A Gentle, Geometric Introduction to Copositive Optimization

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Abstract

This paper illustrates the fundamental connection between nonconvex quadratic optimization and copositive optimization—a connection that allows the reformulation of nonconvex quadratic problems as convex ones in a unified way. We focus on examples having just a few variables or a few constraints for which the quadratic problem can be formulated as a copositive-style problem, which itself can be recast in terms of linear, second-order-cone, and semidefinite optimization. A particular highlight is the role played by the geometry of the feasible set.

1 Introduction

When attempting to solve nonconvex optimization problems globally, convex optimization frequently plays an important role. For example, mixed-integer linear programs are most often solved by the use of linear programming relaxations, and semidefinite optimization can be used to relax quadratic and polynomial problems. A related issue is the generation of valid inequalities, or cuts, that tighten a given relaxation without eliminating any of the underlying solutions of the nonconvex problem. Whether good cuts can be generated efficiently—either in theory or in practice—is closely related to the nonconvex problem's inherent difficulty [16].

Many approaches try to characterize good classes of cuts, and of course, the best outcome characterizes all the necessary cuts to tighten a convex relaxation fully. This can be called *convexifying the problem* because, in principle, once all the required cuts are known, the convex relaxation with all cuts can be solved (no local minima!) to solve the original nonconvex problem. On the other hand, some problems are so difficult that one cannot

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expect to characterize all the required cuts, or even if one could, there would be so many cuts that the tight relaxation would be too big to solve. Nevertheless, much can be learned by *trying* to characterize all the cuts. Indeed, such attempts account for a vast swath of the optimization literature.

Copositive optimization is a relatively new approach for analyzing the specific, difficult case of optimizing a general nonconvex quadratic function over a polyhedron $\{x : Ax = b, x \geq 0\}$ [9]. Briefly defined, copositive optimization is linear optimization over the convex cone of *copositive matrices,* i.e., symmetric matrices Z for which the quadratic form $y^T Zy$ is nonnegative for all input vectors $y \geq 0$. The dual problem is linear optimization over the cone of *completely positive matrices,* i.e., symmatric matrices Y that can be expressed as a sum $Y = \sum_k y^k (y^k)^T$ of rank-1 matrices $y^k (y^k)^T$, where each $y^k \geq 0$. Despite involving different matrix cones, both primal and dual problems are often referred to simply as "copositive programs." Copositive optimization has deep connections with semidefinite and polynomial optimization.

Relative to nonconvex quadratic optimization over $\{x : Ax = b, x \geq 0\}$, one of the primary insights provided by copositive optimization is that the problem can be convexified by a class of cuts called *copositive cuts,* which correspond to copsitive matrices. Moreover, copositive cuts do not depend in any way on the objective or the data (A, b). Rather, they depend only on the condition $x \geq 0$. So, in this sense, copositive cuts can be studied independently as a key to quadratic optimization over polyhedra. Of course, characterizing all copositive cuts will be extremely difficult, but any progress can be applied uniformly to many different problems. This is the promise and the goal of copositive optimization.

Beyond just quadratic optimization over polyhedra, there exist extensions and applications of copositive optimization handling all sorts of objectives, e.g., ones based on polynomial functions and ones involving uncertainty, and all sorts of constraints, e.g., polyhedra with binary variables, ellipsoids, and sets defined by polynomial inequalities.

In this paper, we introduce the reader to copositive optimization from an atypical—yet we hope interesting—viewpoint by investigating several specific types of nonconvex quadratic optimization problems, each of which is characterized by the geometry of its feasible region. Typically, either the number of variables is small or the number of constraints is small (or both). In each case, we show exactly what cuts are required to convexify the problem. In other words, we find the copositive-style cuts required for that particular situation. Geometric insight is stressed throughout.

Throughout the paper, we reiterate a common theme of copositive optimization—that nonconvex quadratic optimization problems can be solved by convexifying using copositive-style cuts and, though duality, completely-positive-style matrices. In fact, the basic ideas

found in the paper [9] for the case of polyhedra are the core required for all results in this paper. Hence, our main contribution is a unified presentation of the various results, and we do so from "easiest to hardest" in hopes of gently introducing the reader to the insights provided by copositive optimization. To complete the connection with copositive optimization, in Section 10 we recapitulate the main results of [9] for nonconvex quadratic optimization over polytopes.

We would also like to emphasize that we do not claim to have identified the best or only way to solve each of the various problem types—or that these are the most practically relevant problem classes. Moreover, we do not compare with other methods, e.g., branch-and-bound techniques, methods that exploit sparsity in the problem, or higher-order lifting approaches. We would simply like to introduce the reader to the area of copositive optimization and to provide interesting insights along the way. Of course, we hope that the ideas and techniques presented will prove useful for solving quadratic problems in practice; some ideas along these lines are mentioned in Section 11.

All results except those in Section 9 have already appeared in the literature, and we cite the relevant papers in context. On the other hand, many of the proofs have been simplified compared to the literature, and we have taken care to present all proofs in a unified way. We also refer the reader to the following excellent survey papers on copositive optimization and related issues: [2, 3, 6, 7, 8, 10, 14, 17, 19]. In addition, we remark that our approach here is closely related to so-called *domain copositive* and *set-semidefinite* approaches; see [15, 23].

1.1 Reuse of notation

We caution the reader that this paper makes heavy reuse of notation throughout. For example, the same symbol $\mathcal{G}$ has a different definition in each section. We do this partly to keep the notation as simple as possible. However, we also hope that this reuse will actually assist the reader in understanding the main points of the paper. This is because, while $\mathcal{G}$ has a different, specific meaning in each section, it plays the same general role always. So, for example, if the reader understands the role of $\mathcal{G}$ in Section 2, then he or she is prepared to understand its role in Sections 4–10 even as the specific definition of $\mathcal{G}$ changes.

2 Intervals

Consider the one-dimensional optimization

$$\begin{array}{rcll} v^* & := & \min & H_{11}x_1^2 + 2g_1x_1 \\ & & \text{s.t.} & -1 \le x_1 \le 1 \end{array} \tag{INT}$$

where $H_{11}, g_1 \in \mathbb{R}$ are given scalars, and let $\mathcal{F} := \{x_1 : -1 \le x_1 \le 1\}$ denote the feasible set. By variable shifting and scaling, quadratic optimization over any interval can be recast in this form. To solve (INT), one can use standard calculus techniques that examine the quadratic objective $H_{11}x_1^2 + 2g_1x_1$ at its critical points in $(-1, 1)$ as well as at the endpoints -1 and 1. However, here we seek a single convex optimization problem that is equivalent to (INT).

The key idea is to introduce a new variable X_{11} equaling x_1^2:

$$\begin{array}{rcll} v^* & = & \min & H_{11}X_{11} + 2g_1x_1 \\ & & \text{s.t.} & x_1 \in \mathcal{F},\ X_{11} = x_1^2. \end{array} \tag{1}$$

Because the objective of (1) is linear, standard convex analysis allows us to express the optimal value in a different way—as the minimization of the linear objective $H_{11}X_{11} + 2g_1x_1$ over the closed convex hull of pairs (x_1, x_1^2) with $x_1 \in \mathcal{F}$. That is, $v^* = \min\{H_{11}X_{11} + 2g_1x_1 : (x_1, X_{11}) \in \mathcal{G}\}$, where

$$\mathcal{G} := \overline{\text{conv}}\left\{(x_1, x_1^2) : x_1 \in \mathcal{F}\right\}.$$

In this case, we can construct an explicit characterization of $\mathcal{G}$ using convex inequalities in (x_1, X_{11}); see Figure 1. So

$$\begin{array}{rcll} v^* & = & \min & H_{11}X_{11} + 2g_1x_1 \\ & & \text{s.t.} & x_1^2 \le X_{11} \le 1. \end{array}$$

Note that the first constraint $x_1^2 \le X_{11}$ is a relaxation of $x_1^2 = X_{11}$, while $X_{11} \le 1$ is the relaxed version of $(1 - x_1)(x_1 + 1) \ge 0 \Leftrightarrow 1 - x_1^2 \ge 0$.

It is important to note that, given an optimal pair (x_1^*, X_{11}^*) for the convex problem, the scalar x_1^* is not necessarily optimal for the original problem. For example, suppose $H_{11} = -1$ and $g_1 = 0$. Then $v^* = -1$, and $(x_1^*, X_{11}^*) = (0, 1)$ is feasible and optimal. However, $x_1^* = 0$ is not optimal for the original problem since its quadratic objective value is 0. In general, one can see that the optimal solution set in (x_1, X_{11}) is the convex hull of the optimal solution

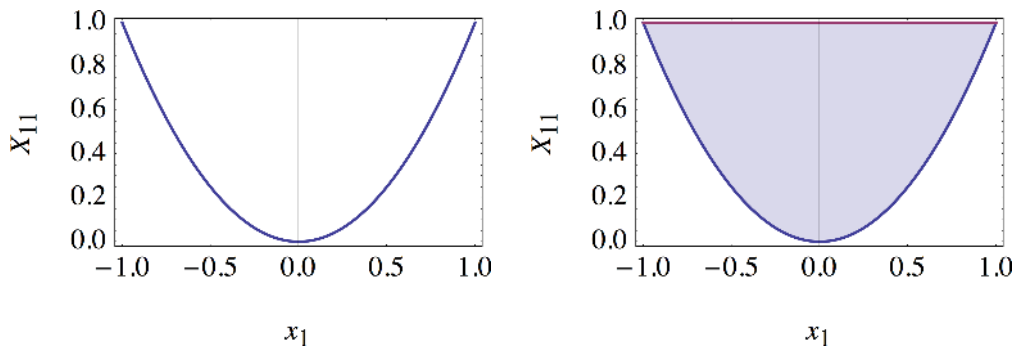

Figure 1: Convexification of the set $\{(x_1, x_1^2) : x_1 \in \mathcal{F}\}$ for $\mathcal{F} = [-1, 1]$. The left plot depicts the set itself, while the right depicts its convex hull with defining convex inequalities $x_1^2 \leq X_{11} \leq x_1$.

set in (x_1, x_1^2). In our example, $(x_1^*, X_{11}^*) = (0, 1) = \frac{1}{2}(-1, 1) + \frac{1}{2}(1, 1)$, where both -1 and 1 are optimal for the original problem.

A sufficient condition for an optimal (x_1^*, X_{11}^*) to yield an optimal x_1^* for the original problem is the equation $X_{11}^* = (x_1^*)^2$, which simply recovers the nonconvex condition $X_{11} = x_1^2$ that was relaxed in the convexification process. It is analogous to recovering an integer solution from a linear relaxation in integer optimization. This is the *rank-1 condition*, that we will see many times throughout the paper.

3 Triangles (and Tetrahedra)

Whereas we have considered quadratic optimization over intervals in Section 2, we now consider the two-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^2} \quad H_{11}x_1^2 + 2H_{21}x_1x_2 + H_{22}x_2^2 + 2g_1x_1 + 2g_2x_2 \\ & \text{s.t.} \quad Ax \leq b \end{aligned} \tag{TRI}$$

where $H_{11}, H_{21}, H_{22}, g_1, g_2 \in \mathbb{R}$ are given scalars and the feasible set $\mathcal{F} := \{x \in \mathbb{R}^2 : Ax \leq b\}$ with $A \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$ defines a proper triangle in the plane. By *proper triangle*, we mean that $\mathcal{F}$ is a three-sided, bounded polygon with interior.

Analogous to the optimization (1) for intervals, we can introduce three new variables and

three quadratic equations to linearize the objective in (TRI):

$$\begin{aligned} v^* \quad := \quad & \min_{x\in\mathbb{R}^2} \quad H_{11}X_{11} + 2H_{21}X_{21} + H_{22}X_{22} + 2g_1x_1 + 2g_2x_2 \\ & \text{s.t.} \quad Ax \le b,\ X_{11} = x_1^2,\ X_{21} = x_1x_2,\ X_{22} = x_2^2. \end{aligned}$$

By defining the symmetric 2×2 matrices $H = (H_{ij})$ and $X = (X_{ij})$, we can simplify the notation to

$$\begin{aligned} v^* \quad := \quad & \min_{x\in\mathbb{R}^2} \quad \langle H, X\rangle + 2\langle g, x\rangle \\ & \text{s.t.} \quad Ax \le b,\ X = xx^T, \end{aligned} \tag{2}$$

where $\langle H, X\rangle := \sum_{i=1}^2 \sum_{j=1}^2 H_{ij}X_{ij}$ is the inner product between symmetric matrices, $\langle g, x\rangle$ is the usual vector inner product. Note that xx^T is symmetric, positive semidefinite, and rank-1 and that, due to symmetry, the 2×2 matrix equation includes two equivalent equations $X_{21} = x_1x_2$ and $X_{12} = x_1x_2$. As with intervals in the previous section, we would like to convexify the set $\{(x, xx^T) : x \in \mathcal{F}\}$. If we can provide a compact, explicit description of this convex hull, then we can optimize any quadratic over a triangle using a single convex optimization problem. Indeed, Theorem 1 below provides the description, which was first proved in [1].

So define $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$, where $\overline{\text{conv}}$ denotes the closure of the convex hull. Our goal is to provide an explicit description of $\mathcal{G}$, but to do so, we first examine a related, yet different, convex hull. Let $C := (b, -A) \in \mathbb{R}^{3\times3}$ be the horizontal concatenation of b and $-A$, so that the vector inequality $Cy \ge 0$ with $y \in \mathbb{R}^3$ defines a polyhedral cone in $\mathbb{R}^3$. We investigate

$$\mathcal{K} := \overline{\text{conv}}\{yy^T : Cy \ge 0\} \subseteq \mathcal{S}^3$$

where $\mathcal{S}^3$ denotes symmetric matrices of size 3×3. Note that, since $Cy \ge 0$ defines a cone, $\mathcal{K}$ also equals the closed *conic* hull of matrices yy^T with $Cy \ge 0$, that is, the closure of all possible sums of such matrices yy^T.

The following lemmas help us characterize $\mathcal{K}$ in Proposition 1 and also $\mathcal{G}$ in Theorem 1.

Lemma 1. *Regarding (TRI), the cone $\{x : Ax \le 0\}$ equals $\{0\}$, the set $\{x : Ax \le -b\}$ is empty, and the matrix $C := (b, -A) \in \mathbb{R}^{3\times3}$ is invertible.*

Proof. By standard polyhedral theory, the boundedness of the nonempty $\mathcal{F}$ is equivalent to $\{x : Ax \le 0\} = \{0\}$. Moreover, since $\mathcal{F}$ is a proper triangle, the set $\{x : Ax = b\}$ is empty, i.e., no point x satisfies all three inequalities simultaneously, and $\text{rank}(A) = 2$. It follows that C is invertible; if not, then $b = Ay$ for some $y \in \mathbb{R}^2$, a contradiction. In addition, the

system $Ax \leq -b$ is infeasible; if not, then $x \in \mathcal{F}$ and $Ay \leq -b$ imply $A(x+y) \leq 0$, which in turn implies $y = -x$ and $A(-x) \leq -b$. Then $Ax = b$, a contradiction. $\square$

Lemma 2 ([20])**.** *Let $d \leq 4$, and let $\mathcal{S}^d$ be the space of $d \times d$ symmetric matrices. If $Z \in \mathcal{S}^d$ is positive semidefinite with nonnegative entries, then there exists a collection of d-dimensional nonnegative vectors $\{z^k\}$ such that $Z = \sum_k z^k (z^k)^T$.*

We remark that Lemma 2 equates completely positive matrices (see the Introduction and Section 10) with matrices that are positive semidefinite and component-wise nonnegative—the so-called *doubly nonnegative matrices*. It establishes a fundamental connection between copositive programming with linear and semidefinite programming in low dimensions.

In the following proposition, $Y \succeq 0$ means that Y is symmetric positive semidefinite, and the constraint $CYC^T \geq 0$ indicates that the matrix CYC^T has all nonnegative entries.

Proposition 1. *Regarding (TRI), $\mathcal{K} := \overline{\text{conv}}\{yy^T : Cy \geq 0\} = \left\{Y \succeq 0 : CYC^T \geq 0\right\}$, where $C := (b, -A)$.*

Proof. For ease of reference, define $\mathcal{R} := \left\{Y \succeq 0 : CYC^T \geq 0\right\}$, which we note is closed and convex. We first argue $\mathcal{K} \subseteq \mathcal{R}$. Because $\mathcal{K}$ is the closure of the convex hull of matrices yy^T with $Cy \geq 0$ and because $\mathcal{R}$ is closed and convex, it suffices to show that such rank-1 matrices yy^T are in $\mathcal{R}$. So take $Y = yy^T$ with $Cy \geq 0$. Then clearly $Y \succeq 0$, and the matrix of linear inequalities $CYC^T \geq 0$ holds because CYC^T equals the rank-1 product $(Cy)(Cy)^T$.

To show the reverse containment, let $Y \in \mathcal{R}$. Note that $Y \succeq 0$ implies $CYC^T \succeq 0$. Then, by Lemma 2, since $CYC^T \in \mathcal{S}^3$ is both positive semidefinite and nonnegative, we can write $CYC^T = \sum_k z^k (z^k)^T$ with $z^k \geq 0$. Defining $y^k := C^{-1} z^k$, where C^{-1} exists by Lemma 1, this implies $Y = \sum_k y^k (y^k)^T$ with $Cy^k \geq 0$, showing $Y \in \text{conv}\{yy^T : Cy \geq 0\} \subseteq \mathcal{K}$. $\square$

We remark that the proof could be expressed in a slightly different way. The first step is to show $\text{conv}\{yy^T : Cy \geq 0\} \subseteq \mathcal{R}$ by arguing $yy^T \in \mathcal{R}$, and the second is to show the reverse containment by appealing to Lemma 2. Then $\text{conv}\{yy^T : Cy \geq 0\} = \mathcal{R}$, and since $\mathcal{R}$ is closed, it must hold further that the convex hull equals the closed convex hull, which proves the proposition. This relationship between the convex hull and its closure occurs multiple times throughout the paper.

To state the characterization of $\mathcal{G}$ in Theorem 1, we introduce some notation that we will use throughout the rest of the paper. For any d, given $(x, X) \in \mathbb{R}^d \times \mathcal{S}^d$, define

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}^{d+1}. \tag{3}$$

In words, $Y(x, X)$ is the matrix that positions x and X within a symmetric $(d+1) \times (d+1)$ matrix with leading entry 1, trailing block X, and border x.

Theorem 1. *Regarding (TRI), it holds that*

$$\begin{aligned} \mathcal{G} &:= \overline{\operatorname{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0,\ CYC^T \geq 0 \end{array} \right\} \end{aligned}$$

with $Y(x, X)$ given by (3) and $C := (b, -A)$.

Proof. For ease of reference, define

$$\mathcal{R} := \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0,\ CYC^T \geq 0 \end{array} \right\},$$

which we note is closed and convex. As in the proof of Proposition 1, the containment $\mathcal{G} \subseteq \mathcal{R}$ holds easily. To prove the reverse containment, let $(x, X) \in \mathcal{R}$, and define $Y := Y(x, X)$. By Proposition 1, we can write $Y = \sum_k y^k (y^k)^T$ with each nonzero $y^k \in \mathbb{R}^3$ satisfying $Cy^k \geq 0$. Decomposing $\binom{\nu_k}{w^k} := y^k$ with $\nu_k \in \mathbb{R}$ and $w^k \in \mathbb{R}^2$, we see $Aw^k \leq b\nu_k$. It cannot hold that $\nu_k < 0$; otherwise, $w^k/|\nu_k|$ would be a solution of the system $Ax \leq -b$, contradicting Lemma 1. If $\nu_k = 0$, then $w^k = 0$ by Lemma 1. So in fact every ν_k is positive. Define $x^k := w^k/\nu_k$. Then

$$Y = \sum_k {\nu_k}^2 \begin{pmatrix} 1 \\ x^k \end{pmatrix} \begin{pmatrix} 1 \\ x^k \end{pmatrix}^T \quad \text{with} \quad Ax^k \leq b \Leftrightarrow x^k \in \mathcal{F}.$$

Since the top-left entry of Y equals 1, we have $\sum_k {\nu_k}^2 = 1$. So Y is a convex combination of rank-1 matrices $\binom{1}{x^k}\binom{1}{x^k}^T$ with $x^k \in \mathcal{F}$. This proves $Y \in \mathcal{G}$ as desired. □

Theorem 1 can be easily extended to tetrahedra in $\mathbb{R}^3$, i.e., when $x \in \mathbb{R}^3$ and $\mathcal{F} = \{x : Ax \leq b\}$ defines a four-sided polyhedron with triangular faces. The extension simply relies on the fact that Lemma 2 also works for $d = 4$. On the other hand, Theorem 1 cannot be extended verbatim to simplices in $\mathbb{R}^4$ or higher precisely because Lemma 2 does not hold for $d \geq 5$.

As discussed, a primary consequence of Theorem 1 is that any two-variable quadratic (nonconvex or otherwise) can be optimized over a triangle in the plane by solving a single, explicit convex program. We now investigate this convex program in a bit more detail, especially to shed light on how it relates to the geometry of the triangle.

Considering (TRI) and Theorem 1 together, we know $v^* = \min\{\langle H, X\rangle + 2\langle g, x\rangle : Y(x, X) \in \mathcal{K}\}$. Keeping the emphasis on (x, X), we expand this as

$$\begin{array}{rcll} v^* & = & \min & \langle H, X\rangle + 2\langle g, x\rangle \\ & & \text{s.t.} & bb^T - Axb^T - bx^TA^T + AXA^T \geq 0 \\ & & & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0. \end{array}$$

Letting the i-th constraint of $Ax \leq b$ be denoted as $\langle a^i, x\rangle \leq b_i$, we can separate the constraint $bb^T - Axb^T - bx^TA^T + AXA^T \geq 0$ into its $9 = 3^2$ individual entries $b_ib_j - b_j\langle a^i, x\rangle - b_i\langle a^j, x\rangle + \langle a^i, Xa^j\rangle \geq 0$. Symmetry of X implies that three entries are redundant, and $X \succeq 0$ implies that, when $i = j$, the left-hand side $b_i^2 - 2b_i\langle a^i, x\rangle + \langle a^i, Xa^i\rangle \geq b_i^2 - 2b_i\langle a^i, x\rangle + \langle a^i, x\rangle^2 = (b_i - \langle a^i, x\rangle)^2$ is already nonnegative due to the fact that $X \succeq xx^T$. So our problem reduces to

$$\begin{array}{rcll} v^* & = & \min & \langle H, X\rangle + 2\langle g, x\rangle \\ & & \text{s.t.} & b_1b_2 - b_2\langle a^1, x\rangle - b_1\langle a_2, x\rangle + \langle a^1, Xa_2\rangle \geq 0 \\ & & & b_1b_3 - b_3\langle a^1, x\rangle - b_1\langle a_3, x\rangle + \langle a^1, Xa_3\rangle \geq 0 \\ & & & b_2b_3 - b_3\langle a_2, x\rangle - b_2\langle a_3, x\rangle + \langle a_2, Xa_3\rangle \geq 0 \\ & & & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \end{array}$$

a convex problem with a single SDP constraint, three linear inequalities, and a single equality (which sets the top-left matrix entry to 1).

The linear inequality $b_ib_j - b_j\langle a^i, x\rangle - b_i\langle a^j, x\rangle + \langle a^i, Xa^j\rangle \geq 0$ has a geometric interpretation relative to the triangle $\mathcal{F}$ in $\mathbb{R}^2$. To see this, imagine that the rank-1 condition $X = xx^T$ holds. Then the linear inequality becomes

$$0 \leq b_ib_j - b_j\langle a^i, x\rangle - b_i\langle a^j, x\rangle + \langle a^i, x\rangle\langle a^j, x\rangle = (b_i - \langle a^i, x\rangle)(b_j - \langle a^j, x\rangle),$$

which is a nonconvex quadratic inequality that is valid on $\mathcal{F}$ because the linear inequalities $\langle a^i, x\rangle \leq b_i$ and $\langle a^j, x\rangle \leq b_j$ define two facets of $\mathcal{F}$. So the linear inequalities in (x, X) can be viewed as relaxed quadratic inequalities, one for each pair of sides of $\mathcal{F}$. These types of constraints are often called *RLT constraints* after the reformulation-linearization technique of [22]; see also [21]. Figure 2 depicts these three quadratic, nonconvex inequalities relative to $\mathcal{F}$.

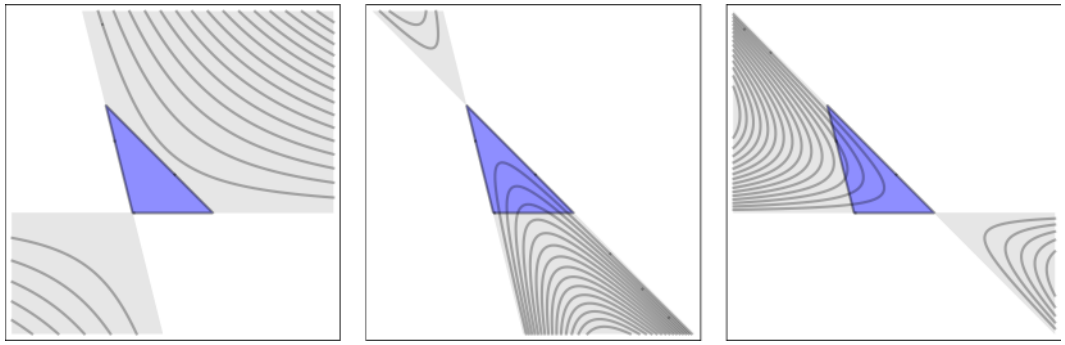

Figure 2: Convexification of the set $\{(x, xx^T) : x \in \mathcal{F}\}$ when $\mathcal{F}$ is a triangle requires a positive semidefiniteness constraint, a single equality constraint, and three linear inequality constraints. The inequality constraints are relaxed versions of nonconvex quadratic constraints of the form $(b_i - \langle a^i, x\rangle)(b_j - \langle a^j, x\rangle) \geq 0$, where $\langle a^i, x\rangle \leq b_i$ and $\langle a^j, x\rangle \leq b_j$ define facets of $\mathcal{F}$. Depicted here are the three quadratics in relation to the triangle $\mathcal{F}$. The lightly shaded regions satisfy the quadratics, and the quadratics' level curves are plotted for reference.

4 Convex Quadrilaterals

Similar to (TRI), we consider the two-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^2} \quad \langle x, Hx\rangle + 2\langle g, x\rangle \\ & \text{s.t.} \quad Ax \leq b \end{aligned} \tag{QUAD}$$

where $H \in \mathcal{S}^2$ and $g \in \mathbb{R}^2$. Here, however, we assume $A \in \mathbb{R}^{4\times 2}$ and $b \in \mathbb{R}^4$ define the feasible set $\mathcal{F} := \{x \in \mathbb{R}^2 : Ax \leq b\}$, which is a proper convex quadrilateral in the plane, i.e., $\mathcal{F}$ is a four-sided, bounded polygon with interior. As before, we would like to determine an explicit description of $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$. Theorem 2 below was proven for a rectangular $\mathcal{F}$ in [1].

As one might suspect, the description and corresponding proof for $\mathcal{G}$ are quite similar to the previous case for triangles. Indeed, we have the following three results (proofs discussed below):

Lemma 3. *Regarding (QUAD), the cone $\{x : Ax \leq 0\}$ equals $\{0\}$, the set $\{x : Ax \leq -b\}$ is empty, and the matrix $C := (b, -A) \in \mathbb{R}^{4\times 3}$ has full column rank.*

Proposition 2. *Regarding (QUAD), $\mathcal{K} := \overline{\text{conv}}\{yy^T : Cy \geq 0\} = \{Y \succeq 0 : CYC^T \geq 0\}$, where $C := (b, -A)$.*

Theorem 2. *Regarding (QUAD), it holds that*

$$\begin{aligned}\mathcal{G} &:= \overline{\operatorname{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{(x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0,\ CYC^T \geq 0 \end{array}\right\}\end{aligned}$$

with $Y(x, X)$ given by (3) and $C := (b, -A)$.

The proof of Lemma 3 is completely analogous to the proof of Lemma 1, and Theorem 2 is proved exactly like Theorem 1 (except that it is based on Lemma 3 and Proposition 2). Only the proof of Proposition 2 takes some additional care.

Proof of Proposition 2. Let $\alpha \in \mathbb{R}^4$ be any vector such that $\hat{A} := (A, \alpha)$ makes $\hat{C} := (b, -\hat{A})$ invertible. Note that $\{y \in \mathbb{R}^3 : Cy \geq 0\}$ is the projection of the cone $\{\hat{y} \in \mathbb{R}^4 : \hat{C}\hat{y} \geq 0, \hat{y}_4 = 0\}$ onto the variables $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$.

We first prove

$$\hat{\mathcal{K}} := \overline{\operatorname{conv}}\{\hat{y}\hat{y}^T : \hat{C}\hat{y} \geq 0, \hat{y}_4 = 0\} = \{\hat{Y} \succeq 0 : \hat{C}\hat{Y}\hat{C}^T \geq 0, \hat{Y}_{44} = 0\} =: \hat{\mathcal{R}}.$$

Note that $\hat{\mathcal{R}}$ is closed and convex. The containment $\hat{\mathcal{K}} \subseteq \hat{\mathcal{R}}$ is clear. So let $\hat{Y} \in \hat{\mathcal{R}}$. Since $\hat{C}\hat{Y}\hat{C}^T \in \mathcal{S}^4$ is both positive semidefinite and entry-wise nonnegative, we can use Lemma 2 to write $\hat{C}\hat{Y}\hat{C}^T = \sum_k \hat{z}^k(\hat{z}^k)^T$ with $\hat{z}^k \geq 0$. Defining $\hat{y}^k := \hat{C}^{-1}\hat{z}^k$, this implies $\hat{Y} = \sum_k \hat{y}^k(\hat{y}^k)^T$ with $\hat{C}\hat{y}^k \geq 0$. In addition, $0 = \hat{Y}_{44} = \sum_k (\hat{y}_4^k)^2$ ensures $\hat{y}_4^k = 0$ for each k. In total, this establishes $\hat{Y} \in \hat{K}$.

To prove $\mathcal{K} = \left\{Y \succeq 0 : CYC^T \geq 0\right\}$, again the containment $\subseteq$ is clear. For the reverse, $Y \succeq 0$ with $CYC^T \geq 0$ can be embedded in $\hat{Y} \in \hat{\mathcal{R}}$ by appending a zero row and a zero column. Then, by the previous paragraph, $\hat{Y}$ is the sum of terms $\hat{y}^k(\hat{y}^k)^T$ with $\hat{C}\hat{y}^k \geq 0$ and $\hat{y}_4^k = 0$. By the projection mentioned in the first paragraph of this proof, Y is thus the sum of terms $y^k(y^k)^T$ with $Cy^k \geq 0$. So $Y \in \mathcal{K}$. □

Similar to the case for triangles, Theorem 2 implies that (QUAD) can be solved by a convex program with one positive semidefinite constraint, one equality constraint (setting the top-left matrix entry to 1), and several linear inequality constraints that are derived from nonconvex quadratic inequality constraints. Each of those quadratic constraints is $(b_i - \langle a^i, x\rangle)(b_j - \langle a^j, x\rangle) \geq 0$ for two facets of the quadrilateral. In this case, there are six such quadratics, which are depicted in Figure 3.

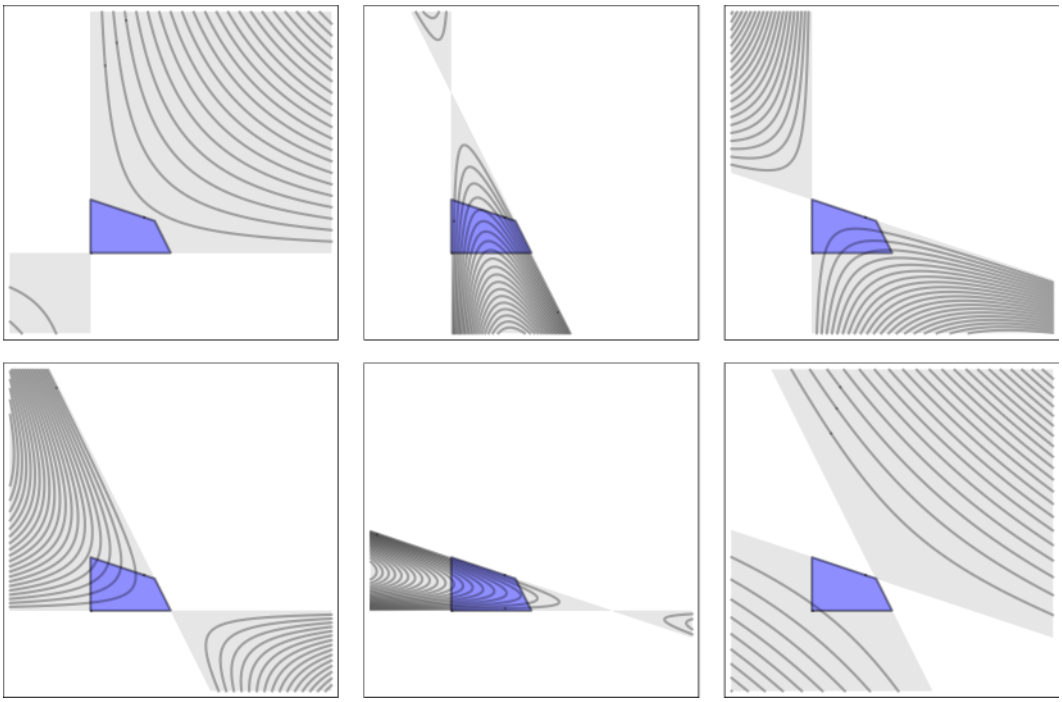

Figure 3: Convexification of the set $\{(x, xx^T) : x \in \mathcal{F}\}$ when $\mathcal{F}$ is a convex quadrilateral requires six linear inequalities derived from quadratics $(b_i - \langle a^i, x\rangle)(b_j - \langle a^j, x\rangle) \geq 0$, where $\langle a^i, x\rangle \leq b_i$ and $\langle a^j, x\rangle \leq b_j$ define facets of $\mathcal{F}$. Depicted here are the six quadratics in relation to $\mathcal{F}$.

5 Ellipsoids (in Any Dimension)

Thus far in Sections 2–4, we have considered several types of polytopes in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ (intervals, triangles, tetrahedra, and convex quadrilaterals). We now consider a different type of shape in $\mathbb{R}^n$, that of an ellipsoid. Relying on an affine transformation, we may assume the ellipsoid is simply the unit ball defined by the inequality $\|x\| \leq 1$, where $\|\cdot\|$ is the Euclidean norm. Then consider the n-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^n} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad \|x\| \leq 1 \end{aligned} \tag{ELL}$$

where $H \in \mathcal{S}^n$, $g \in \mathbb{R}^n$, and the feasible set $\mathcal{F} := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is the unit ball in $\mathbb{R}^n$. We wish to describe the set $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$. Problem (ELL) is equivalent to the *trust-region subproblem* studied in nonlinear programming (see [13] for example), and the specific results in this section first appeared in [23].

We first introduce the second-order (or Lorentz) cone in $\mathbb{R}^{n+1}$:

$$\begin{aligned} \mathcal{L} &:= \left\{ y \in \mathbb{R}^{n+1} : \sqrt{y_2^2 + \cdots + y_{n+1}^2} \leq y_1 \right\} \\ &= \left\{ y \in \mathbb{R}^{n+1} : y_1 \geq 0,\ y_2^2 + \cdots + y_{n+1}^2 \leq y_1^2 \right\} \\ &= \left\{ y \in \mathbb{R}^{n+1} : y_1 \geq 0,\ \langle y, Ly \rangle \geq 0 \right\} \end{aligned}$$

where the matrix $L \in \mathcal{S}^{n+1}$ is a diagonal matrix with entries $(1, -1, \ldots, -1)$. The next result characterizes the convex hull of matrices yy^T with $y \in \mathcal{L}$.

Proposition 3. *Regarding (ELL),* $\mathcal{K} := \overline{\text{conv}}\left\{yy^T : y \in \mathcal{L}\right\} = \{Y \succeq 0 : \langle L, Y \rangle \geq 0\}$.

Proof. For ease of reference, define $\mathcal{R} := \{Y \succeq 0 : \langle L, Y \rangle \geq 0\}$, which we note is a closed, convex cone. The containment $\mathcal{K} \subseteq \mathcal{R}$ is clear. To prove the reverse, let Y be an extreme ray of $\mathcal{R}$. We will show that Y is rank-1, that is, $Y = yy^T$ for some y, which will establish $Y \in \mathcal{K}$.

The spectral decomposition of Y provides a factorization $Y = \sum_k y^k (y^k)^T$. By negating y^k if necessary, we may assume $y_1^k \geq 0$. Then, if $\langle y^k, Ly^k \rangle \geq 0$ for each k, we see $Y \in \mathcal{K}$. On the other hand, if some $y^i \notin \mathcal{L}$, then the cumulative inequality $0 \leq \langle L, Y \rangle = \sum_k \langle y^k, Ly^k \rangle$ implies that some $y^j \in \text{int}(\mathcal{L})$. In other words, some $\langle y^j, Ly^j \rangle > 0$ must balance $\langle y^i, Ly^i \rangle < 0$. Let z be a (non-trivial) convex combination of y^i and y^j such that $z \in \text{bd}(\mathcal{L})$—that is, z satisfies $\langle z, Lz \rangle = 0$. Note that: $z \neq 0$ because $z_1 > 0$ due to the fact that $y_1^i \geq 0$ and $y_1^j > 0$; and $z \in \text{Range}(Y)$ because both y^i and y^j are in $\text{Range}(Y)$.

Consider $Y_\epsilon := Y - \epsilon\, zz^T$ for $\epsilon > 0$. Because $Y \succeq 0$ and $z \in \text{Range}(Y)$, it is well known that $Y_\epsilon \succeq 0$ for small ϵ. In addition, $\langle L, Y_\epsilon \rangle = \langle L, Y \rangle - \epsilon \langle z, Lz \rangle = \langle L, Y \rangle \geq 0$. So $Y_\epsilon \in \mathcal{R}$ for ϵ small. Note that $z \in \text{bd}(\mathcal{L})$ implies $zz^T \in \mathcal{R}$ also. Then $Y = Y_\epsilon + \epsilon\, zz^T$ where both Y_ϵ and zz^T are nonzero elements in $\mathcal{R}$. Because Y is an extreme ray of $\mathcal{R}$, it must be the case that Y_ϵ is parallel to zz^T, and hence Y is a positive multiple of zz^T. So Y is rank-1. □

We can now use Proposition 3 to characterize $\mathcal{G}$, the convex hull of terms (x, xx^T) for $x \in \mathcal{F}$.

Theorem 3. *Regarding (ELL),*

$$\begin{aligned}\mathcal{G} &:= \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0,\ \langle L, Y \rangle \geq 0 \end{array} \right\}.\end{aligned}$$

with $Y(x, X)$ given by (3) and $L := \text{Diag}(1, -1, \ldots, -1) \in \mathcal{S}^{n+1}$.

Proof. For ease of reference, define

$$\mathcal{R} := \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0,\ \langle L, Y \rangle \geq 0 \end{array} \right\},$$

which is closed and convex. The containment $\mathcal{G} \subseteq \mathcal{R}$ is clear, and so to prove the reverse, let $Y \in \mathcal{R}$. By Proposition 3, we can write $Y = \sum_k y^k (y^k)^T$ with each nonzero $y^k \in \mathcal{L}$. Decomposing $\binom{\nu_k}{w^k} := y^k$ with $\nu_k \in \mathbb{R}$ and $w^k \in \mathbb{R}^n$, we see $\|w^k\| \leq \nu_k$. If $\nu_k = 0$, then $w^k = 0$. So in fact every ν_k is positive. Define $x^k := w^k / \nu_k$. Then

$$Y = \sum_k {\nu_k}^2 \begin{pmatrix} 1 \\ x^k \end{pmatrix} \begin{pmatrix} 1 \\ x^k \end{pmatrix}^T \quad \text{with} \quad \|x^k\| \leq 1.$$

Since the top-left entry of Y equals 1 via the equation $Y = Y(x, X)$, we have $\sum_k {\nu_k}^2 = 1$. So Y is a convex combination of rank-1 matrices $\binom{1}{x^k}\binom{1}{x^k}^T$ with $x^k \in \mathcal{F}$. This proves $Y \in \mathcal{G}$ as desired. □

Focusing on x and X, an alternative expression for $\mathcal{G}$ is

$$\mathcal{G} := \left\{ (x, X) : \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0,\ \text{tr}(X) \leq 1 \right\},$$

where the linear inequality $\text{tr}(X) \leq 1$ is a relaxed version of $\|x\| \leq 1 \Leftrightarrow \text{tr}(xx^T) = \langle x, x \rangle \leq 1$.

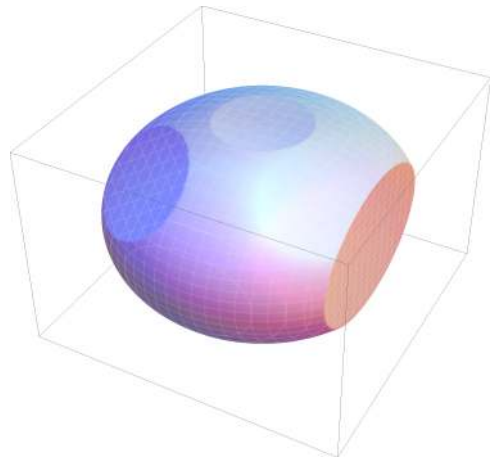

Figure 4: An ellipsoid in $\mathbb{R}^3$ missing three separate caps.

6 Ellipsoids Without Caps (in Any Dimension)

This section integrates—at least partially—the polyhedral aspects of Sections 2–4 with the non-polyhedral features of Section 5. The geometric object that we study is an *ellipsoid without caps*, i.e., a full-dimensional ellipsoid in $\mathbb{R}^n$ from which one or more caps are removed. An important assumption, which holds throughout the section, is that the caps themselves do not intersect; see Figure 4. This yields an object that locally looks like either an ellipsoid, a halfspace, or the intersection of an ellipsoid with a single halfspace. Nowhere does it locally look like the intersection of two or more halfspaces. Related papers include [5, 11, 12, 18, 23, 25]. In particular, [11] establishes that the non-intersecting assumption on the caps is necessary for the results of this section to hold, and [5] shows the solvability of such quadratic problems via an approach that does not make use of convex relaxation.

In Section 6.1, we examine the case of a single missing cap, while Section 6.2 considers the case of multiple missing caps.

6.1 Missing one cap

Consider the n-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^n} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad \|x\| \leq 1 \\ & \qquad\quad \langle a^1, x \rangle \leq b_1 \end{aligned} \tag{CAP}$$

where $H \in \mathcal{S}^n$, $g \in \mathbb{R}^n$, $a^1 \in \mathbb{R}^n$, and $b_1 \in \mathbb{R}$. The feasible set $\mathcal{F} := \{x \in \mathbb{R}^n : \|x\| \leq 1, \langle a^1, x\rangle \leq b_1\}$ is the unit ball in $\mathbb{R}^n$ with a single cap deleted. After an affine transformation, any quadratic over an ellipsoid without a cap can be modeled this way.

As usual, we want to characterize $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$, and the proof style matches earlier sections. Proposition 4 is the key result necessary for Theorem 4 below. These results first appeared in [23].

Proposition 4. *Regarding (CAP),*

$$\mathcal{K} := \overline{\text{conv}} \left\{ yy^T : \begin{array}{c} y \in \mathcal{L} \\ \langle c^1, y\rangle \geq 0 \end{array} \right\} = \left\{ Y \succeq 0 : \begin{array}{c} \langle L, Y\rangle \geq 0 \\ Yc^1 \in \mathcal{L} \end{array} \right\},$$

where $L := \text{Diag}(1, -1, \ldots, -1) \in \mathcal{S}^{n+1}$ *and* $c^1 := \binom{b_1}{-a^1}$.

Proof. For ease of reference, define

$$\mathcal{R} := \left\{ Y \succeq 0 : \begin{array}{c} \langle L, Y\rangle \geq 0 \\ Yc^1 \in \mathcal{L} \end{array} \right\},$$

which is closed and convex, and for notational convenience, define $c := c^1$. The containment $\mathcal{K} \subseteq \mathcal{R}$ holds because $yy^T \succeq 0$, $\langle L, yy^T\rangle = \langle y, Ly\rangle \geq 0$, and $(yy^T)c = \langle c, y\rangle\, y \in \mathcal{L}$.

So we need to prove the reverse. Let Y be an extreme ray of $\mathcal{R}$. We argue that Y must be rank-1 by considering three cases. For the first case, suppose $Yc \in \text{int}(\mathcal{L})$. Then Y is also extreme for the cone $\{Y \succeq 0 : \langle L, Y\rangle \geq 0\}$, and hence Y is rank-1 by Proposition 3.

For the second case, suppose $\langle c, Yc\rangle = 0$, and let $Y = \sum_k y^k (y^k)^T$ be a decomposition provided by Proposition 3, i.e., with every $y^k \in \mathcal{L}$. Then $0 = \langle c, Yc\rangle = \sum_k \langle c, y^k\rangle^2$, and so every y^k satisfies $\langle c, y^k\rangle = 0$ as well. It follows that each $y^k(y^k)^T$ is an element of $\mathcal{K}$. Since Y is extreme, all $y^k(y^k)^T$ must be parallel, and Y is rank-1.

For the last case, suppose $z := Yc \in \text{bd}(\mathcal{L})$ with $\langle c, z\rangle = \langle c, Yc\rangle > 0$; in particular, $z \neq 0$. Mimicking the proof of Proposition 3, we can show $Y_\epsilon := Y - \epsilon\, zz^T$ satisfies $Y_\epsilon \succeq 0$ and $\langle L, Y_\epsilon\rangle \geq 0$ for small $\epsilon > 0$. In addition,

$$Y_\epsilon\, c = (Y - \epsilon\, zz^T)c = Yc - \epsilon\langle c, z\rangle z = (1 - \epsilon\langle c, z\rangle)z \in \text{bd}(\mathcal{L}).$$

We have thus shown $Y_\epsilon \in \mathcal{R}$ for sufficiently small $\epsilon > 0$. As in the proof of Proposition 3, this establishes $\text{rank}(Y) = 1$. $\square$

Theorem 4. *Regarding (CAP),*

$$\begin{aligned}\mathcal{G} &:= \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \langle L, Y \rangle \geq 0 \\ Yc^1 \in \mathcal{L} \end{array} \right\}\end{aligned}$$

with $Y(x, X)$ *given by (3),* $L := \text{Diag}(1, -1, \ldots, -1) \in \mathcal{S}^{n+1}$*, and* $c^1 := \binom{b_1}{-a^1}$.

Proof. The proof matches exactly the proof of Theorem 3. □

It is instructive to write the constraints definining $\mathcal{G}$ in terms of x and X only:

$$\begin{aligned}&\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ &\text{tr}(X) \leq 1 \\ &\|b_1 x - X a^1\| \leq b_1 - \langle a^1, x \rangle.\end{aligned}$$

So $\mathcal{G}$ is described by one semidefinite constraint, one linear inequality, one linear equality (setting the top-left matrix entry to 1), and one second-order-cone constraint. As discussed in Section 5, $\text{tr}(X) \leq 1$ is the relaxed version of the convex quadratic constraint $\langle x, x \rangle \leq 1$. For the second-order-cone constraint, which is new in this section, reintroducing xx^T for X, we see that it is a relaxed version of

$$\|(b_1 - \langle a^1, x \rangle)x\| \leq b_1 - \langle a^1, x \rangle,$$

which is valid because it is the multiplication of $\|x\| \leq 1$ with $b_1 - \langle a^1, x \rangle \geq 0$. This constraint, called an *SOCRLT constraint* in [11], is highly nonlinear in x but convex in (x, X) after xx^T is relaxed to X. Figure 5 depicts the constraints $\langle x, x \rangle \leq 1$ and $\|(b_1 - \langle a^1, x \rangle)x\| \leq b_1 - \langle a^1, x \rangle$ relative to $\mathcal{F}$ in $\mathbb{R}^2$.

6.2 Missing multiple caps

Extending Section 6.1, now consider the n-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^n} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad \|x\| \leq 1 \\ & \qquad\quad Ax \leq b \end{aligned} \tag{CAPS}$$

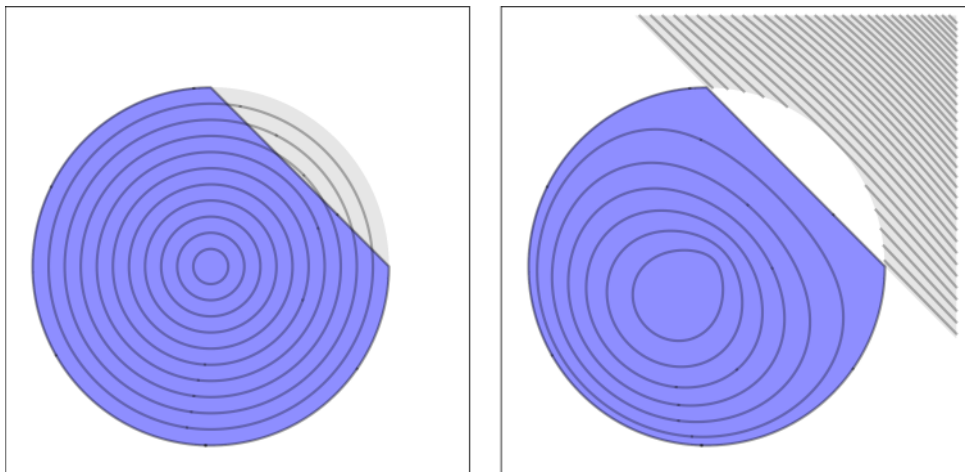

Figure 5: Convexification of the set $\{(x, xx^T) : x \in \mathcal{F}\}$, when $\mathcal{F}$ is an ellipsoid without a cap, requires four constraints: one positive-semidefinite, one linear equality, one linear inequality, and one second-order-cone. The linear inequality is a relaxed version of $\langle x, x\rangle \leq 1$, which is depicted in the left picture. The second-order-cone constraint is a relaxed version of the valid nonconvex constraint $\|(b_1 - \langle a^1, x\rangle)x\| \leq b_1 - \langle a^1, x\rangle$, depicted on the right.

where $H \in \mathcal{S}^n$, $g \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. The feasible set $\mathcal{F} := \{x \in \mathbb{R}^n : \|x\| \leq 1, Ax \leq b\}$ is the unit ball in $\mathbb{R}^n$ with m caps deleted. As mentioned at the beginning of Section 6, we assume that the deleted caps are themselves non-intersecting; see Figure 4. Breaking $Ax \leq b$ into its individual constraints $\langle a^i, x\rangle \leq b_i$, we also assume that no $\langle a^i, x\rangle \leq b_i$ is redundant on $\mathcal{F}$. Geometrically, this allows us to state the following condition.

Condition 1. *If $x \in \mathbb{R}^n$ satisfies $\|x\| \leq 1$ and $\langle a^i, x\rangle = b_i$ for some i, then $x \in \mathcal{F}$.*

To characterize $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$, we first analyze $\mathcal{K} := \overline{\text{conv}}\{yy^T : y \in \mathcal{L}, Cy \geq 0\} \subseteq \mathcal{S}^{n+1}$, where $C := (b, -A)$. Also define $c^i := \binom{b_i}{-a^i}$. We need to restate Condition 1 in terms of y and prove two lemmas in preparation for Proposition 5 and Theorem 5 below.

Condition 2. *If $y \in \mathbb{R}^{n+1}$ satisfies $y \in \mathcal{L}$ and $\langle c^i, y\rangle = 0$ for some i, then $Cy \geq 0$.*

Lemma 4. *If $y \in \text{int}(\mathcal{L})$, then under Condition 2, at most one term $\langle c^i, y\rangle$ equals 0.*

Proof. Suppose $\langle c^i, y\rangle = \langle c^j, y\rangle = 0$ for $i \neq j$, and let d be any vector with $\langle c^i, d\rangle < 0$ and $\langle c^j, d\rangle = 0$. Such a d exists because c^i and c^j are linearly independent as we have assumed that the constraints $\langle a^i, x\rangle \leq b_i$ and $\langle a^j, x\rangle \leq b_j$ are not redundant for $\mathcal{F}$. Then, for small $\epsilon > 0$, $y_\epsilon := y + \epsilon\, d$ satisfies $y_\epsilon \in \mathcal{L}$ and $\langle c^j, y_\epsilon\rangle = 0$, and yet $Cy_\epsilon \ngeq 0$ because $\langle c^i, y_\epsilon\rangle < 0$. This violates Condition 2. $\square$

The following result is a generic result about convex cones in any dimension. It will be applied in the proof of Proposition 5 for matrix cones, but the proof uses lower-case vector notation.

Lemma 5. *Let $\mathcal{P}$ be a closed convex cone, and let $\mathcal{Q}$ be a half-space containing the origin. Every extreme ray of $\mathcal{P} \cap \mathcal{Q}$ is either an extreme ray of $\mathcal{P}$ or can be expressed as the sum of two extreme rays of $\mathcal{P}$.*

Proof. Suppose $\mathcal{Q}$ is defined by $\langle q, x \rangle \geq 0$, and let $\bar{x}$ be an extreme ray of $\mathcal{P} \cap \mathcal{Q}$. Since $\bar{x} \in \mathcal{P}$, we can write $\bar{x} = \sum_k \bar{x}^k$, where each $\bar{x}^k$ is an extreme ray of $\mathcal{P}$.

For a vector variable $\lambda = (\lambda_k)$, define $x(\lambda) := \sum_k \lambda_k \bar{x}^k$. For example, $x(e) = \bar{x}$, where e is the all-ones vector. Also define the polyhedral cone $\Lambda := \{\lambda \geq 0 : \langle q, x(\lambda) \rangle \geq 0\}$, which satisfies $x(\Lambda) \subseteq \mathcal{P} \cap \mathcal{Q}$ by construction. In addition, standard polyhedral theory guarantees that every extreme ray $\lambda \in \Lambda$ has at most two positive entries.

As noted, $e \in \Lambda$. Hence, we can write $e = \sum_j \lambda^j$, where each λ^j is an extreme ray of Λ. Expanding $\bar{x} = x(e)$, we have $\bar{x} = \sum_j x(\lambda^j)$ with each $x(\lambda^j) \in \mathcal{P} \cap \mathcal{Q}$. Since $\bar{x}$ is extreme in $\mathcal{P} \cap \mathcal{Q}$ by assumption, every $x(\lambda^j)$ is a positive multiple of $\bar{x}$. Since λ^j has at most two positive entries, this completes the proof. $\square$

Proposition 5. *Regarding (CAPS),*

$$\mathcal{K} := \overline{\text{conv}} \left\{ yy^T : \begin{array}{c} y \in \mathcal{L} \\ Cy \geq 0 \end{array} \right\} = \left\{ Y \succeq 0 : \begin{array}{c} \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \\ Yc^i \in \mathcal{L} \ \ \forall\, i \end{array} \right\}.$$

with $L := \text{Diag}(1, -1, \ldots, -1) \in \mathcal{S}^{n+1}$, $C := (b, -A)$ and $c^i := \binom{b_i}{-a^i}$ for all $i = 1, \ldots, m$.

Proof. The proof is by induction. Proposition 4 covers the base case when only one cap is missing, and we suppose the result holds for $m - 1$ caps. Then for m caps, let

$$\mathcal{R} := \left\{ Y \succeq 0 : \begin{array}{c} \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \\ Yc^i \in \mathcal{L} \ \ \forall\, i \end{array} \right\},$$

which is closed and convex. As usual, we have $\mathcal{K} \subseteq \mathcal{R}$. We need to prove $\mathcal{K} \supseteq \mathcal{R}$. So let $Y \in \mathcal{R}$ be extreme; our goal is to show that Y is rank-1. Also define $z^i := Yc^i \in \mathcal{L}$ for each i, and note that the i-th column of the matrix constraint $CYC^T \geq 0$ equivalently expresses $Cz^i \geq 0$. We consider two primary cases: (i) $z^i \in \text{bd}(\mathcal{L})$ for some i; and (ii) $z^i \in \text{int}(\mathcal{L})$ for all i.

For case (i), suppose first that $z^i = 0$, and take $i = m$ without loss of generality. Then $\langle c^m, Yc^m\rangle = \langle c^m, z^m\rangle = 0$. Based on $Y \succeq 0$ and $\langle L, Y\rangle \geq 0$, we use Proposition 3 to write $Y = \sum_k y^k(y^k)^T$ with $y^k \in \mathcal{L}$. Then $\langle c^m, Yc^m\rangle = 0$ implies $\langle c^m, y^k\rangle = 0$ for each k. So $Cy^k \geq 0$ by Condition 2, and thus $Y \in \mathcal{K}$. Now suppose $0 \neq z^i \in \mathrm{bd}(\mathcal{L})$, but this time take $i = 1$ without loss of generality. Then Proposition 4 implies

$$Y = \sum_k y^k(y^k)^T \qquad \text{with} \qquad \langle c^1, y^k\rangle \geq 0,\ y^k \in \mathcal{L}$$

which implies $z^1 = Yc^1 = \sum_k \langle c^1, y^k\rangle\, y^k$. For a given k, if $\langle c^1, y^k\rangle > 0$, then y^k must be a positive multiple of z^1 as z^1 is a nonzero element of the boundary of $\mathcal{L}$. Otherwise, if $\langle c^1, y^k\rangle = 0$, then $Cy^k \geq 0$ by Condition 2. Either way, we have $y^k(y^k)^T \in \mathcal{K} \subseteq \mathcal{R}$. Since Y is extreme in $\mathcal{R}$, this implies that every $y^k(y^k)^T$ is a positive multiple of Y, which in turn shows that Y is rank-1.

For case (ii) with $z^i \in \mathrm{int}(\mathcal{L})$ for all i, suppose $CYC^T > 0$. Then Y is extreme for the cone $\{Y \succeq 0 : \langle L, Y\rangle \geq 0\}$ and hence rank-1 by Proposition 3. So suppose some $\langle c^i, Yc^j\rangle = \langle c^i, z^j\rangle = \langle z^i, c^j\rangle = 0$. Without loss of generality, we may assume $i \neq j$. If not, then a diagonal entry of $CYC^T \succeq 0$ equals 0, and hence an entire row equals 0 from which we may choose an entry. Moreover, by a simple relabeling of indices, take $i = 1$ and $j = m$ and for ease of notation, define $s := c^1$ and $t := c^m$.

By Lemma 4, we know that $i = 1$ is the only index satisfying $\langle c^i, z^m\rangle = 0$. Similarly, $j = m$ is the only index satisfying $\langle z^1, c^j\rangle = 0$. In particular, $\langle s, Ys\rangle = \langle s, z^1\rangle > 0$ and $\langle t, Yt\rangle = \langle t, z^m\rangle > 0$. Defining

$$\mathcal{P} := \left\{ Y \succeq 0 : \begin{array}{c} \langle L, Y\rangle \geq 0 \\ \langle c^i, Yc^j\rangle \geq 0 \quad \forall\, i, j \leq m-1 \\ Yc^i \in \mathcal{L} \qquad \forall\, i \leq m-1 \end{array} \right\}, \qquad \mathcal{Q} := \{Y : \langle c^1, Yc^m\rangle \geq 0\},$$

we also see that Y is extreme in $\mathcal{P} \cap \mathcal{Q}$. Applying the induction hypothesis on $\mathcal{P}$ and Lemma 5 on $\mathcal{P} \cap \mathcal{Q}$, we conclude that $\mathrm{rank}(Y) \leq 2$. If the rank equals 1, we are done. So assume $\mathrm{rank}(Y) = 2$. We derive a contradiction to complete the proof. Consider the equation

$$W := \begin{pmatrix} s^T \\ t^T \\ I \end{pmatrix} Y \begin{pmatrix} s & t & I \end{pmatrix} = \begin{pmatrix} \langle s, Ys\rangle & \langle s, Yt\rangle & s^TY \\ \langle t, Ys\rangle & \langle t, Yt\rangle & t^TY \\ Ys & Yt & Y \end{pmatrix} = \begin{pmatrix} \langle s, Ys\rangle & 0 & (z^1)^T \\ 0 & \langle t, Yt\rangle & (z^m)^T \\ z^1 & z^m & Y \end{pmatrix}.$$

It holds that $W \succeq 0$ with $\mathrm{rank}(W) \leq \mathrm{rank}(Y) = 2$, and the Schur complement theorem implies $M := Y - \sigma z^1(z^1)^T - \tau z^m(z^m)^T \succeq 0$, where $\sigma := (s^TYs)^{-1} > 0$ and $\tau := (t^TYt)^{-1} > 0$,

with $\operatorname{rank}(M) = \operatorname{rank}(W) - 2$. So $\operatorname{rank}(M) \leq 0$ and $M = 0$, that is, $Y = \sigma z^1 (z^1)^T + \tau z^m (z^m)^T$, contradicting the assumption that Y is extreme. □

Theorem 5. *Regarding (CAPS),*

$$\begin{aligned}\mathcal{G} &:= \overline{\operatorname{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \\ Yc^i \in \mathcal{L} \ \ \forall\, i \end{array} \right\}\end{aligned}$$

with $Y(x, X)$ given by (3), $L := \operatorname{Diag}(1, -1, \ldots, -1) \in \mathcal{S}^{n+1}$, $C := (b, -A)$, and $c^i := \binom{b_i}{-a^i}$ for all $i = 1, \ldots, m$.

Proof. Based on Proposition 5, the proof follows exactly the proof of Theorem 3. □

Theorem 5 first appeared in [12]. However, we remark that, in that paper, the base case of the induction proof was incorrectly stated to correspond to $m = 0$ linear constraints. As written here, the correct base case is $m = 1$, and the result of [12] can be fixed by simply changing $m = 0$ to $m = 1$.

7 Elliptic Convex Polygons

An *elliptic convex polygon* in $\mathbb{R}^2$ is a convex polygon whose vertices all lie on a single ellipse (the boundary of an ellipsoid in $\mathbb{R}^2$). Said differently, an elliptic convex polygon is one that is inscribed in an ellipsoid. After an affine transformation, every elliptic convex polygon is equivalent to a *cyclic* convex polygon, i.e., one inscribed in the unit ball. The results of Section 6—and of Theorem 5 in particular—can be specialized to cover this case; see [12].

Consider the 2-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^2} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad Ax \leq b \end{aligned} \tag{INSC}$$

where $H \in \mathcal{S}^2$, $g \in \mathbb{R}^2$, $A \in \mathbb{R}^{m \times 2}$, and $b \in \mathbb{R}^m$, and the feasible set $\mathcal{F} := \{x \in \mathbb{R}^2 : Ax \leq b\}$ defines a cyclic convex polygon. Note that that the constraint $\|x\| \leq 1$ is redundant for $\mathcal{F}$

and that $\mathcal{F}$ satisfies Condition 1 in Section 6.2. As a result, Theorem 5 applies and

$$\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \\ Yc^i \in \mathcal{L} \ \ \forall\, i \end{array} \right\}.$$

However, the redundancy of $\|x\| \leq 1$ allows one to streamline the description of $\mathcal{G}$.

Theorem 6. *Regarding (INSC),*

$$\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \end{array} \right\}$$

with $Y(x, X)$ given by (3), $L := \text{Diag}(1, -1, -1) \in \mathcal{S}^3$, and $C := (b, -A)$.

Proof. For ease of reference, define

$$\mathcal{R} := \left\{ (x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \langle L, Y \rangle \geq 0 \\ CYC^T \geq 0 \end{array} \right\}$$

which we note is closed and convex. Based on the discussion preceding the theorem, we need only show that $(x, X) \in \mathcal{R}$ implies $z^i := Yc^i \in \mathcal{L}$, where $Y = Y(x, X)$. So let $Y = Y(x, X) \succeq 0$ satisfy $\langle L, Y \rangle \geq 0$ and $C^T YC \geq 0$. The i-th column of $CYC^T \geq 0$ is equivalent to $Cz^i \geq 0$. Decomposing $\binom{\nu_i}{w^i} := z^i$, we thus see $Aw^i \leq \nu_i b$. Similar to the proofs of Theorems 1 and 2, we must have $\nu_i \geq 0$. If $\nu_i = 0$, then $w^i = 0$ and $z^i \in \mathcal{L}$. If $\nu_i > 0$, then define $x^i := w^i/\nu_i$ so that $Ax^i \leq b$, which implies $\|x^i\| \leq 1$, which in turn shows that $z^i \in \mathcal{L}$. □

8 Intersections of Two Ellipsoids (in Dimension 2)

Sections 2–6 have examined geometries arising from polyhedral and ellipsoidal constraints. Although we have examined several linear constraints at a time, so far we have only included at most one ellipsoidal constraint. In this section, we summarize (but do not prove) recent progress that has been made on characterizing $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$ when $\mathcal{F}$ is the intersection of two ellipsoids in $\mathbb{R}^2$. These results come from [24]. By an affine transformation, we can assume that one of the ellipsoids is the unit ball.

Consider the 2-dimensional optimization problem

$$\begin{array}{rcll} v^* & := & \min\limits_{x\in\mathbb{R}^2} & \langle x, Hx\rangle + 2\langle g, x\rangle \\ & & \text{s.t.} & \|x\| \le 1 \\ & & & \|D(x-d)\| \le 1 \end{array} \tag{ELL2}$$

where $H \in \mathcal{S}^2$, $g \in \mathbb{R}^2$, $d \in \mathbb{R}^2$, and $D \in \mathcal{S}^2$ is positive definite. So the feasible set $\mathcal{F}$ is the intersection of two ellipsoids, where in particular d is the center of the second ellipsoid. We remark that the paper [4] establishes the polynomial-time solvability of (ELL2) and its extension to general n via techniques that do not require convex relaxation. It will be helpful to write the optimization in homogenized form:

$$\begin{array}{rcll} v^* & := & \min\limits_{y\in\mathbb{R}^3} & \langle y, \hat{H}y\rangle \\ & & \text{s.t.} & y \in \mathcal{L} := \{y \in \mathbb{R}^3 : y_1 \ge 0, \langle y, Ly\rangle \ge 0\} \\ & & & y \in \mathcal{M} := \{y \in \mathbb{R}^3 : y_1 \ge 0, \langle y, My\rangle \ge 0\} \\ & & & y_1 = 1 \end{array}$$

where $L := \mathrm{Diag}(1, -1, -1) \in \mathcal{S}^3$,

$$M := \begin{pmatrix} 1 - \|Dd\|^2 & (D^2d)^T \\ D^2d & -D^2 \end{pmatrix} \in \mathcal{S}^3, \qquad \hat{H} := \begin{pmatrix} 0 & g^T \\ g & H \end{pmatrix} \in \mathcal{S}^3.$$

Define $\mathcal{K} := \overline{\mathrm{conv}}\{yy^T : y \in \mathcal{L} \cap \mathcal{M}\}$. It is clear that $Y \succeq 0$, $\langle L, Y\rangle \ge 0$, and $\langle M, Y\rangle \ge 0$ are valid for $\mathcal{K}$. In addition, for any (possibly infinite) collection of valid linear inequalities $Cy \ge 0$ for $\mathcal{L} \cap \mathcal{M}$, we know that $CYC^T \ge 0$ is valid for $\mathcal{K}$. For an individual valid linear inequality $\langle c, y\rangle \ge 0$, the cone constraint $Yc \in \mathcal{L} \cap \mathcal{M}$ is also valid for $\mathcal{K}$. These are the types of valid inequalities that we have seen in Sections 2–7, but as it turns out, they are not enough to capture $\mathcal{K}$. We need something stronger.

To describe what is needed for $\mathcal{K}$, we first note that all constraints of the types $CYC^T \ge 0$ and $Yc \in \mathcal{L} \cap \mathcal{M}$ just mentioned can be reduced to a single type of constraint. Let $\mathcal{L}^*$ and $\mathcal{M}^*$ be the dual cones, and consider $(p, q) \in \mathcal{L}^* \times \mathcal{M}^*$ so that the quadratic inequality $\langle p, y\rangle\langle q, y\rangle \ge 0$ is valid on $\mathcal{L} \cap \mathcal{M}$. Then it is not difficult to see that $\langle pq^T, Y\rangle \ge 0$ is valid for $\mathcal{K}$ and that the entire collection of constraints $\{\langle pq^T, Y\rangle \ge 0 : (p, q) \in \mathcal{L}^* \times \mathcal{M}^*\}$ captures ones such as $CYC^T \ge 0$ and $Yc \in \mathcal{L} \cap \mathcal{M}$.

Given $(p, q) \in \mathcal{L}^* \times \mathcal{M}^*$, we are interested in valid quadratics on $\mathcal{L} \cap \mathcal{M}$ of the form $\langle p, y\rangle\langle q, y\rangle \ge \langle r, y\rangle^2$, where $r \in \mathbb{R}^3$. It is not initially clear that such valid constraints actually

exist with $r \neq 0$, but if they do, then they clearly imply the constraints $\langle p, y\rangle\langle q, y\rangle \geq 0$ of the previous paragraph. To establish notation, define

$$\mathcal{V} := \left\{ (p, q, r) \in \mathcal{L}^* \times \mathcal{M}^* \times \mathbb{R}^3 : \begin{array}{c} \langle p, y\rangle\langle q, y\rangle \geq \langle r, y\rangle^2 \\ \text{is valid for } \mathcal{L} \cap \mathcal{M} \end{array} \right\}.$$

It is interesting to note that, although a constraint $\langle p, y\rangle\langle q, y\rangle \geq \langle r, y\rangle^2$ is generally nonconvex over $y \in \mathbb{R}^3$, it is actually convex over $y \in \mathcal{L}\cap\mathcal{M}$. This is because $y \in \mathcal{L}\cap\mathcal{M}$ implies $\langle p, y\rangle \geq 0$ and $\langle q, y\rangle \geq 0$, in which case the quadratic can be modeled using a rotated second-order cone.

It turns out that $\mathcal{V}$ does indeed provide an interesting, non-trivial class of valid inequalities that strengthen the previously known simpler ones $\langle p, y\rangle\langle q, y\rangle \geq 0$. In addition, the resulting valid inequalities $\langle pq^T, Y\rangle \geq \langle rr^T, Y\rangle$ for $\mathcal{K}$ are precisely what is needed to capture $\mathcal{K}$. This leads to the following description for $\mathcal{G}$ (see corollary 1 and the surrounding discussion in [24]):

Theorem 7. *Regarding (ELL2),*

$$\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\} = \left\{ (x, X) : \begin{array}{c} Y = Y(x, X),\ Y \succeq 0 \\ \langle L, Y\rangle \geq 0,\ \langle M, Y\rangle \geq 0 \\ \langle pq^T - rr^T, Y\rangle \geq 0 \ \ \forall\ (p, q, r) \in \mathcal{V} \end{array} \right\}.$$

In practice, to solve (ELL2), the separation problem over the constraints indexed by $\mathcal{V}$ is critical since $\mathcal{V}$ is semi-infinite and cannot be listed explicitly. Indeed, this separation can be handled reasonably well in computation even in higher dimensions (i.e., when $n \geq 2$). One important observation is that, for a fixed $(p, q, r) \in \mathcal{L}^* \times \mathcal{M}^* \times \mathbb{R}^3$ not necessarily in $\mathcal{V}$, there is always a maximum $\lambda \geq 0$, say $\lambda_{\max}$, such that $(p, q, \sqrt{\lambda}r)$ is in $\mathcal{V}$. In essence, $\lambda_{\max}$ corresponds to the strongest *valid* constraint of the form $\langle p, y\rangle\langle q, y\rangle \geq \lambda\langle r, y\rangle^2$. In practice, it is critical to identify valid triples $(p, q, \sqrt{\lambda_{\max}}r)$ to get the strongest cuts.

As an example (completely in terms of $x \in \mathbb{R}^2$, not $y \in \mathbb{R}^3$), consider the intersection of the two ellipsoids defined by $x_1^2 + x_2^2 \leq 1$ and $2x_1^2 + \frac{1}{2}x_2^2 \leq 1$, which is depicted in Figure 6. The feasible point $x^1 := \binom{0}{1}$ is supported by the valid inequality $x_2 \leq 1$, and the feasible point $x^2 := \binom{1/\sqrt{2}}{0}$ is supported by the valid inequality $x_1 \leq \frac{1}{\sqrt{2}}$. Moreover, the line defined by $\sqrt{2}x + y = 1$ passes through both points x^1 and x^2. For $\lambda \geq 0$, consider the quadratic inequality

$$(1 - x_2)(\tfrac{1}{\sqrt{2}} - x_1) \geq \lambda(1 - \sqrt{2}x_1 - x_2)^2.$$

Related to the discussion above, $1 - x_2$ plays the role of $\langle p, y\rangle$, $\frac{1}{\sqrt{2}} - x_1$ plays the role of $\langle q, y\rangle$, and $1 - \sqrt{2}x_1 - x_2$ plays the role of $\langle r, y\rangle$. One can determine numerically that the above

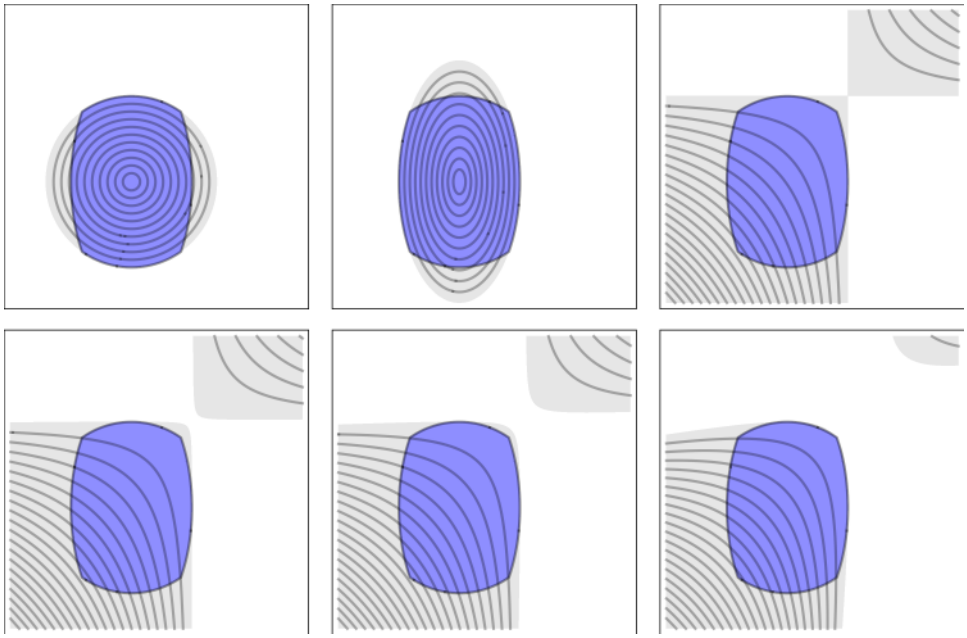

Figure 6: Convexification of the set $\{(x, xx^T) : x \in \mathcal{F}\}$, when $\mathcal{F}$ is the intersection of two ellipsoids, requires a semidefiniteness constraint, a linear equality constraint, two constraints based on the defining convex quadratics (first two plots), and an entire family of linear constraints derived from valid quadratic inequalities that "hug" the feasible region (final four plots).

quadratic inequality is valid on $\mathcal{F}$ up until a maximum value of approximately $\lambda_{\max} = 0.07$.

It is also worth mentioning that, given our choice of x^1 and x^2 above, we had to choose $1 - \sqrt{2}x_1 - x_2$ for the right-hand-side quadratic (or a scaled version of it) in order to achieve a positive $\lambda_{\max}$. This is because the zeros of the right-hand side must contain the zeros of the left-hand side; otherwise, $\lambda_{\max}$ will be zero.

In Figure 6, we depict six valid quadratic inequalities relative to $\mathcal{F}$. Proceeding left-to-right and top-to-bottom, the first two pictures show the quadratics $x_1^2 + x_2^2 \leq 1$ and $2x_1^2 + \frac{1}{2}x_2^2 \leq 1$, respectively. The next four show the quadratic of the previous paragraph for the following increasingly larger choices of λ: 0, $\frac{1}{16}\lambda_{\max}$, $\frac{1}{4}\lambda_{\max}$, and $\lambda_{\max}$. One can see how the quadratic "hugs" the feasible region $\mathcal{F}$ tighter and tighter as λ increases.

9 Cuts for the Case of General Convex Polygons

In the preceding Sections 2–8, we have explored many examples of geometries $\mathcal{F}$—most in $\mathbb{R}^2$, some in $\mathbb{R}^n$—for which we can describe the convex hull $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$, thus allowing us to optimize arbitrary quadratic functions over $\mathcal{F}$. Except for the case of the intersection of two ellipsoids, the description of $\mathcal{G}$ has been efficient in the sense that it can be written explicitly using a polynomial number of semidefinite, second-order-cone, and linear constraints. For the intersection of two ellipsoids, $\mathcal{G}$ requires a semi-infinite class of linear inequalities that exploit the special, curved geometry of $\mathcal{F}$.

In this section, we attempt to tackle general convex polygons in $\mathbb{R}^2$ using the insight gained thus far. So consider

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^2} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad Ax \leq b \end{aligned} \tag{POLY}$$

where $H \in \mathcal{S}^2$, $g \in \mathcal{S}^2$, $A \in \mathbb{R}^{m \times 2}$, and $b \in \mathbb{R}^m$. We assume that $\mathcal{F} := \{x \in \mathbb{R}^2 : Ax \leq b\}$ is m-sided and bounded with interior. To describe $\mathcal{G}$, we will certainly need $Y = Y(x, X) \succeq 0$ and $CYC^T \geq 0$, where $C := (b, -A)$. As we saw in Sections 3 and 4, these are enough to handle triangles ($m = 3$) and quadrilaterals ($m = 4$). On the other hand, Section 7 tells us that more is needed when $m \geq 5$. For example, if we know that $\mathcal{F}$ is inscribed in an ellipsoid defined by $\|D(x - d)\| \leq 1$, then adding the constraint

$$\langle D^2, X \rangle - 2\langle D^2 d, x \rangle \leq 1 - \|Dd\|^2$$

captures $\mathcal{G}$. So the question remains: what additional constraints are necessary for general

polygons $\mathcal{F}$ that are *not* inscribed in an ellipsoid?

We propose to search for quadratic constraints valid on $\mathcal{F}$ that define shapes $\mathcal{H}$ in which $\mathcal{F}$ is partially inscribed. By *partially inscribed*, we mean that $\mathcal{F} \subseteq \mathcal{H}$ and some of the vertices of $\mathcal{F}$ lie in $\mathrm{bd}(\mathcal{H})$. For example, we might find an ellipsoid $\mathcal{H} \supseteq \mathcal{F}$ such that $\mathcal{F}$ has four vertices in $\mathrm{bd}(\mathcal{H})$, or we might find a hyperbolic region $\mathcal{H}$ such that one branch contains $\mathcal{F}$ with three vertices in its boundary.

Suppose that a shape $\mathcal{H}$ partially inscribing $\mathcal{F}$ is defined by the quadratic $\langle x, Sx\rangle + 2\langle s, x\rangle \leq \sigma$. The containment $\mathcal{F} \subseteq \mathcal{H}$ ensures $\langle S, X\rangle + 2\langle s, x\rangle + \sigma \leq 0$ is valid for $\mathcal{G}$, and the fact that $\mathcal{F}$ is partially inscribed in $\mathcal{H}$ lessens the chance that $\langle x, Sx\rangle + 2\langle s, x\rangle \leq \sigma$ could be strengthened. For example, if $\mathcal{F} \subseteq \mathrm{int}(\mathcal{H})$, then σ could be decreased to tighten the constraint, a type of situation we would like to avoid.

In particular, our search is motivated by the types of quadratics discussed in Section 8. Let x^1 and x^2 be two distinct vertices of $\mathcal{F}$, and let $\langle \alpha^1, x\rangle \leq \beta_1$ and $\langle \alpha^2, x\rangle \leq \beta_2$ be any two inequalities valid for $\mathcal{F}$, which support x^1 and x^2, respectively. For example, $\langle \alpha^j, x\rangle \leq \beta^j$ may be a proper convex combination of the two original constraints of the form $\langle a^i, x\rangle \leq b_i$ supporting x^j. Also let $\langle \gamma, x\rangle = \delta$ be the equation of the line connecting x^1 and x^2 in $\mathbb{R}^2$, which is unique up to scaling. We then consider the class of quadratic constraints $q_\lambda(x) \geq 0$, where $\lambda \geq 0$ and

$$q_\lambda(x) := (\beta_1 - \langle \alpha^1, x\rangle)(\beta_2 - \langle \alpha^2, x\rangle) - \lambda(\langle \gamma, x\rangle - \delta)^2.$$

Also define the shape $\mathcal{H}_\lambda := \{x \in \mathbb{R}^2 : q_\lambda(x) \geq 0\}$. For illustration, Figure 7 depicts a polygon $\mathcal{F}$, the three lines defining the quadratic $q_\lambda(x)$, and a particular shape $\mathcal{H}_\lambda$ in which $\mathcal{F}$ is partially inscribed. In this case, $\mathcal{H}_\lambda$ is an ellipsoid and two vertices of $\mathcal{F}$ lie in $\mathrm{bd}(\mathcal{H}_\lambda)$.

By construction, $\mathcal{F}$ is partially inscribed in $\mathcal{H}_0$, and only the vertices x^1 and x^2 are in $\mathrm{bd}(\mathcal{H}_0)$. Indeed, x^1 and x^2 are in $\mathrm{bd}(\mathcal{H}_\lambda)$ for all λ. On the other hand, one can see that increasing λ eventually violates $\mathcal{F} \subseteq \mathcal{H}_\lambda$. Using the continuity of S_λ in λ (not difficult to prove), it follows that there exists a largest $\lambda_{\max}$ such that $\mathcal{F}$ is partially inscribed in $\mathcal{H}_{\lambda_{\max}}$. In addition, as discussed in Section 8, quadratic inequalities such as $q_\lambda(x) \geq 0$ are convex on $\mathcal{F}$. This means that $\mathcal{H}_{\lambda_{\max}}$ must necessarily contain a third vertex x^j in its boundary. Indeed, this observation allows us to deduce the following formula for $\lambda_{\max}$:

$$\lambda_{\max} := \min \left\{ \frac{(\beta_1 - \langle \alpha^1, x^j\rangle)(\beta_2 - \langle \alpha^2, x^j\rangle)}{(\langle \gamma, x^j\rangle - \delta)^2} : \begin{array}{c} x^j \text{ is a vertex of } \mathcal{F}, \\ \text{not equal to } x^1 \text{ or } x^2 \end{array} \right\}.$$

The minimizing vertex x^j in this definition is the third vertex in $\mathrm{bd}(\mathcal{H}_{\lambda_{\max}})$, but there can be more vertices if there are multiple vertex minimizers. Note that the specific λ in Figure 7

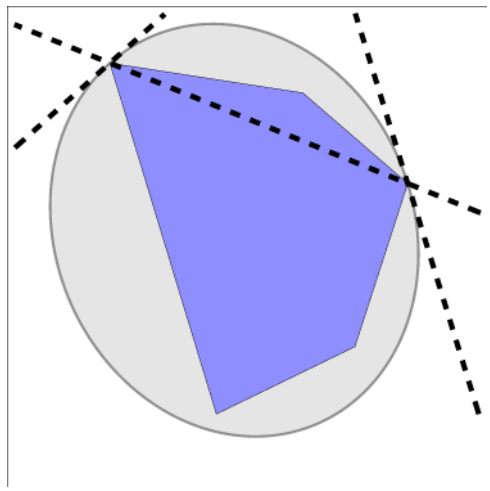

Figure 7: A depiction of the three lines that are used to construct a quadratically defined shape in which $\mathcal{F}$ is partially inscribed. Two lines support two vertices, while the third connects the vertices.

is less than $\lambda_{\max}$ because only x^1 and x^2 are in the boundary of $\mathcal{H}_\lambda$.

Summarizing, our search for a *single* valid quadratic relies on the specification of $\langle \alpha^1, x\rangle \leq \beta_1$ supporting x^1, $\langle \alpha^2, x\rangle \leq \beta_2$ supporting x^2, and $\langle \gamma, x\rangle = \delta$ connecting x^1 and x^2. Then $\lambda_{\max}$ is calculated using the other vertices, which gives the valid $q_{\lambda_{\max}}(x) \geq 0$ and the shape $\mathcal{H}_{\lambda_{\max}}$ in which $\mathcal{F}$ is partially inscribed at three or more vertices. We can also search for *multiple* valid quadratics by considering multiple combinations of x^1, x^2, $\langle \alpha^1, x\rangle \leq \beta_1$, and $\langle \alpha^2, x\rangle \leq \beta_2$. We do not claim that searching for multiple valid quadratics is efficient in a theoretical sense, but it can be automated quite easily in practice given the defining inequalities and vertices of $\mathcal{F}$.

At this time, we do not have a proof that all valid quadratics of the type described are enough to capture $\mathcal{G}$, but we conjecture that they are. To support this conjecture, we randomly generated 6,000 instances of (POLY). The values for m were taken from $\{5, 6, 7, 8, 9, 10\}$, and each value of m had 1,000 instances. In addition, for each instance, both the objective $\langle x, Hx\rangle + 2\langle g, x\rangle$ and constraints $Ax \leq b$ were generated randomly, i.e., no objectives or feasible sets were repeated from one instance to another. We then solved each instance using just the constraints $Y = Y(x, X) \succeq 0$ and $CYC^T \geq 0$, where $C := (b, -A)$. We considered an instance *globally solved* if the optimal Y of the relaxation was numerically rank-1. Otherwise, the instance was *globally unsolved*. This left 4,835 globally unsolved instances.

For those 4,835 unsolved instances, we then re-solved each instance, but this time separating the quadratics discussed above. (Again, we do not claim that our separation was

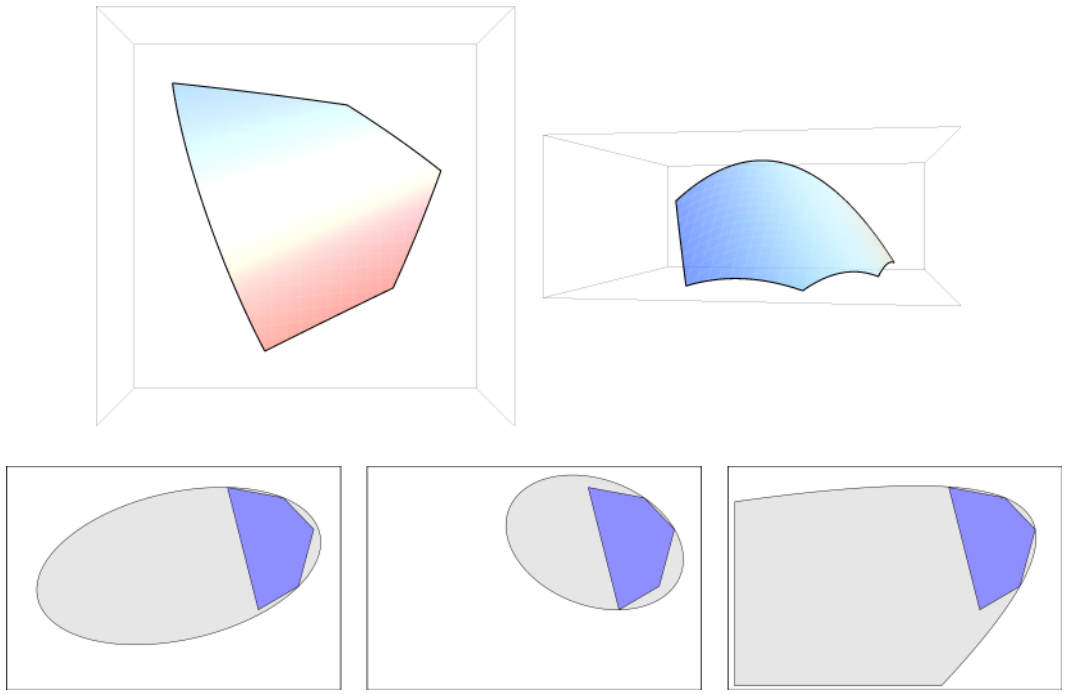

Figure 8: An instance of (POLY) with $m = 5$. The first two pictures show the quadratic objective plotted over the feasible region in 3D (both top and side views). The final three pictures show in 2D the shapes of the three cuts required to solve this instance globally.

efficient in theory, but it was quick and thorough in practice.) We found that all 4,835 instances were globally solved by this procedure, and a large majority (85% of the 4,835) required just 1 or 2 cuts, although one instance did require 10 cuts. In Figure 8, we illustrate one of the instances with $m = 5$, showing both the function and the shapes of the three cuts required to globally solve this instance.

10 Polytopes (in Any Dimension)

To complete the connection of the prior sections of this paper with the field of copositive optimization, we now recapitulate the main—but here simplified—result of [9], which considers the n-dimensional optimization problem

$$\begin{aligned} v^* \quad := \quad & \min_{x \in \mathbb{R}^n} \quad \langle x, Hx \rangle + 2\langle g, x \rangle \\ & \text{s.t.} \quad Ax = b \\ & \qquad\quad x \geq 0, \end{aligned} \tag{TOPE}$$

where $H \in \mathcal{S}^n$, $g \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. We assume that the feasible set $\mathcal{F} := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is nonempty and bounded, i.e., $\mathcal{F}$ is a non-trivial polytope. As before, we would like to determine an explicit description of $\mathcal{G} := \overline{\text{conv}}\{(x, xx^T) : x \in \mathcal{F}\}$, which will allow us to calculate v^* as the optimal value of a convex optimization problem.

In this case, however, we will need more than just linear, second-order-cone, and semidefinite programmming. Let $\mathcal{S}^n$ denote the set of $n \times n$ symmetric matrices and $\mathbb{R}^n_+$ the set of all nonnegative n-dimensional column vectors, and define

$$\mathcal{COP} := \left\{Z \in \mathcal{S}^n : y^T Z y \geq 0 \text{ for all } y \in \mathbb{R}^n_+\right\},$$
$$\mathcal{CP} := \left\{Y \in \mathcal{S}^n : Y = \sum_k y^k (y^k)^T \text{ for some finite collection } \{y^k\}_k \subseteq \mathbb{R}^n_+ \setminus \{0\}\right\},$$

which are, respectively, the set of copostive and completely positive matrices mentioned in the Introduction. Although the results below use only $\mathcal{CP}$ explicitly, we define $\mathcal{COP}$ as well because $\mathcal{CP}$ and $\mathcal{COP}$ are dual cones and because copositive optimization gets its name from the copositive matrices.

As we have done in Sections 2–9, we first prove a proposition for $\mathcal{K}$ involving the homogenization of the feasible set $\mathcal{F}$, which directly leads to the main theorem classifying $\mathcal{G}$ for $\mathcal{F}$.

Proposition 6. *Regarding (TOPE),*

$$\mathcal{K} := \overline{\text{conv}}\{yy^T : Cy = 0, y \geq 0\} = \left\{Y \in \mathcal{CP} : CYC^T = 0\right\},$$

where $C := (b, -A)$.

Proof. For ease of reference, define $\mathcal{R} := \left\{Y \in \mathcal{CP} : CYC^T = 0\right\}$, which we note is closed and convex. We first argue $\mathcal{K} \subseteq \mathcal{R}$. Because $\mathcal{K}$ is the closure of the convex hull of matrices yy^T with $Cy = 0, y \geq 0$ and because $\mathcal{R}$ is closed and convex, it suffices to show that such rank-1 matrices yy^T are in $\mathcal{R}$. So take $Y = yy^T$ with $Cy = 0, y \geq 0$. Then clearly $Y \in \mathcal{CP}$, and the matrix of linear inequalities $CYC^T = 0$ holds because CYC^T equals the rank-1 product $(Cy)(Cy)^T$.

To show the reverse containment, let $Y \in \mathcal{R}$. Note that $Y \in \mathcal{CP}$ implies $Y = \sum_k y^k (y^k)^T$ for some collection of vectors $\{y^k\}$ with each $y^k \geq 0$. In addition, the equation $CYC^T = 0$ is equivalent to $\sum_k (Cy^k)(Cy^k)^T = 0$, which implies each $Cy^k = 0$ since $(Cy^k)(Cy^k)^T \succeq 0$. So every $y^k (y^k)^T \in \mathcal{K}$, and thus $Y \in \mathcal{K}$. □

We remark that, for $Y \in \mathcal{CP}$, the matrix condition $CYC^T = 0$ is equivalent to the vector

equation $\operatorname{diag}(CYC^T) = 0$ because $Y \in \mathcal{CP}$ implies $Y \succeq 0$, which in turn implies $CYC^T \succeq 0$. The use of $\operatorname{diag}(CYC^T) = 0$ is closer to the original development in [9].

Theorem 8. *Regarding (TOPE), it holds that*

$$\begin{aligned}\mathcal{G} &:= \overline{\operatorname{conv}}\{(x, xx^T) : x \in \mathcal{F}\} \\ &= \{(x, X) : Y(x, X) \in \mathcal{K}\} = \left\{(x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \in \mathcal{CP},\ CYC^T = 0 \end{array}\right\}\end{aligned}$$

with $Y(x, X)$ given by (3) and $C := (b, -A)$.

Proof. For ease of reference, define

$$\mathcal{R} := \left\{(x, X) : \begin{array}{c} Y = Y(x, X) \\ Y \in \mathcal{CP},\ CYC^T = 0 \end{array}\right\},$$

which we note is closed and convex. As in the proof of Proposition 6, the containment $\mathcal{G} \subseteq \mathcal{R}$ holds easily. To prove the reverse containment, let $(x, X) \in \mathcal{R}$, and define $Y := Y(x, X)$. By Proposition 6, we can write $Y = \sum_k y^k (y^k)^T$ with each nonzero $y^k \in \mathbb{R}^{n+1}$ satisfying $Cy^k = 0, y^k \geq 0$. Decomposing $\binom{\nu_k}{w^k} := y^k$ with scalar $\nu_k \geq 0$ and vector $w^k \geq 0$, we see $Aw^k = b\nu_k$. If $\nu_k = 0$, then $w^k = 0$ because the system $\{x \geq 0 : Ax = 0\} = \{0\}$ because $\mathcal{F}$ is nonempty and bounded. So in fact every $\nu_k > 0$. Define $x^k := w^k/\nu_k$. Then

$$Y = \sum_k \nu_k^2 \begin{pmatrix} 1 \\ x^k \end{pmatrix} \begin{pmatrix} 1 \\ x^k \end{pmatrix}^T \quad \text{with} \quad Ax^k \leq b \Leftrightarrow x^k \in \mathcal{F}.$$

Since the top-left entry of Y equals 1, we have $\sum_k \nu_k^2 = 1$. So Y is a convex combination of rank-1 matrices $\binom{1}{x^k}\binom{1}{x^k}^T$ with $x^k \in \mathcal{F}$. This proves $Y \in \mathcal{G}$ as desired. □

11 Conclusion

In this paper, we have introduced and illustrated some of the basic ideas of copositive optimization using a number of specific examples, and we have tried to demonstrate how the geometry of the feasible region plays a critical role. Although we have looked at specific examples, many of the basic ideas and proof techniques hold in more general settings of copositive optimization, and so we hope this paper prepares the reader for further investigations into this exciting area.

As indicated in the Introduction, part of the value of this type of research comes from its characterization of classes of valid cuts for nonconvex problems, and it will be interesting to

investigate the practical applications of the various cut classes. In particular, the different classes could be applied anytime the corresponding geometry appears as a substructure in a given problem. For example, any problem with linear inequalities and ellipsoidal constraints could potentially benefit from the classes $CYC^T \geq 0$ and $Yc^i \in \mathcal{L}$ in Section 6. In addition, for a problem with $x \in \mathbb{R}^n$, one could examine pairs of variables at a time, say, x_i and x_j. If the projection of the feasible region onto (x_i, x_j) is a polygon and its facets and vertices can be worked out explicitly, then the ideas of Section 9 would be applicable.

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