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A GEOMETRIC APPROACH TO SOME COEFFICIENT INEQUALITIES FOR UNIVALENT FUNCTIONS

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I. Introduction.

The relationships between extremal problems for univalent functions and the theory of quadratic differentials are now known and well-understood and the reader will find a systematic application of the theory of quadratic differentials to univalent functions in the monograph [1] by J. A. Jenkins.

One of the main difficulties in the solution of extremal problems for univalent functions can be explained as follows.

Let P denote an extremal problem for univalent functions; by a general principle of Teichmüller, there is associated a quadratic differential $Q(z)\,dz^2$ on some Riemann surface R, such that the critical trajectories of this differential are tied up with the extremal functions for P. Unfortunately, the quadratic differential $Q(z)\,dz^2$ depends also on the unknown extremal functions for the problem P, thus making the determination of the critical trajectories of $Q(z)\,dz^2$ very difficult. Only a special class of problems P can be treated directly in this way.

Another approch, which has proved to be fruitful, is giving first the quadratic differential $Q(z)\,dz^2$ (or at least « a part » of it) and then finding which problems P are associated to $Q(z)\,dz^2$. Possibly the most general result in this direction of ideas is provided by the General Coefficient Theorem [1], [2], of J. A. Jenkins, which contains as very special cases most of the known inequalities for univalent functions.

Recently Z. Charzynski and M. Schiffer [3], [4] proved the Bieberbach conjecture $|a_4| \leq 4$ in an ingenious and indirect way, namely: they proved first an auxiliary inequality, which combined with other known inequalities implied the desired result.

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Their second proof [4] of $|a_4| \leq 4$ was based on a new inequality obtained in the following steps:

(i) by a geometric analysis of the associated quadratic differential, one proves that unless the differential itself is of a simple type, the part of the trajectories one is interested in lie in some half-plane;

(ii) a weighted mean of these trajctories falls in the opposite half-plane, a contradiction;

(iii) having thus proved that the quadratic differential $Q(z)\,dz^2$ must be of a simple type, one is able to find the trajectories and thus the extremal functions for the inequality considered;

(iv) having found the extremals it is a simple matter to find the maximum.

Other instances of this method are in the paper [7] by P. R. Garabedian and M. Schiffer.

The purpose of this paper is to prove a general result about critical trajectories of a quadratic differential $Q(z)\,dz^2$ on the z-sphere, arising from the following problem. Let be given a quadratic differential $Q(z)\,dz^2$ on the z-sphere, a « good » subset T_0 of the set $\overline{T}$ of critical trajectories of $Q(z)\,dz^2$, a continuously differentiable Jordan arc J on R.

Under which conditions on J can we assert that $J \cap T_0$ is either empty or a single point?

Our answer is (see Theorem 1 later)

Condition (α): T_0 is good if a certain connectedness condition is satisfied;

Condition (β): $Q(z)\,dz^2$ has at most three poles and only one pole of multiplicity ≥ 2;

Condition (γ): $\operatorname{Im}\{Q(z)\,dz^2\} \neq 0$ along J.

We give two applications of our Theorem 1 (a third one may be considered the proof [4] by Charzynski and Schiffer of $|a_4| \leq 4$).

In Theorem 3 we prove a new coefficient inequality which may be helpful for a proof of $|a_5| \leq 5$; this inequality, as a special case, proves that $\operatorname{Re}(a_5) \leq 5$ if $a_2 =$ real, a result obtained, and generalized to higher coefficients, by A. Obrock [5]

Next we give a simplified proof of the inequality $|b_3| \leq \frac{1}{2} + e^{-6}$ which, while still along the lines of the first proof [7] by P. R. Garabedian and M. Schiffer, avoids the numerical computations of that paper.

Finally, we remark that there is a close analogy between our Theorem 3 and a recent inequality by M. Ozawa [6]. The Ozawa inequality has been generalized by Garabedian in [8] and perhaps there is a new inequality, analogous to Garabedian's, generalizing our Theorem 3.

The next section contains definitions and known results (here without proofs) about quadratic differentials, which are collected here for reader's convenience. The reader will find these definitions and results (with proofs) in the monograph [1] by J. A. Jenkins Chapter III.

II. Quadratic differentials.

Let R be a finite oriented Riemann surface, $Q(z)\,dz^2$ a (meromorphic) quadratic differential on it. The quadratic differential $Q(z)\,dz^2$ is called positive if

$$Q(\zeta)\,d\zeta^2 > 0 \text{ on } \partial R,$$

where ζ is a boundary uniformising parameter of R, except possibly for at most a finite number of zeros.

The set of zeros and simple poles of $Q(z)\,dz^2$ will be denoted by C and called the set of critical points of the quadratic differential; by H is meant the set of all poles of $Q(z)\,dz^2$ of order at least 2.

A trajectory of $Q(z)\,dz^2$ is a maximal regular analytic curve Γ along which

$$Q(z)\,dz^2 > 0\,;$$

this trajectory is an integral curve of the differential equation

$$Q(z)\left(\frac{dz}{d\tau}\right)^2 = 1,$$

where τ is a suitable real parameter. It follows that through an interior point of $R - C - H$ there is one and only one trajectory.

A trajectory having a limiting end point at a point of C will be called critical; there is only a finite number of critical trajectories, and T will denote their union.

If U is a set on R, then $\bar{U}$ will denote its closure, while the $\widehat{\ }$-closure $\widehat{U}$ of U will mean the set of interior points of $\bar{U}$.

We shall state the for convenience of the reader the main results and definitions related to quadratic differentials. The Global Structure Theorem and the Three Pole Theorem which will follow are due to J. A. Jenkins [1] as well as some of the following definitions.

A set K on a finite oriented Riemann surface R is an F — set with respect to the quadratic differential $Q(z)\,dz^2$ if every trajectory of $Q(z)\,dz^2$ which meets K lies entirely in K.

Let T denote the union of all critical trajectories of $Q(z)\,dz^2$, and consider $R - \overline{T}$. One of the main results of the theory is the fact that $R - \overline{T}$ consists of a finite number of domains which are either simply-connected or doubly-connected, called end, strip, circle and ring domains.

The precise definition of these domains is as follows.

DEFINITION 1. *An end domain E on R is a maximal connected open set E with the following properties:*

i) *E is an F — set:*

ii) *E contains no critical point of $Q(z)\,dz^2$;*

iii) *through any point of E there is one and only one trajectory, which starts and terminates at a pole of $Q(z)\,dz^2$ of order at least 2;*

iv) *the metric $dw^2 = Q(z)\,dz^2$ maps E conformally onto an upper or lower half-plane in the w — plane.*

REMARK. It is possible to show that the order of the pole considered in condition (iii) is at least 3.

DEFINITION 2. *A strip domain S on R is a maximal connected open set S with the following properties.*

i) *S is an F — set;*

ii) *S contains no critical point of $Q(z)\;dz^2$;*

iii) *through any point of S there is one and only one trajectory, which starts and terminates at two poles of $Q(z)\,dz^2$, possibly coincident, of order at least 2;*

iv) *the metric $dw^2 = Q(z)\,dz^2$ maps S conformally onto a strip $a < \mathrm{Im}(w) < b$ in the w — plane.*

DEFINITION 3. *A circle domain C on R is a maximal connected open set C with the following properties*

i) *C is an F — set;*

ii) *C contains a single double pole A of $Q(z)\;dz^2$;*

iii) *through any point of $C - A$ there is one and only trajectory, which is a closed Jordan curve separating A from the boundary of C;*

iv) *there is a purely immaginary constant c such that the metric $dw^2/w^2 = c^2\,Q(z)\,dz^2$ maps B conformally onto a circle $|w| < r$, the point A going into $w = 0$, in the w — plane.*

DEFINITION 4. *A ring domain R is a maximal connected open set R with the following properties:*

i) *R is an F — set;*

ii) *R contains no critical point of $Q(z)\,dz^2$;*

iii) *through any point of R there is one and only one trajectory, which is a closed Jordan curve;*

iv) *there is a purely immaginary constant c such that the metric $dw^2/w^2 = c^2 Q(z)\,dz^2$ maps R conformally onto a circular ring $r_1 < |w| < r_2$ in the w — plane.*

Now we are ready to state the

Globlal Structure Theorem (J. A. Jenkins [1]).

Let R be a finite oriented Riemann Surface, $Q(z)\,dz$ a positive quadratic differential on it.

Let T denote the union of all trajectories $Q(z)\,dz^2$ which have one end point at a zero or a simple pole of $Q(z)\,dz^2$, let $\overline{T}$ be the closure of T and let $\widehat{T}$ be the interior of $\overline{T}$. Suppose also that no one of the following situations happens, up to conformal equivalence:

a) $Q(z)\,dz^2 = dz^2$, *R the z — sphere;*

b) $Q(z)\,dz^2 = b\,dz^2/z^2$, *R the z — sphere;*

c) $Q(z)\,dz^2$ *holomorphic, R a torus.*

Then we have

i) *$R - \overline{T}$ consists of a finite number of end, strip, circle, and ring domains;*

ii) *each such domain is bounded by a finite number of trajectories together wich the points at which the latter meet; every boundary component of such a domain contains a zero or a simple pole of $Q(z)\,dz^2$, except that a boundary component of a circle or a riug domain may coincide with a boundary component of R; for a strip domain the two boundary elements, arising from points of the set of poles of $Q(z)\,dz^2$ of order at least 2, divide the boundary into two parts each of which contains a zero or a simple pole of $Q(z)\,dz^2$;*

iii) *every pole of $Q(z)\,dz^2$ of order m greater than 2 has a neighborhood covered by the $\widehat{\ }$-closure of $m - 2$ end domains and finite number of strip domains;*

iv) *every pole of $Q(z)\,dz^2$ of order 2 has a neighborhood covered by the $\widehat{\ }$-closure of the union of a finite number of strip domains or has a neighborhood contained in a circle domain.*

For later developments we shall need a condition under which a quadratic differential $Q(z)\,dz^2$ on the z — sphere is such that the $\widehat{\ }$-closure $\widehat{T}$ of T is empty. The answer is provided by Jenkins' (see [1]):

Three Pole Theorem. Let R be the z — sphere, $Q(z)\,dz^2$ a quadratic differential with at most three distint poles.

Then $\widehat{T}$ is empty.

III. Quadratic differentials with only three poles.

The aim of this section is to establish the following results.

THEOREM 1. *Let R be the z — sphere, $Q(z)\,dz^2$ a quadratic differential on it with at most three distint poles, only one of which, say B is of order at least 2.*

Let T_0 be a connected component of $\overline{T} - B$, and let J be a continuosly differentiable Jordan arc on R not containing poles of $Q(z)\,dz^2$, and such that $B \notin \overline{J}$ and

$$\operatorname{Im}(Q(z)\,dz^2) \neq 0 \text{ on } J.$$

Then $\overline{J}$ can meet T_0 at most in one point.

PROOF. If A, A' are two points of J, we shall denote by $J_{AA'}$, the open subarc of J having A, A' as its end points.

Suppose $T_0 \cap J$ contains at least two points. The hypothesis

$$\operatorname{Im}(Q(z)\,dz^2) \neq 0 \text{ on } J,$$

shows that J is nowhere tangent to a trajectory of $Q(z)\,dz^2$. Hence there is an open subarc $J_{PP'}$, of J not meeting T_0 and whose and points, P, P' belong to T_0, this because $\widehat{T}$ and hence $\widehat{T_0}$ is empty by the Three Pole Theorem.

We note that the quadratic differential $Q(z)\,dz^2$ is certainly holomorphic and non-zero on $J_{PP'}$, because no critical point of $Q(z)\,dz^2$ belongs to J.

The set $\overline{T}_0 = T_0 + B$ is connected, whence $R - \overline{T}_0$ consists of a finite number of simply-connected domains one of which, say D_0, will contain $J_{PP'}$. Now from the fact that $T_0 = \overline{T}_0 - B$ is connected we see that $\partial(D_0 - B)$ is connected too. We shall denote by Δ_0 the simply connected domain whose boundary consists of $J_{PP'}$ and of the connected part of $\partial(D_0 - B)$ joining the points P, P', and such that $B \notin \overline{\Delta}_0$. This last condition can be fulfilled because $B \notin \overline{J}_{PP'}$ by hypothesis.

Let us suppose that $Q(z)\,dz^2$ is not conformally equivalent to dz^2. Using the fact that $\widehat{T}$ is empty and the Global Structure Theorem we see that there is a sufficiently small neighborhood N of P such that $N \cap \Delta_0$ is contained in a domain K which is an end, strip, circle or a ring domain.

Let Q be a point of $J_{PP'} \cap N$. The quadratic differential $Q(z)\,dz^2$ is holomorphic and non-zero at Q whence there is an unique trajectory Γ of $Q(z)\,dz^2$ through Q. We note that Γ is trasversal to $J_{PP'}$ because

$$\operatorname{Im}\,(Q(z)\,dz^2) \neq 0$$

on $J_{PP'}$.

We assert that Γ intersects $J_{PP'}$ at least twice, and that there is a subarc Γ^* of Γ lying in Δ_0 and having its end points on $J_{PP'}$. Otherwise, let $\tilde{\Gamma}$ be that part of Γ lying in Δ_0. $\tilde{\Gamma}$ cannot end at Q, otherwise Q would be a critical point of $Q(z)\,dz^2$, which is not the case. Also, $\tilde{\Gamma}$ cannot end on $\partial\Delta_0 - J_{PP'}$ or inside Δ_0 because if this were the case then either $\tilde{\Gamma}$ would be a critical trajectory or $\bar{\Delta}_0$ would contain a pole of $Q(z)\,dz^2$ of order at least two. This is impossible, the former alternative because Γ would belong to $\bar{T}_0$, the latter because $Q(z)\,dz^2$ would have at least two poles of order at least 2, by the fact that $B \notin \bar{\Delta}_0$.

It is clear that $\partial\Delta_0 - J_{PP'}$ is part of the boundary ∂K of K. It follows from the previous discussion that there is a point Q_1 on $J_{PP'} \cap N$ and a simple subarc Γ_1^* of a trajectory, having Q_1, Q_1', where $Q_1' \in J_{PP'}$, as its end points, such that if Δ_1 denotes the simply-connected subdomain of Δ_0 whose boundary is $\Gamma_1^* + J_{Q_1Q_1'}$, we have

$$\Delta_0 - \bar{\Delta}_1 \subset K.$$

Now let Q_2 be a point of J_{PQ_1} and let Γ_2^* be the simple subarc of the trajectory through Q_2, contained in Δ_0 and ending at a point Q_2' of $J_{PP'}$. Obviously Γ_2^* cannot cross $\partial\Delta_0 - J_{PP'}$ nor Γ_1^*, and it follows that $Q_2' \in J_{Q_1'P'}$. Letting Δ_2 be the simply-connected subdomain of Δ_0 whose boundary is $\Gamma_2^* + J_{Q_2Q_2'}$, we deduce that $\Delta_1 \subset \Delta_2$ and

$$\Delta = \Delta_2 - \bar{\Delta}_1 \subset K.$$

Clearly Δ is a simply-connected subdomain of K, whose boundary is a simple, closed Jordan curve consisting of $J_{Q_2Q_1} + \Gamma_1^* + J_{Q_1'Q_2'} + \Gamma_2^*$, and the quadratic differential $Q(z)\,dz^2$ is holomorphic and non-zero inside and on $\bar{\Delta}$. It follows from the Cauchy Residue Theorem that

$$\int_{\partial\Delta} Q(z)^{1/2}\,dz = 0,$$

whence

$$\int_{Q_2}^{Q_1} Q(z)^{1/2}\,dz + \int_{Q_1'}^{Q_2'} Q(z)^{1/2}\,dz =$$

$$= \int_{\Gamma_1^*} Q(z)^{1/2}\,dz + \int_{\Gamma_2^*} Q(z)^{1/2}\,dz,$$

when the integrals on the left hand side of this equation are taken along J, while the integrals in the right hand side are taken moving along Γ_1^* from Q_1' to Q_1 and moving along Γ_2^* from Q_2 to Q_2'.

On every trajectory Γ of $Q(z)\,dz^2$ we have $Q(z)\,dz^2 > 0$, thus:

$$Q(z)^{1/2}\,dz \quad \text{is real on } \Gamma_1^* \text{ and } \Gamma_2^*.$$

Hence

$$\operatorname{Im}\left(\int_{Q_2}^{Q_1} Q(z)^{1/2}\,dz + \int_{Q_1'}^{Q_2'} Q(z)^{1/2}\,dz\right) = 0.$$

Finally, by hypothesis $\operatorname{Im}(Q(z)\,dz^2) \neq 0$ on J, and $Q(z)\,dz^2$ was holomorphic and non-zero on J. Hence $\operatorname{Im}(Q(z)^{1/2}\,dz)$ has a constant sign on J. Now, as $Q_2 \in J_{PQ_1}$ and $Q_2' \in J_{Q_1'P'}$ we see that $\operatorname{Im}(Q(z)^{1/2}\,dz)$ has always the same sign on both subarcs $J_{Q_2Q_1}$ and $J_{Q_1'Q_2'}$ of J.

Hence

$$\operatorname{Im}\left(\int_{Q_1}^{Q_2} Q(z)^{1/2}\,dz + \int_{Q_1'}^{Q_2} Q(z)^{1/2}\,dz\right) \neq 0,$$

contradicting the previous equation and the hypothesis that $T_0 \cap \bar{J}$ contained at least two points.

It remain for consideration the case in which $Q(z)\,dz^2$ is conformally equivalent to dz^2. However in this case the set T is empty and Theorem 1 becomes trivial.

This completes the proof of Theorem 1.

COROLLARY. *Theorem* 1 *remains true if J contains one simple pole A of $Q(z)\,dz^2$, provided T_0 is the connected component of $\bar{T} - B$ containing A,* *and we have*

$$T_0 \cap \bar{J} = A.$$

PROOF. In fact $J - A$ consists of two disjoint components J_1, J_2; to each of them we may apply the result of Theorem 1.

REMARK. *Theorem* 1 *and Theorem* 1, *Corollary remain true if the condition*

$$\operatorname{Im} (Q(z)\, dz^2) \neq 0 \quad on \ J,$$

is weakened to

$$\operatorname{Re} (Q(z)\, dz^2) < 0$$

at every point where $\operatorname{Im} (Q(z)\, dz^2) = 0$ *on* J.

PROOF. In fact it follows again that J is nowhere tangent to a trajectory of the quadratic differential $Q(z)\, dz^2$; now let

$$Q(z)^{1/2}\, dz = u + iv.$$

We have

$$Q(z)\, dz^2 = u^2 - v^2 + 2iuv$$

and if $v = 0$ then

$$Q(z)\, dz^2 > 0.$$

Hence $\operatorname{Im} (Q(z)^{1/2}\, dz)$ has a constant sign on J and the proof of Theorem 1 still applies.

IV. Applications to the theory of univalent functions.

Let S denote the family of functions

$$f(w) = w + a_2 w^2 + \dots$$

which are regular and univalent in the unit disk $|w| < 1$, also let Σ denote the family of functions

$$g(w) = \frac{1}{w} + b_0 + b_1 w + \dots$$

which are regular and univalent in the punctured disk $0 < |w| < 1$, with a simple pole at the origin.

It is clear that if $f(w) \in S$, then

$$\frac{1}{f(w)} = \frac{1}{w} - a_2 + (a_2^2 - a_3) w + \ldots \in \Sigma .$$

We shall identify the closed z-plane to the z-sphere, and call it R.

The relationship between extremal problems for functions of classes S and Σ and the theory of quadratic differentials comes from the Teichmüller principle which asserts that any such extremal function maps $|w| < 1$ onto the z-spere R slit along a subset Γ of the set $\overline{T}$ of critical trajectories of an appropriate quadratic differential on R. For functions in S, the exact formulation is the following.

For any $f(w) = w + a_2 w^2 + \ldots \in S$ we put $a_n = x_n + i y_n$, and consider the region V_n of points $(x_2, y_2, \ldots, x_n, y_n)$ in $(2n-2)$-dimensional real space, for varying f in S. It is easily proved that V_n is closed and bounded.

PROPOSITION 1. *Let $F = F(x_2, \ldots, x_n ; y_2, \ldots, y_n)$ be real and continuous with its first derivatives in an open set U containing V_n, also suppose $|\operatorname{grad} F| > 0$ in U.*

Let $f(w) = w + a_2 w^2 + \ldots \in S$ be extremal for the problem of maximizing F within S. Let

$$f^m(w) = \sum_m^\infty a_k^{(m)} w^k$$

and let

$$A_\nu = \sum_{k=\nu+1}^{n} a_k^{(\nu+1)} \left(\frac{\partial F}{\partial x_k} - i \frac{\partial F}{\partial y_k} \right).$$

Then the function

$$z = \frac{1}{f(w)} \in \Sigma$$

maps $|w| < 1$ univalently onto $R - \Gamma$ where Γ is a subcontinuum, containing the origin and of mapping radius 1, of the closure $\overline{T}$ of the set T of critical trajectories of the quadratic differential on R

$$- \left(\sum_1^{n-1} A_\nu z^{\nu-2} \right) dz^2 .$$

PROOF. It is an immediate consequence of [9], Lemma VII and Theorem III (this last result is needed to prove that Γ is a subcontinuum of $\overline{T}$).

We shall prove

THEOREM 2. *Let $f(w) = w + a_2 w^2 + \ldots \in S$ and have a real second* *coefficient* a_2. *Then*

$$\mathrm{Re}\,(a_5) \leq 5,$$

with equality only when $f(w) = \dfrac{w}{(1-w)^2}$ *and* $f(w) = \dfrac{w}{(1+w)^2}$.

This result is a consequence of the following more general inequality

THEOREM 3. *Let* $f(w) = w + a_2 w^2 + \ldots \in S$ *and let* $\alpha > 0$. *Then we have*

$$x_5 + 2y_2 y_4 + \frac{3}{2}\, y_3^2 - 4x_2\, y_2\, y_3 + [2x_2^2 - \alpha - x_3]\, y_2^2 \leq 5,$$

with equality only when $f(w) = \dfrac{w}{(1-w)^2}$ *and* $f(w) = \dfrac{w}{(1+w)^2}$.

LEMMA 1. *Let* F *be the functional considered in Theorem 2. Then any extremal function* $f(w)$ *for* F *has* $y_2 = 0$.

PROOF. By Proposition 1 we find using straight-forward algebra that the associated quadratic differential is

$$Q(z)\, dz^2 = \left(A_4 z^2 + A_3 z + A_2 + \frac{A_1}{z}\right) dz^2,$$

where

$$A_4 = 1,$$

$$A_3 = 4x_2 + i\,2y_2,$$

$$A_2 = 3x_2^2 + 3x_3 + 2y_2^2 + i\,4x_2\, y_2,$$

$$A_1 = 2x_2\, x_3 + 2x_4 + 4y_2\, y_3 - 2\, x_2\, y_2^2 + i\,(2x_2^2 + 2\alpha)\, y_2.$$

We apply Theorem 1 taking for J any segment of real axis containing the origin. Suppose $y_2 \neq 0$, so that $A_1 \neq 0$ and $Q(z)\, dz^2$ has a simple pole at the origin; let T_0 be the connected component of $\bar{T} - B$ where B is the point at ∞, containing the origin.

We have on J that dz^2 is real whence

$$\mathrm{Im}\,(Q(z)\, dz^2) = -\frac{2y_2}{z}\,[(z + x_2)^2 + \alpha]\, dz^2 \neq 0$$

because $\alpha > 0$ and z is real. By Theorem 1, Corollary, we deduce that

$T_0 \cap J$ is the origin, and in particular that T_0 lies entirely either in the upper half-plane or in the lower half-plane. Now a simple calculation shows that the tangent vector to T_0 at the origin has argument $-\arg(-A_1)$ whence T_0 and Γ lie in the same half-plane as a_2 does.

On the other hand, from

$$\frac{1}{f(w)} = \frac{1}{w} - a_2 + (a_2^2 - a_3)\, w + \dots$$

putting $w = e^{i\theta}$, $z = z(e^{i\theta}) = \dfrac{1}{f(e^{i\theta})}$, we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} z\, d\theta = -\, a_2 .$$

We may think of $d\mu = \dfrac{d\theta}{2\pi}$ as a non-negative measure on Γ of unit total measure $\int_\Gamma d\mu = 1$, and we get

$$\int_\Gamma z\, d\mu = -\, a_2 .$$

Thus a mean of Γ with respect to a non-negative measure lies in the opposite half-plane where Γ lies, a contradiction whence $y_2 = 0$.

COROLLARY 1. $\max_S F = \max' \left\{ x_5 + \dfrac{3}{2} y_3^2 \right\}$, *where* max′ *is taken in the subclass of S consisting of functions with real second coefficient a_2.*

COROLLARY 2. *The closure $\overline{T}$ of the set T of critical trajectories of the quadratic differential $Q(z)\, dz^2$ is symmetrical about the real axis.*

PROOF. As $y_2 = 0$, we obtain

$$A_4 = 1,$$

$$A_3 = 4x_2,$$

$$A_2 = 3x_2^2 + 3x_3,$$

$$A_1 = 2x_2\, x_3 + 2x_4$$

and $Q(z)\, dz^2$ is real on the real axis; Corollary 2 follows at once from this.

Lemma 2. *If $f(w)$ maximizes F, the origin cannot be a zero of $Q(z)\,dz^2$.*

Proof. If $z=0$ is a zero of $Q(z)\,dz^2$ we would have

$$x_2^2+x_3=0,$$

$$x_2 x_3+x_4=0$$

whence

$$x_3=-x_2^2 \quad \text{and} \quad x_4=x_2^3.$$

By the Area Theorem for $\dfrac{1}{f(w)}$, we have

$$|a_2^2-a_3|^2+2\,|a_2^3-2a_2a_3+a_4|^2+3\,|a_2^4+a_3^2-3a_2^2a_3+2a_2a_4-a_5|^2\leq 1.$$

Now $a_2=x_2$ is real whence from the previous inequality we get a fortiori

$$4x_2^2+y_3^2+3\,(7x_2^4-y_3^2-x_5)^2\leq 1.$$

Hence $x_2^2\leq\frac{1}{2}$, $y_3^2\leq 1$, and $x_5+y_3^2\leq\frac{1}{\sqrt{3}}+7x_2^4$, $x_5+\frac{3}{2}\,y_3^2\leq\frac{1}{\sqrt{3}}+ + 7x_2^4+\frac{1}{2}<4$, which is impossible because max $F\geq 5$ as we can see trivially from the Koebe function.

Proof of Theorem 3. By Lemma 2, T_0 is unforked at the origin and there is exactly one trajectory Λ of T_0 whose closure contains the origin. By Lemma 1, Corollary 2 this trajectory must lie either on the real axis or on the immaginary axis. In the latter case from $Q(z)\,dz^2>0$ we deduce $A_3=0$, i. e. $a_2=0$ and this by the Area Theorem would give

$$|a_3|^2+3\,|a_3^2-a_5|^2\leq 1$$

and again $x_5+\frac{3}{2}\,y_3^2<5$, contradicting the fact that $f(w)$ maximizes F. Hence this trajectory must lie on the real axis.

Now suppose Γ contains a zero of the quadratic differential $Q(z)\,dz^2$. Then Γ must contain at least one end point of the trajectory Λ other than the origin and thus Γ contains a real zero z_0 of $Q(z)\,dz^2$.

Consider the variation in the large of $f(w)$ given by

$$f^*(w)=\frac{f(w)}{1-z_0 f(w)}.$$

We have $z_0\in\Gamma$ whence $f^*(w)\in S$.

Putting $f^*(w) = z + a_2^* z^2 + \dots$ and $a_n^* = x_n^* + i y_n^*$ a simple computation gives

$$a_2^* = a_2 + z_0 ,$$

$$a_3^* = a_3 + 2a_2 z_0 + z_0^2,$$

$$a_4^* = a_4 + (a_2^2 + 2a_3) z_0 + 3a_2 z_0^2 + z_0^3,$$

$$a_5^* = a_5 + (2a_2 a_3 + 2a_4) z_0 + (3a_2^2 + 3a_3) z_0^2 + 4a_2 z_0^3 + z_0^4.$$

It follows that a_2^* is real, also

$$y_3^* = y_3$$

and

$$x_5^* = x_5 + (2x_2 x_3 + 2x_4) z_0 + (3x_2^2 + 3x_3) z_0^2 + 4x_2 z_0^3 + z_0^4 =$$

$$= x_5 + z_0^2 \, Q(z_0) = x_5 .$$

Hence

$$x_5^* + \frac{3}{2} y_3^{*2} = x_5 + \frac{3}{2} y_3^2$$

so that by Lemma 1, Corollary 1 $f^*(w)$ maximizes F. Hence the function $\dfrac{1}{f^*(w)}$ maps $|w| < 1$ onto the z^*-sphere slit along a subcontinuum Γ^* of the closure $\overline{T}^*$ of the union T^* of critical trajectories of a quadratic differential

$$Q^*(z^*) \, dz^{*2}$$

where

$$Q^*(z^*) = -\left(z^{*2} + A_3^* z^* + A_2^* + \frac{A_3^*}{z^*}\right) dz^{*2}.$$

Obviously $f^*(w)$ has the same general properties of $f(w)$ and we obtain in particular that the set Γ^* is unforked at the origin and coincides with a segment of the real axis in a neighborhood of the origin.

On the other hand,

$$\frac{1}{f^*(w)} = \frac{1}{f(w)} - z_0 ,$$

thus on Γ we have

$$z - z_0 = z^*.$$

It follows that Γ satisfies both differential equations

$$Q(z)\,dz^2 > 0$$

and

$$Q^*(z - z_0)\,dz^2 > 0,$$

and belongs to the associated $\bar{T}$ sets of both quadratic differentials. By our previous discussion, the $\bar{T}$ set of $Q(z)\,dz^2$ is a segment of the real axis in a neighborhood of the origin, while the $\bar{T}$ set of $Q^*(z - z_0)\,dz^2$ is a segment of the real axis in a neighborhood of z_0.

It follows that if Γ can be continued past the point z_0, it must still lie on the real axis. Hence Γ is a segment of the real axis.

If Γ does not contain any zero of $Q(z)\,dz^2$, we have $\Gamma \subset A$ and again Γ is a segment of the real axis.

Thus we have proved that Γ is a real segment. As Γ has mapping radius 1, this segment must be of length 4 and $f(w)$ a function

$$f(w) = \frac{w}{1 - 2cw + w^2}, \qquad -1 \leq c \leq 1$$

Our functional F in this case is

$$F = 1 - 12c^2 + 16c^4$$

which is maximum and equal to 5 when $c^2 = 1$. This completes the proof of Theorems 2 and 3.

Our next result will be a simplified proof of the inequality $|b_3| \leq \frac{1}{2} + e^{-6}$ for functions of Σ, first proved by Garabedian and Schiffer [7]. Our basic contribution will be to show that the properties of the extremal functions can be deduced from a geometric study of the trajectories of the associated quadratic differential. In particular we shall avoid the complicated arguments of section 4 as well as the numerical computations of section 5 of [7].

THEOREM 4. *Let* $g(w) = \frac{1}{w} + b_0 + b_1 w + \ldots \in \Sigma$.

Then

$$\operatorname{Re}(b_3) - \frac{1}{2}\{\operatorname{Im}(b_1)\}^2 \geq -\frac{1}{2} - e^{-6},$$

and this inequality is sharp.

In particular, $|b_3| \leq \frac{1}{2} + e^{-6}$ *and this inequality is again sharp.*

We first determine the quadratic differential with the problem of finding the minimum of the functional in Theorem 4. This is a routine matter in variational methods for univalent functions, but for the convenience of the reader we shall give an indication on how to find it.

We note that without loss of generality we may suppose $b_0 = 0$ by making a translation of $g(w)$.

By Teichmüller's principle, an extremal $g(w)$ maps $|w| < 1$ univalently onto $R - \Gamma$ where Γ is a closed subcontinuum of mapping radius 1 of the set $\bar{T}$ associated to some quadratic differential $Q(z)\,dz^2$. In particular, Γ consists of a finite number of analytic arcs. Let $z_0 \in \Gamma$ be not an end point of these arcs; then for sufficiently small ϱ and for each complex number B_1 with $|B_1| \leq 1$ there is a varied function $g^*(w)$ with

$$g^*(w) = g(w) - z_0 + C_0\,\varrho + \frac{(C_1 + B_1)\,\varrho^2}{g(w) - z_0} + o(\varrho^2)$$

and where

$$\lim_{\varrho \to +0} C_1 = -\left(\frac{dz_0}{|dz_0|}\right)^2;$$

for this result, see [7], pag. 120 eq. 27.

We get

$$b_1^* = b_1 + (C_1 + B_1)\,\varrho^2 + o(\varrho^2),$$

$$b_3^* = b_3 + (z_0^2 - b_1)(C_1 + B_1)\,\varrho^2 + o(\varrho^2),$$

and writing $b_n = \beta_n + i\gamma_n$ we get from the extremal property of $g(w)$:

$$\beta_3 - \frac{1}{2}\gamma_1^2 \leq \beta_3^* - \frac{1}{2}\gamma_1^{*2} = \beta_3 - \frac{1}{2}\gamma_1^2 + \mathrm{Re}\,\{(z_0^2 - b_1)(C_1 + B_1)\,\varrho^2\} -$$

$$- \mathrm{Im}\,\{\gamma_1 (C_1 + B_1)\,\varrho^2\} + o(\varrho^2) =$$

$$= \beta_3 - \frac{1}{2}\gamma_1^2 + \varrho^2\,\mathrm{Re}\,\{(z_0^2 - \beta_1)(C_1 + B_1)\} + o(\varrho^2).$$

Letting $\varrho \to +0$ we see that this inequality is possible for each $|B_1| \leq 1$ only if

$$-(z_0^2 - \beta_1)\,dz_0^2 \geq 0.$$

Thus we have proved that $g(w)$ maps $|w| < 1$ onto $R - \Gamma$, when Γ is a closed subcontinuum of the clósure $\bar{T}$ of the set T of critical trajectories of the quadratic differential

$$-(z^2 - \beta_1)\, dz^2 ,$$

where

$$\beta_1 = \operatorname{Re}(b_1).$$

By making a rotation of an angle $\frac{\pi}{2}$, we may suppose that $\beta_1 \geq 0$.

LEMMA 3. *Γ contains all critical points of the quadratic differential* $-(z^2 - \beta_1)\, dz^2$.

PROOF. It is clear from $\beta_1 \geq 0$ that $\overline{T}$ then contains the line segment $-\beta_1^{1/2} \leq z \leq \beta_1^{1/2}$ joining the two critical points $\pm \beta_1^{1/2}$ of $Q(z)\, dz^2$, so that $T_0 = \overline{T} - B$, where B is the pole of $Q(z)\, dz^2$ at $z = \infty$, is connected. Clearly T_0 is still connected in case $\beta_1 = 0$.

From the fact that $g(w)$ has constant coefficient zero we obtain

$$\int_\Gamma z d\mu = 0.$$

where $d\mu = \frac{1}{2\pi} d\theta$ is the measure induced on Γ by $z = g(e^{i\theta})$. It follows that Γ has points both in the right and left half-planes.

Now T_0 meets the immaginary axis only in the origin, as we can see on applying Theorem 1, Remark which is possible because

$$Q(z)\, dz^2 < 0$$

on the immaginary axis except at the origin.

Hence Γ contains the origin.

If $\beta_1 = 0$, Lemma 3 is proved.

Now suppose $\beta_1 > 0$ and that Γ contains no zeros of $Q(z)\, dz^2$. By our previous discussion, Γ is a line segment contained in $-\beta_1^{1/2} \leq z \leq \beta_1^{1/2}$, hence of length at most $2\beta_1^{1/2}$, hence of length at most 2 by the well-known elementary inequality $|b_1| \leq 1$. This however is impossible because Γ has mapping radius 1 and so it would have length 4.

Next, suppose Γ contains exactly one critical point, say $\beta_1^{1/2}$. Then Γ consists of a line segment L given by $-a \leq z \leq \beta_1^{1/2}$, where $a < \beta_1^{1/2}$, plus possibly two arcs issuing from the point $\beta_1^{1/2}$ and lying in the right half-plane. Note that $a > 0$ because otherwise $\int_\Gamma z\, d\mu$ would have a positive real part.

To exclude this case, and thus prove Lemma 3, we use a simple argument of Charzynski and Schiffer [4], Lemma 1.

Let L_a be the line segment $-a \leq z \leq a$ and let M be a line segment on the immaginary axis. Let

$$h(\zeta) = c\zeta + c_0 + \frac{c_1}{\zeta} + \dots, \; c > 0$$

be the development near $\zeta = \infty$ of the univalent function mapping conformally $R - M$ onto $R - L_a$; also let

$$H = h^{-1}(\Gamma - L_a).$$

Let further

$$t(w) = \frac{d}{w} + d_0 + d_1 w + \dots, \; d > 0$$

map $|w| < 1$ conformally onto $R - (M + H)$.

The set $M + H$ has no points in the left half-plane by the symmetry of the mapping $h(\zeta)$ about the immaginary axis and because $\Gamma - L_a$ lies in the righ half-plane. It follows that

$$\operatorname{Re}(d_0) = \operatorname{Re}\left(\int_{M+H} z d\mu\right) > 0$$

because the set H is non-empty and has a positive real part.

Now

$$g(w) = h[t(w)] = \frac{1}{w} + b_0 + b_1 w + \dots \in \Sigma$$

and maps $|w| < 1$ onto $R - \Gamma$. As $b_0 = 0$ we obtain

$$b_0 = cd_0 + c_0 = 0.$$

By the symmetry of $h(\zeta)$ about the immaginary axis we obtain

$$\operatorname{Re}(c_0) = 0,$$

which is a contrad ction to the equation $cd_0 + c_0 = 0$, because $c > 0$ and $\operatorname{Re}(d_0) > 0$.

This completes the proof of Lemma 3.

PROOF OF THEOREM 4. Let $g(w)$ be extremal for our problem. On the circle $|w| = 1$, $g(w) = z$ satisfies

$$-(z^2 - \beta_1)\,dz^2 \geq 0,$$

whence

$$(wg'(w))^2\,[g(w)^2 - \beta_1) = \frac{1}{w^4} - \frac{\beta_1}{w^2} - \frac{2b_2}{w} - (4b_3 + 2b_1^2 - 2b_1\beta_1) + \dots$$

is real and non-negative on $|w| = 1$. By the Schwarz reflection principle, it can be continued across $|w| = 1$ and we get the differential equation

$$(wg'(w))^2\,[g(w)^2 - \beta_1] =$$

$$= \frac{1}{w^4} - \frac{\beta_1}{w^2} - \frac{2b_2}{w} - (4b_3 + 2b_1^2 - 2b_1\beta_1) - 2\bar{b}_2 w - \beta_1 w^2 + w^4,$$

where the right hand side of this equation is real and non-negative on $|w| = 1$.

By Lemma 3, Γ contains the critical points of $-(z^2 - \beta_1)\,dz^2$. It follows that the right hand side of the differential equation for $g(w)$ has four double roots on $|w| = 1$. Hence it is a perfect square, and

$$\frac{1}{w^4} - \frac{\beta_1}{w^2} - \frac{2b_2}{w} - (4b_3 + 2b_1^2 - 2b_1\beta_1) - 2\bar{b}_2 w - \beta_1 w^2 + w^4 =$$

$$= \left(\frac{1}{w^2} - \frac{\beta_1}{2} + w^2\right)^2,$$

because the left hand side of this equation is real and non-negative on $|w| = 1$.

It follows that

$$b_2 = 0,$$

$$\beta_3 - \frac{1}{2}\gamma_1^2 = -\frac{1}{2} - \frac{1}{16}\beta_1^2,$$

$$\gamma_3 = -\frac{1}{2}\beta_1\gamma_1$$

where we have written $b_n = \beta_n + i\gamma_n$.

Our differential equation becomes on taking square roots

$$[g(w)^2 - \beta_1]^{1/2}\,g'(w) = \frac{1}{w^3} - \frac{\beta_1}{2}\frac{1}{w} + w$$

and integrating we find

$$\frac{g\,(g^2-\beta_1)^{1/2}}{2}-\frac{\beta_1}{2}\log\frac{g+(g^2-\beta_1)^{1/2}}{\beta_1^{1/2}}=-\frac{1}{2}\frac{1}{w^2}+\frac{1}{2}w^2-\frac{\beta_1}{2}\log w+K,$$

where K is a costant of integration.

In order to find K, one expands both members of this equation near $w=0$ and gets

$$K=-\left(b_1-\frac{\beta_1}{4}\right)+\frac{\beta_1}{4}\log\frac{4}{\beta_1}.$$

On the other hand, $\beta_1^{1/2}\in\Gamma$ thus there is w_0 with $|w_0|=1$ such that $g^2(w_0)=\beta_1$. This gives

$$\frac{1}{2}(w_0^2-w_0^{-2})-\frac{\beta_1}{2}\log w_0+K=0,$$

which proves that K is purely immaginary. Hence

$$\mathrm{Re}\left(-\left(b_1-\frac{\beta_1}{4}\right)+\frac{\beta_1}{4}\log\frac{4}{\beta_1}\right)=-\frac{3\beta_1}{4}+\frac{\beta_1}{4}\log\frac{4}{\beta_1}=0.$$

It follows that

$$K=-i\gamma_1$$

and that either $\beta_1=0$ or $\beta_1=\frac{4}{e^3}$.

But then $\beta_3-\frac{1}{2}\gamma_1^2=-\frac{1}{2}-\frac{1}{16}\beta_1^2$ is a minimum when $\beta_1=\frac{4}{e^3}$ and the minimum is $-\frac{1}{2}-e^{-6}$.

Equality is attained when g satisfies

$$\frac{g\,(g^2-\beta_1)^{1/2}}{2}-\frac{\beta_1}{2}\log\frac{g+(g^2-\beta_1)^{1/2}}{\beta_1^{1/2}}=-\frac{1}{2}\frac{1}{w^2}+\frac{1}{2}w^2-\frac{\beta_1}{2}\log w-i\gamma_1,$$

where now $\beta_1=\frac{4}{e^3}$, and $g(w)$ has expansion near $w=0$ given by

$$g(w)=\frac{1}{w}+(\beta_1+i\gamma_1)\,w+b_3\,w^3+\dots,$$

and γ_1 is such that $g(w)$ remains holomorphic and univalent in $0<|w|<1$.

This completes the proof of Theorem 4.

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