A Geometric Approach to Robustness in Complex Networks

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***Abstract*—We explore the geometry of networks in terms of an n-dimensional Euclidean embedding represented by the Moore-Penrose pseudo-inverse of the combinatorial laplacian. The length of the position vector in this n-dimensional space yields a structural centrality index $(\mathcal{C}^*)$ for the nodes that captures the detour overhead incurred when the random walk between a pair of nodes is forced to go through the node in question, averaged over all source-destination pairs. We demonstrate how this overhead is related to the number of alternative/redundant paths between the node in question and the rest of the network; thereby reflecting the immunity/vulnerability of a node to random edge failures. Through empirical evaluation over example and real world networks, we demonstrate how the structural centrality of nodes captures their structural roles in the network and is suitably sensitive to perturbations/rewirings in the network.**

I. Introduction

Unlike traditional studies on network robustness, that typically treat networks as combinatoric objects and rely primarily on classical graph-theoretic concepts (e.g. minimum cuts) to characterize network robustness, we explore a geometric approach which enables us to employ more advanced theories and techniques, quantify and compare robustness of networks in terms of their local and global structures.

Robustness of nodes to failures in complex networks is dependent on their overall *connectedness* in the network. Several centralities, that characterize connectedness of nodes in complex networks in varying ways, have been proposed in literature. Perhaps the simplest of all is degree — the number of edges incident on a node. Except in *scale free* networks that display *rich club connectivity* [2], [7], [8], degree is essentially a *local* measure and does not determine the overall connectedness of a node. A more sophisticated measure of centrality is geodesic closeness [10], [11]. It is defined as the (reciprocal of) average shortest-path distance of a node from all other nodes in the network. However, communication in networks is not always confined to shortest paths alone. Therefore, geodesic based centralities only partially capture connectedness of nodes. Recently, subgraph centrality — the number of subgraphs of a graph that a node participates in — has also been proposed [6]. In principle, a node with high subgraph centrality, should be better connected to other nodes in the network through redundant paths. Alas, subgraph centrality is computationally intractable and the proposed index in [6] approximates subgraph centrality by the sum of lengths of all *closed* walks, weighed in inverse proportions by the factorial of their lengths; which inevitably introduces local connectivity bias.

In this work, we study a geometric embedding of networks using the Moore-Penrose (pseudo) inverse of the graph Laplacian for the network, denoted henceforth by $\mathbf{L}^+$. We show that the diagonal entries of $\mathbf{L}^+$, that represent the distance of each node to the origin in the $n-dimensional$ Euclidean space of the network embedding, provide a robust structural centrality measure $(\mathcal{C}^*)$ for the nodes in the network. Moreover, the trace of $\mathbf{L}^+$, $Tr(\mathbf{L}^+)$, also called the *Kirchoff index* $(\mathcal{K})$, provides a structural robustness measure for the network as a whole.

Through both rigorous mathematical arguments as well as numerical simulations using synthetic and realistic network topologies, we demonstrate that our new indices better characterize robustness of nodes in network as compared to other existing metrics (e.g. node centrality measured based on degree, shortest paths etc.). A rank-order of nodes in terms of their structural centralities helps distinguish them in terms of their structural roles (such as core, gateway etc.). Also, structural centrality and the Kirchoff index, are both appropriately sensitive to local perturbations in the network, a property not displayed by other centralities in literature.

The rest of the paper is organized as follows: We begin by describing a geometric embedding of the network using the eigen space of $\mathbf{L}^+$ and introduce structural centrality and Kirchoff index as measures of robustness in §II. §III demonstrates how structural centrality of a node reflects the average detour overhead in random walks through a particular node in question, §IV presents comparative empirical analysis and in §V the paper is concluded.

II. Geometric Embedding of Networks using $\mathbf{L}^+$ and Structural Centrality

In studying the *geometry* of networks, we first need to embed a network (e.g. represented abstractly as a graph) into an appropriate geometric space endowed with a metric function (mathematically, a metric space). In this section we describe an n-dimensional embedding of the complex network using, the Moore-Penrose pseudo-inverse of the combinatorial laplacian ($\mathbf{L}^+$). The squared length of the position vector for a node in this space yields a geometric measure of centrality for the node while the sum of the squared lengths of the position vectors of all nodes, or the trace of $\mathbf{L}^+$, yields an overall robustness index for the graph. But first we need to introduce some basic notations.

Given a complex network, its topology is in general represented as a (weighted) graph, $G = (V, E, W)$, where $V(G)$ is the set of nodes representing, say, switches, routers or end systems in the network; $E = \{e_{uv} : u, v \in V\}$ is the set of edges connecting pairs of nodes representing, for example, the (physical or logical) communication links between the pair of nodes; and $W = w_{uv} \in \Re^+ : e_{uv} \in E(G)$ is a set of weights assigned to each edge of the graph (here $\Re^+$ denotes the set of nonnegative real numbers). These weights can be used to represent, for example, the capacity, latency, or geographical distance, or an (administrative) routing cost associated with the edge (communication link) e_{uv}. Note that if w_{uv} is simply 0 or 1, we have a simple and unweighted graph.

Given $G = (V, E, W)$, we introduce an $n \times n$ *affinity* matrix $\mathbf{A} = [a_{ij}]$ associated with G, where $n = |V(G)|$ is the number of nodes in G (the *order* of G), and $a_{ij} \geq 0$ is some function of the weight w_{ij}. For a simple graph where $w_{ij} \in \{0, 1\}$, setting $a_{ij} = w_{ij}$ yields the standard adjacency matrix of the graph G. In general, each entry a_{ij} captures some measure of affinity between nodes i and j: the larger a_{ij} is, nodes i and j are in a sense *closer* or more *strongly connected*. Hence in general, we refer to $\mathbf{A}$ as an affinity matrix associated with G. We assume that $a_{ij} = a_{ji}$, i.e. $\mathbf{A}$ is symmetric. For $1 \leq i \leq n$, define $d(i) = \sum_j a_{ij}$, and refer to $d(i)$ as the (generalized) degree of node i. (Note that if G is a simple unweighted graph j and $\mathbf{A}$ is its adjacency matrix, then $d(i)$ is the degree of node i.)

The *combinatorial Laplacian* of $\mathbf{A}$ (or the associated graph G), is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D} = [d_{ii}] = d(i)$ is a diagonal matrix with $d(i)$'s on the diagonal. The Laplacian is a positive semidefinite matrix, and thus has n non-negative Eigen values λ_i's. For $1 \leq i \leq n$, let $\mathbf{u}_i$ be the corresponding eigenvector of λ_i such that $||\mathbf{u}_i||_2^2 = \mathbf{u}_i'\mathbf{u}_i$. We assume that the eigenvalues λ_i's are ordered such that $\lambda_1 \geq ... \geq \lambda_n = 0$. Then the matrix formed by the corresponding eigenvectors $\mathbf{u}_i's$, $\mathbf{U} = [\mathbf{u}_1, ..., \mathbf{u}_n]$, is orthogonal i.e. $\mathbf{U}'\mathbf{U} = \mathbf{I}$, the identity matrix. More importantly, $\mathbf{L}$ admits an eigen decomposition $\mathbf{L} = \mathbf{U}\Lambda\mathbf{U}'$, where Λ is the diagonal matrix $\Lambda = [\lambda_{ii}] = \lambda_i$.

Like $\mathbf{L}$, its Moore-Penrose (pseudo) inverse $\mathbf{L}^+$ is also positive semi-definite, and admits an eigen decomposition of the form, $\mathbf{L}^+ = \mathbf{U}'\Lambda^{-1}\mathbf{U}$, where Λ^{-1} is a diagonal matrix consisting of λ^{-1} if $\lambda_i > 0$, and 0 if $\lambda_i = 0$ (for simplicity of notation, in the following we will use the convention $\lambda_i^{-1} = 0$ if $\lambda_i = 0$). Define $\mathbf{X} = \Lambda^{-1/2}\mathbf{U}$. Hence, $\mathbf{L}^+ = \mathbf{X}'\mathbf{X}$ which means that the network can be embedded into the Euclidean space $\Re^n$ where the coordinates of node i are given by $\mathbf{x}_i$, the i^{th} column of $\mathbf{X}$. As the centroid of the position vectors lies at the origin in this n-dimensional space [9], the squared distance of node i from the origin is exactly the corresponding diagonal entry of $\mathbf{L}^+$ i.e. $||\mathbf{x}_i||_2^2 = l_{ii}^+$ and the squared distance between two nodes $i, j \in V(G), ||\mathbf{x}_i - \mathbf{x}_j||_2^2 = Vol(G)^{-1}C_{ij}$ where $Vol(G) = \sum_{i=1}^n d(i)$ is called the *volume* of the graph (a constant for the graph) and C_{ij} is called the *commute time* defined as the expected length of commute in a random walk between i and j in the network [4].

Based on the geometric embedding of the graph using $\mathbf{L}^+$ described above, we now put forth two new robustness metrics. First, a rank order for individual nodes in terms of their relative robustness properties called *structural centrality*, defined as $\mathcal{C}^*(i) = 1/l_{ii}^+$, for $i \in V(G)$. Specifically, closer a node is to the origin in this n-dimensional space, more structurally central it is and vice versa. Next, the sum of the squared lengths of the position vectors of all nodes $Tr(\mathbf{L}^+) = \sum_{i=1}^n l_{ii}^+$, called the Kirchoff index ($\mathcal{K}$), is a measure of the overall robustness of the network. Geometrically, more compact the embedding is, or equivalently lower the value of $\mathcal{K}(G)$, more robust the network G is. We can therefore use Kirchoff index to compare the robustness of two graphs with the same order and volume.

In what follows, we demonstrate how these two metrics indeed reflect robustness of nodes and the overall graph respectively, first through rigorous mathematical analysis and then with empirical evaluations.

III. Structural Centrality, Random Walks and Electrical Voltages

To show that structural centrality ($\mathcal{C}^*$) and Kirchoff index ($\mathcal{K}$) indeed provide a measures of robustness, we relate them to the lengths of random walks on the graph.

In §III-A, we demonstrate how $\mathcal{C}^*(k)$ for node captures an overhead in random *detours* through node k as a *transit* vertex. Next in §III-B, we provide an electrical interpretation for the same.

A. Detours in Random Walks

A simple random walk $(i \rightarrow j)$, is a discrete stochastic process that starts at a node i, the source, visits other nodes in the graph G and stops on reaching the destination j [12]. In contrast, we define a *random detour* as:

Definition 1: Random Detour $(i \rightarrow k \rightarrow j)$: A random walk starting from a source node i, that must visit a transit node k, before it reaches the destination j and stops.

Effectively, such a random detour is a combination of two simple random walks: $(i \rightarrow k)$ followed by $(k \rightarrow j)$. We quantify the difference between the random detour $(i \rightarrow k \rightarrow j)$ and the simple random walk $(i \rightarrow j)$ in terms of the number of steps required to complete each of the two processes given by hitting time.

Definition 2: Hitting Time (H_{ij}): The expected number of steps in a random walk starting at node i before it reaches node j for the first time.

Clearly, $H_{ik} + H_{kj}$ is the expected number of steps in the random detour $(i \rightarrow k \rightarrow j)$. Therefore, the overhead incurred is:

$$\Delta H^{i \rightarrow k \rightarrow j} = H_{ik} + H_{kj} - H_{ij} \tag{1}$$

Intuitively, more peripheral transit k is, greater the overhead in (1). The overall peripherality of k is captured by the following average:

$$\Delta H^{(k)} = \frac{1}{n^2\ Vol(G)} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta H^{i \rightarrow k \rightarrow j} \tag{2}$$

Alas, hitting time is not a Euclidean distance as $H_{ij} \neq H_{ji}$ in general. An alternative is to use commute time $C_{ij} = H_{ij} + H_{ji} = C_{ji}$, a metric, instead. More importantly [14],

$$C_{ij} = Vol(G)(l^+_{ii} + l^+_{jj} - l^+_{ij} - l^+_{ji}) \tag{3}$$

and in the overhead form (1), (non-metric) hitting and (metric) commute times are in fact equivalent (see propositions $9-58$ in [13] and Theorem 1 in [18]):

$$\Delta H^{i \rightarrow k \rightarrow j} = (C_{ik} + C_{kj} - C_{ij})/2 = \Delta H^{j \rightarrow k \rightarrow i} \tag{4}$$

We now exploit this equivalence to equate the cumulative detour overhead through transit k from (2) to l^+_{kk} in the following theorem.

Theorem 1: $\Delta H^{(k)} = l^+_{kk}$

Proof: Using $\Delta H^{i \rightarrow k \rightarrow j} = (C_{ik} + C_{kj} - C_{ij})/2$:

$$\Delta H^{(k)} = \frac{1}{2n^2\ Vol(G)} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ik} + C_{kj} - C_{ij}$$

Observing $C_{xy} = Vol(G)\ (l^+_{xx} + l^+_{yy} - 2l^+_{xy})$ [14] and that $\mathbf{L}^+$ is doubly centered (all rows and columns sum to 0) [9], we obtain the proof.
□

Therefore, a low value of $\Delta H^{(k)}$ implies higher $\mathcal{C}^*(k)$ and more structurally central node k is in the network. Theorem 1 is interesting for several reasons. First and foremost, note that:

$$\sum_{j=1}^{n} C_{kj} = Vol(G)\ (n\ l^+_{kk} + Tr(\mathbf{L}^+)) \tag{5}$$

As $Tr(\mathbf{L}^+)$ is a constant for a given graph and an invariant with respect to the set $V(G)$, we obtain $l^+_{kk} \propto \sum_{j=1}^{n} C_{kj}$; lower l^+_{kk} or equivalently higher $\mathcal{C}^*(k)$, implies shorter average commute times between k and the rest of the nodes in the graph on an average. It is well understood that low C_{kj} reflects greater number of alternative (redundant) paths between nodes k and j; which in turn shows better connectivity between the two nodes [4]. Therefore, lower the value of $\mathcal{C}^*(k)$, greater the number of redundant paths between the node k and the rest of the network and consequently more immune is node k to random failures in the network. Moreover,

$$\mathcal{K}(G) = Tr(\mathbf{L}^+) = \sum_{k=1}^{n} l^+_{kk} = \frac{1}{2nVol(G)} \sum_{k=1}^{n} \sum_{j=1}^{n} C_{kj} \tag{6}$$

As $\mathcal{K}(G)$ reflects the average commute time between any pair of nodes in the network, it is a measure of overall structural robustness of G. For two networks of the same order (n) and volume $(Vol(G))$, the one with lower $\mathcal{K}(G)$ has a greater number of redundant paths between any pair of nodes in the network and hence is more immune to random edge failures.

B. An Electrical Interpretation and Recurrence

Interestingly, the detour overhead in (1) is related to *recurrence* in random walks — the expected number of times a random walk $(i \rightarrow j)$ returns to the source i [5]. We now explore how recurrence in detours related to structural centrality of nodes. But first we need to introduce some terminology.

The equivalent electrical network (EEN) [5] for $G(V, E, W)$ is formed by replacing an edge $e_{ij} \in E(G)$ with a resistance equal to w^{-1}_{ij}). The *effective resistance* (Ω_{ij}) is defined as the voltage developed across a pair of terminals i and j when a unit current is injected at

i and is extracted from j, or vice versa. In the EEN, let V_k^{ij} be the voltage of node k when a unit current is injected at i and a unit current is extracted from j. From [19], $U_k^{ij} = d(k)V_k^{ij}$. Substituting $k = i$ we get, $U_i^{ij} = d(i)V_i^{ij}$; the expected number of times a random walk $(i \rightarrow j)$ returns to the source i. For a finite graph G, $U_i^{ij} > 0$. The following theorem connects recurrence to the detour overhead.

Theorem 2:

$$\Delta H^{i \to k \to j} = \frac{Vol(G)\ (U_i^{ik} + U_i^{kj} - U_i^{ij})}{d(i)}$$

Proof: From [19] we have, $\Delta H^{i \to k \to j} = d(i)^{-1}\ Vol(G)\ U_i^{jk}$. The rest of this proof follows by proving $U_i^{jk} = U_i^{ik} + U_i^{kj} - U_i^{ij}$.

From the *superposition principle* of electrical current, we have $V_x^{xz} = V_y^{xz} + V_y^{zx}$. Therefore,

$$V_i^{ik} + V_i^{kj} - V_i^{ij} = (V_j^{ik} + V_j^{ki}) + V_i^{kj} - (V_k^{ij} + V_k^{ji})$$

Rearranging the terms in the RHS,

$$V_i^{ik} + V_i^{kj} - V_i^{ij} = V_j^{ik} + (V_j^{ki} + V_i^{kj} - V_k^{ij} - V_k^{ji})$$

From the *reciprocity principle*, $V_z^{xy} = V_x^{zy}$. Therefore, $V_i^{ik} + V_i^{kj} - V_i^{ij} = V_i^{jk}$. Multiplying by $d(i)$ on both sides we obtain the proof.
$\square$

The term $(U_i^{ik} + U_i^{kj}) - U_i^{ij}$ can be interpreted as the expected extra number of times a random walk returns to the source i in the random detour $(i \rightarrow k \rightarrow j)$ as compared to the simple random walk $(i \rightarrow j)$. Each instance of the random process that returns to the source, must effectively start all over again. Therefore, more often the walk returns to the source greater the expected number of steps required to complete the process and less central the transit k is, with respect to the source-destination pair (i, j).

IV. Empirical Evaluations

We now empirically study the properties of structural centrality ($\mathcal{C}^*$) and Kirchoff index (we use $\mathcal{K}^* = \mathcal{K}^{-1}$ to maintain *higher is better*). We first show in §IV-A, how structural centrality can capture the structural roles played by nodes in the network and then in §IV-B demonstrate how it, along with Kirchoff index, is appropriately sensitivity to rewiring and local perturbations in the network.

A. Identifying Structural Roles of Nodes

Consider the router level topology of the Abilene network (FIG. 1(a)) [1]. At the core of this topology, is a ring of 11 POP's, spread across mainland US, through which several networks interconnect. Clearly, the connectedness of such a network is dependent heavily on the low degree nodes on the ring. For illustration, we mimic the Abilene topology, with a simulated network (FIG. 1(b)) which has a 4-node core $\{v_1, ..., v_4\}$ that connects 10 networks through gateway nodes $\{v_5, ..., v_{14}\}$ (FIG.1(b)). FIG. 2 shows the (max-normalized) values of SC and $\mathcal{C}^*$ for the core $\{v_1, ..., v_4\}$, gateway $\{v_5, ..., v_{14}\}$ and nine other nodes $\{v_{15}, ..., v_{23}\}$ in topology (FIG.1(b)). Notice that v_5, v_6 are the highest degree nodes $(d(v_5) = d(v_6) = 10)$ in the network while v_{14} has the highest SC. In contrast, $\mathcal{C}^*$ rank the core nodes higher than the gateway nodes with v_1 at the top. The relative peripherality of v_5, v_6 and v_{14} as compared to the core nodes requires no elaboration. As far as GC is concerned it ranks all the nodes $v_{15} - v_{23}$ as equally well connected whereas in fact v_{22}, v_{23} have redundant connectivity to the network through each other and are, ever so slightly, better connected than the others in the subnet abstracted by v_5.

We see similar characterization of nodes in the social network of co-authorships [16], as shown through a color scheme based on $\mathcal{C}^*$ values in FIG. 3. Core-nodes connecting different subcommunities of authors are recognized effectively by structural centrality as being more central than several higher degree peripheral nodes.

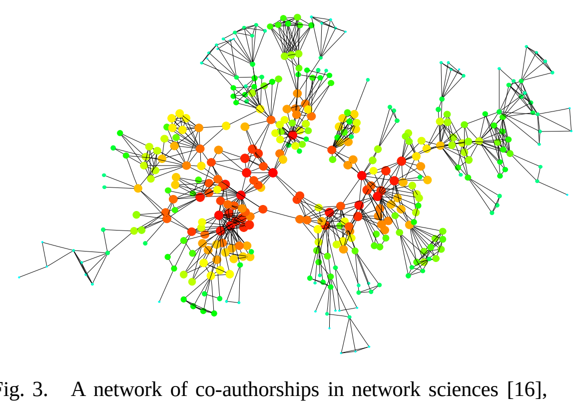
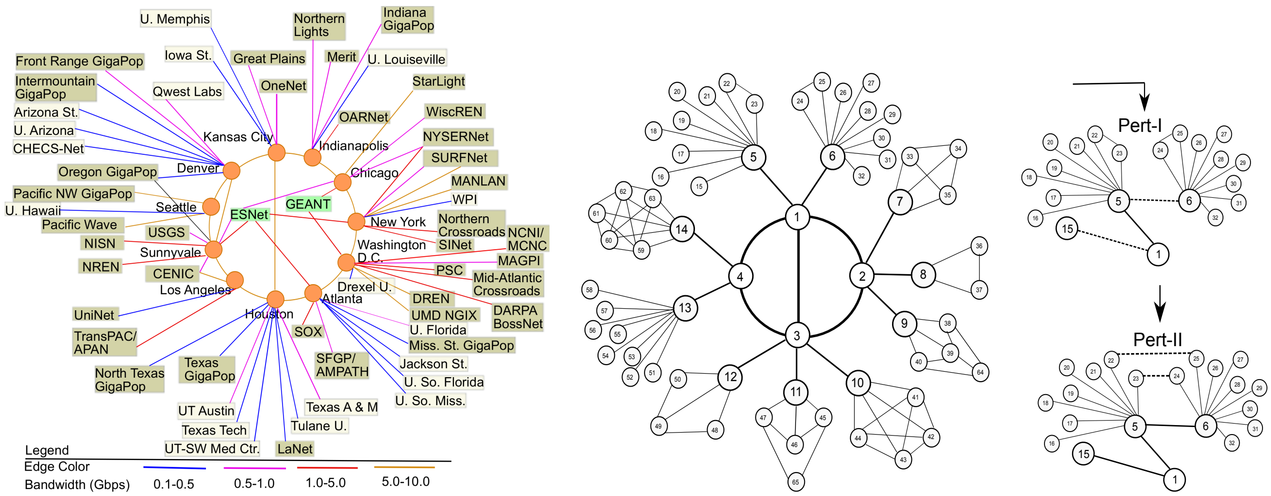


Fig. 3. A network of co-authorships in network sciences [16], $Red \rightarrow Turqoise$ reducing order of $\mathcal{C}^*$.

B. Sensitivity to Local Perturbations

An important property of centrality measures is their sensitivity to perturbations in network structure. Traditionally, structural properties in real world networks have been equated to average statistical properties like



(a) Abilene Topology (b) Simulated topology (c) Perturbations

Fig. 1. Abilene Network and a simulated topology.

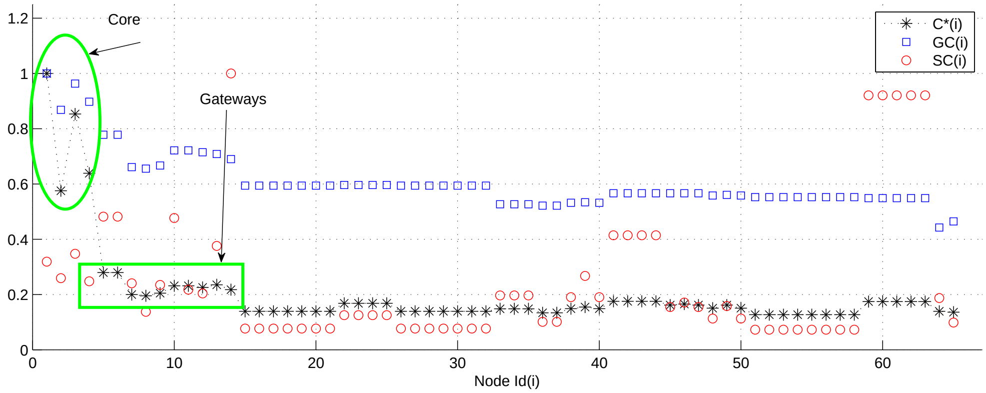


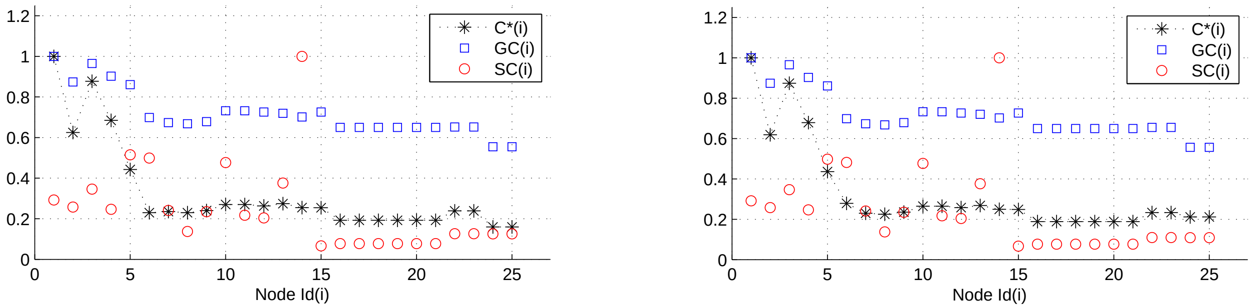
Fig. 2. Max-normalized centralities for simulated topology.

power-law/scale-free degree distributions and rich club connectivity [2], [7], [8]. However, the same degree sequence $D = \{d(1) \geq d(2) \geq ... \geq d(n)\}$, can result in graphs of significantly varying topologies. Let $\mathcal{G}(D)$ be the set of all connected graphs with scaling sequence D. The generalized Randic index $R_1(G)$ [3], [17]:

$$R_1(G) = \sum_{e_{ij} \in E(G)} d(i)d(j) \quad (7)$$

where $G \in \mathcal{G}(D)$, is considered to be a measure of overall connectedness of G as higher $R_1(G)$ suggests *rich club connectivity* (RCC) in G [15]. We now examine the sensitivity of each index with respect to local perturbations in the subnetwork abstracted by the core node v_1 and its two gateway neighbors v_5 and v_6.

First, we rewire edges $e_{15,5}$ and $e_{6,1}$ to $e_{15,1}$ and $e_{6,5}$ respectively (PERT-I FIG. 1(c)). PERT-I is a degree preserving rewiring which only alters local connectivities. FIG. 4(a) shows the altered values of centralities after PERT-I. Note, v_{15} is now directly connected to v_1 which makes $\mathcal{C}^*(v_{15})$ comparable to other gateway nodes while $SC(v_{15})$ seems to be entirely unaffected. PERT-I also results in v_6 losing its direct link to the core, reflected in the decrease in $\mathcal{C}^*(v_6)$ and a corresponding increase in $\mathcal{C}^*(v_5)$. $\mathcal{C}^*$, however, still ranks the core nodes higher than v_5 because PERT-I is local and should not affect nodes outside the sub-network — v_1 continues to abstract the same sub-networks from the rest of the topology. We, therefore, observe that $\mathcal{C}^*$ is appropriately sensitive to the changes in connectedness of nodes in the event of local perturbations. But what about the network on a whole?



(a) After PERT-I (b) After PERT-II

Fig. 4. Max-normalized values of centralities for core, gateway and some other nodes.

Let G and G_1 be the topologies before and after PERT-I. G_1 is less well connected overall than G as the failure of $e_{5,1}$ in G_1 disconnects 19 nodes from the rest of the network as compared to 10 nodes in G. However,

$$\Delta R_1(G \to G_1) = \frac{R_1(G_1) - R_1(G)}{R_1(G)} = 0.029$$

as the two highest degree nodes (v_5 and v_6) are directly connected in G_1. In contrast, $\Delta\mathcal{K}^*(G \to G_1) = -0.045$, which rightly reflects the depreciation in overall connectedness after PERT-I. A subsequent perturbation PERT-II of G_1, rewiring $e_{22,23}$ and $e_{24,25}$ to $e_{22,25}$ and $e_{23,24}$, to obtain G_2 significantly improves local connectivities in the sub-network safeguarding against the failure of edge $e_{5,6}$. However, $\Delta R_1(G_1 \to G_2) = 0$ while $\Delta\mathcal{K}^*(G_1 \to G_2) = 0.036$ which once again shows the efficacy of Kirchoff index as a measure of global connectedness of networks.

V. Conclusion and Future Work

In this work we presented a geometric perspective on robustness in complex networks. We proposed structural centrality and Kirchoff index respectively as measures of robustness of individual nodes and the overall network against random edge failures in the network. Both indices reflect the number of redundant/ alternative paths in the network thereby capturing global connectedness. We also demonstrated that these indices are suitably sensitive to perturbations/rewirings in the network. In future, we aim at investigating similar metrics for the case of strongly connected weighted directed graphs to further generalize our work.

VI. Acknowledgment

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