),
\end{aligned}$$

where

$$\begin{aligned}
(R\cos\theta_3, R\sin\theta_3) &\equiv (r_1\cos\theta_1, r_1\sin\theta_1) \\
&\quad - (r_2\cos\theta_2, r_2\sin\theta_2),
\end{aligned}$$

which, according to Eq. (26) has the expansion

$$\sum_{\nu=-\infty}^{\infty} \mathrm{i}^\nu J_\nu(kR)\, \mathrm{e}^{\mathrm{i}\nu\theta_3}\, \mathrm{e}^{-\mathrm{i}\nu u}$$

thus,

$$J_\nu(kR)\, \mathrm{e}^{\mathrm{i}\nu\psi} = \sum_{m'=-\infty}^{\infty} J_{\nu+m'}(kr_1) J_{m'}(kr_2)\, \mathrm{e}^{\mathrm{i}m'\theta},$$

where $\psi \equiv \theta_3 - \theta_1$, $\theta \equiv \theta_1 - \theta_2$, and $R = ({r_1}^2 + {r_2}^2 - 2r_1r_2\cos\theta)^{1/2}$ [see, *e.g.*, Ref. 14, Eq. (5.12.5)].

1. J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
2. D.J. Griffiths, *Introduction to Quantum Mechanics* (Prentice Hall, Upper Saddle River, NJ, 1995).
3. D.M. Brink and G.R. Satchler, *Angular Momentum*, 3rd ed. (Oxford University Press, Oxford, 1993).
4. H. Hochstadt, *The Functions of Mathematical Physics* (Dover, New York, 1986). Chap. 6.
5. D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
6. M. Bander and C. Itzykson, *Rev. Mod. Phys.* **38** (1966) 330.
7. A.M. Perelomov and Ya.B. Zel'dovich, *Quantum Mechanics: Selected Topics* (World Scientific, Singapore, 1998).
8. G.F. Torres del Castillo and J.L. Calvario Acócal, *Rev. Mex. Fis.* **53** (2007) 407.
9. J.N. Goldberg, A.J. Macfarlane, E.T. Newman, F. Rohrlich and E.C.G. Sudarshan, *J. Math. Phys.* **8** (1967) 2155.
10. G.F. Torres del Castillo, *3-D Spinors, Spin-weighted Functions and their Applications* (Birkhäuser, Boston, 2003).
11. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1996).
12. W.W. Bell, *Special Functions for Scientists and Engineers* (Dover, New York, 2004). Sect. 3.3
13. C. Frye and C.J. Efthimiou, arXiv:1205.3548v1 [math.CA] 2012.
14. N.N. Lebedev, *Special Functions and their Applications* (Dover, New York, 1972).