Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

A GEOMETRIC CHARACTERISATION OF THE BLOCKS OF THE BRAUER ALGEBRA

ANTON COX AND MAUD DE VISSCHER AND PAUL MARTIN

ABSTRACT

We give a geometric description of the blocks of the Brauer algebra in characteristic zero as orbits of the Weyl group of type D_n. We show how the corresponding affine Weyl group controls the block decomposition of the Brauer algebra in positive characteristic, with orbits corresponding to unions of blocks.

1. *Introduction*

Classical Schur-Weyl duality relates the representation theory of the symmetric and general linear groups by realising each as the centraliser algebra of the action of the other on a certain tensor space. The Brauer algebra $B_n(\delta)$ was introduced to provide a corresponding duality for the symplectic and orthogonal groups [**Bra37**]. The abstract k-algebra is defined for each $\delta \in k$, however for Brauer the key case is $k = \mathbb{C}$ with δ integral, when the action of $B_n(\delta)$ on $(\mathbb{C}^{|\delta|})^{\otimes n}$ can be identified with the centraliser algebra for the corresponding group action of $\mathrm{O}(\delta, \mathbb{C})$ for δ positive, and with $\mathrm{Sp}(-\delta, \mathbb{C})$ for δ negative). In characteristic p, the natural algebra in correspondence to the centraliser algebra for δ negative is the symplectic Schur algebra [**Don87**, **Dot98**, **Oeh01**, **DDH08**].

For $|\delta| < n$ the centraliser algebra is a proper quotient of the Brauer algebra. Thus, despite the fact that the symplectic and orthogonal groups, and hence the centraliser, are semisimple over $\mathbb{C}$, the Brauer algebra can have a non-trivial cohomological structure in such cases.

Brown [**Bro55**] showed that the Brauer algebra is semisimple over $\mathbb{C}$ for generic values of δ. Wenzl proved that $B_n(\delta)$ is semisimple over $\mathbb{C}$ for all non-integer δ [**Wen88**]. It was not until very recently that any progress was made in positive characteristic. A necessary and sufficient condition for semisimplicity (valid over an arbitrary field) was given by Rui [**Rui05**]. The blocks were determined in characteristic zero [**CDM05**] by the authors.

The block result uses the theory of towers of recollement from [**CMPX06**], and builds on work by Doran, Hanlon and Wales [**DWH99**]. The approach was combinatorial, using the language of partitions and tableaux, and depended also on a careful analysis of the action of the symmetric group Σ_n, realised as a subalgebra of the Brauer algebra. However, we speculated in [**CDM05**] that there could be an alcove geometric version, in the language of algebraic Lie theory [**Jan03**] (despite the absence of an obvious Lie-theoretic context) . This should replace the combinatorics of partitions by the action of a suitable reflection group on a weight space, so that the blocks correspond to orbits under this action. In this paper we will give such a geometric description of this block result.

A priori there is no specific evidence from algebraic Lie theory to suggest that such a reflection group action will exist (beyond certain similarities with the partition algebra case, where there is a reflection group of infinite type A [**MW98**]). As already noted, the obvious link to Lie theory (via the duality with symplectic and orthogonal groups) in characteristic zero only corresponds to a semisimple quotient.

Remarkably however, we will show that there is a Weyl group W of type D which does control the representation theory. To obtain a natural action of this group, we will find that it is easier to work with the transpose of the usual partition notation. (This is reminiscent of the relation under Ringel duality

---

2000 *Mathematics Subject Classification* 20G05.

between the combinatorics of the symmetric and general linear groups, although we do not have a candidate for a corresponding dual object in this case.)

Our proof of the geometric block result in characteristic 0 is entirely combinatorial, as we show that the action of W corresponds to the combinatorial description of blocks in [**CDM05**]. However, having done this, it is natural to consider extending these results to arbitrary fields.

As the algebras and (cell) modules under consideration can all be defined 'integrally' (over $\mathbb{Z}[\delta]$), one might hope that some aspects of the characteristic 0 theory could be translated to other characteristics by a reduction mod p argument. If this were the case then, for consistency between different values of δ which are congruent modulo p, we might expect that the role of the Weyl group would be replaced by the corresponding affine Weyl group, so that blocks again lie within orbits.

We will extend certain basic results in [**DWH99**] to arbitrary characteristic, and then show that orbits of the affine Weyl group do indeed correspond to (possibly non-trivial) unions of blocks of the Brauer algebra.

In Section 2 we review some basic properties of the Brauer algebra, following [**CDM05**]. Sections 3 and 5 review the Weyl and affine Weyl groups of type D, and give a combinatorial description of their orbits on a weight space. Using this description we prove in Section 4 that we can restate the block result from [**CDM05**] using Weyl group orbits. Section 6 generalises certain representation theoretic results from [**DWH99**] and [**CDM05**] to positive characteristic, which are then used to give a necessary condition for two weights to lie in the same block in terms of the affine Weyl group.

In Section 7 we describe how abacus notation [**JK81**] can be applied to the Brauer algebra, and use this to show that the orbits of the affine Weyl group do not give a sufficient condition for two weights to lie in the same block.

2. *The Brauer algebra*

We begin with a very brief review of the basic theory of Brauer algebras; details can be found in [**CDM05**]. Fix a field k of characteristic $p \geq 0$, and some $\delta \in k$. For $n \in \mathbb{N}$ the Brauer algebra $B_n(\delta)$ can be defined in terms of a basis of partitions of $\{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$ into pairs. To determine the product AB of two basis elements, represent each by a graph on $2n$ points, and identify the vertices $\bar{1},\ldots,\bar{n}$ of A with the vertices $1,\ldots n$ of B respectively. The graph thus obtained may contain some number (t say) of closed loops; the product AB is then defined to be $\delta^t C$, where C is the basis element corresponding to the graph arising after these closed loops are removed (ignoring intermediate vertices in connected components).

Usually we represent basis elements graphically by a diagram with n northern nodes numbered 1 to n from left to right, and n southern nodes numbered $\bar{1}$ to $\bar{n}$ from left to right, where each node is connected to precisely one other by a line. Edges joining northern nodes to southern nodes of a diagram are called propagating lines, the remainder are called northern or southern arcs. An example of the product of two diagrams is given in Figure 1.

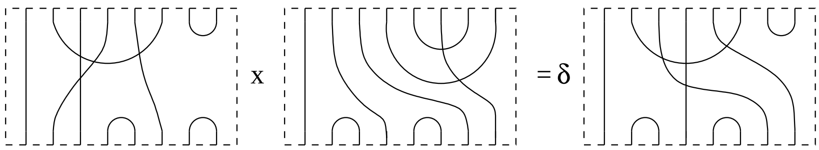

FIGURE 1. *The product of two diagrams in $B_n(\delta)$.*

With this convention, and assuming that $\delta \neq 0$, we have for each $n \geq 2$ an idempotent e_n as illustrated in Figure 2. We will discuss the case $\delta \neq 0$ in what follows; details of the modifications required when $\delta = 0$ can be found in [**CDM05**, Section 8].

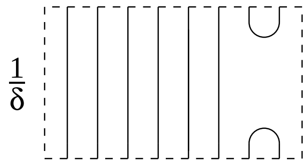

FIGURE 2. *The element e_8.*

The idempotents e_n induce algebra isomorphisms

$$\Phi_n : B_{n-2}(\delta) \longrightarrow e_n B_n(\delta) e_n \tag{2.1}$$

which take a diagram in B_{n-2} to the diagram in B_n obtained by adding an extra northern and southern arc to the right-hand end. From this we obtain, following [**Gre80**], an exact localisation functor

$$\begin{aligned} F_n \,:\, B_n(\delta)\text{-mod} &\longrightarrow B_{n-2}(\delta)\text{-mod} \\ M &\longmapsto e_n M \end{aligned}$$

and a right exact globalisation functor

$$\begin{aligned} G_n \,:\, B_n(\delta)\text{-mod} &\longrightarrow B_{n+2}(\delta)\text{-mod} \\ M &\longmapsto B_{n+2}e_{n+2} \otimes_{B_n} M. \end{aligned}$$

Note that $F_{n+2}G_n(M) \cong M$ for all $M \in B_n$-mod, and hence G_n is a full embedding. As

$$B_n(\delta)/B_n(\delta)e_nB_n(\delta) \cong k\Sigma_n \tag{2.2}$$

the group algebra of the symmetric group on n symbols, it follows from [**Gre80**] and (2.1) that when $\delta \neq 0$ the simple B_n-modules are indexed by the set

$$\Lambda_n = \Lambda^n \sqcup \Lambda_{n-2} = \Lambda^n \sqcup \Lambda^{n-2} \sqcup \cdots \sqcup \Lambda^{\min} \tag{2.3}$$

where Λ^n denotes an indexing set for the simple $k\Sigma_n$-modules, and $\min = 0$ or 1 depending on the parity of n. (When $\delta = 0$ a slight modification of this construction is needed; see [**HP06**] or [**CDM05**, Section 8].) If $\delta \neq 0$ and either $p = 0$ or $p > n$ then the set Λ^n corresponds to the set of partitions of n; we write $\lambda \vdash n$ if λ is such a partition.

If $\delta \neq 0$ and $p = 0$ or $p > n$ then the algebra $B_n(\delta)$ is quasihereditary — in general however it is only cellular [**GL96**]. In all cases however we can explicitly construct a standard/cell module $\Delta_n(\lambda)$ for each partition λ of m where $m \leq n$ with $m-n$ even (by arguing as in [**DWH99**, Section 2]). When λ is a partition of n the module $\Delta_n(\lambda)$ is just the lift of the Specht module S^λ from $k\Sigma_n$ via (2.2). (An explicit construction of Specht modules can be found in [**JK81**]; we note here that they can be defined over $\mathbb{Z}$.) In the quasihereditary case with λ a partition of $n-2t$ we obtain the standard module $\Delta_n(\lambda)$ as

$$\Delta_n(\lambda) \cong G_{n-2}G_{n-4}\cdots G_{n-2t}\Delta_{n-2t}(\lambda).$$

It is easy to give an explicit basis for this module (see [**DWH99**]) and check that it makes sense even over $\mathbb{Z}[\delta]$ with δ an indeterminate. The general cell module construction then follows via base change.

In the quasihereditary case, the heads $L_n(\lambda)$ of the standard modules $\Delta_n(\lambda)$ are simple, and provide a full set of simple $B_n(\delta)$-modules. In the general cellular case, a proper subset of the heads of the cell modules is sufficient to provide such a full set of simples. The key result which we will need is that in all cases, the blocks of the algebra correspond to the equivalence classes of simple modules generated by the relation of occurring in the same cell or standard module [**GL96**, (3.9) Remarks].

3. *Orbits of weights for the Weyl group of type D*

We review some basic results about the Weyl group of type D, following [**Bou68**, Plate IV]. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a set of formal symbols. We set

$$X\ (=X_n) = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$$

which will play the role of a weight lattice. We denote an element

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$$

in X by any tuple of the form $(\lambda_1, \dots, \lambda_m)$, with $m \leq n$, where $\lambda_i = 0$ for $i > m$. The set of dominant weights is given by

$$X^+ = \{\lambda \in X : \lambda = \lambda_1\varepsilon_1 + \cdots \lambda_n\varepsilon_n \text{ with } \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

Define an inner product on $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ by setting

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}$$

and extending by linearity.

Consider the root system of type D_n:

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq n\}$$

For each root $\beta \in \Phi$ we define a corresponding reflection s_β on E by

$$s_\beta(\lambda) = \lambda - (\lambda, \beta)\beta \tag{3.1}$$

for all $\lambda \in E$, and let W be the group generated by these reflections. Fix $\delta \in \mathbb{Z}$ and define $\rho\ (=\rho(\delta)) \in E$ by

$$\rho = (-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, \dots, -\frac{\delta}{2} - (n-1)).$$

We consider the dot action of W on E given by

$$w.\lambda = w(\lambda + \rho) - \rho$$

for all $w \in W$ and $\lambda \in E$. (Note that this preserves the lattice X.) This is the action which we will consider henceforth.

It will be convenient to have an explicit description of the dot action of W on X. Let Σ_n denote the group of permutations of $\mathbf{n} = \{1, \dots, n\}$. Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ in X, we have $\mu = w.\lambda$ for some $w \in W$ if and only if

$$\mu_i + \rho_i = \sigma(i)(\lambda_{\pi(i)} + \rho_{\pi(i)})$$

for all $1 \leq i \leq n$ and some $\pi \in \Sigma_n$ and $\sigma : \mathbf{n} \longrightarrow \{\pm 1\}$ with

$$d(\sigma) = |\{i : \sigma(i) = -1\}|$$

even. (See [**Bou68**, IV.4.8].) Thus $\mu = w.\lambda$ if and only if there exists $\pi \in \Sigma_n$ such that for all $1 \leq i \leq n$ we have either

$$\mu_i - i = \lambda_{\pi(i)} - \pi(i) \tag{3.2}$$

or

$$\mu_i + \lambda_{\pi(i)} - i - \pi(i) = \delta - 2 \tag{3.3}$$

and (3.3) occurs only for an even number of i.

For example, if $n = 5$ and $\lambda = (6, 4, -2, 3, 5)$ then $\mu = (-4, \delta, 5, \delta - 3, 4)$ is in the same orbit under the dot action of W, taking $\pi(1) = 3$, $\pi(2) = 5$, $\pi(3) = 2$, $\pi(4) = 1$, $\pi(5) = 4$, and $\sigma(i) = 1$ for i odd and $\sigma(i) = -1$ for i even.

We will also need to have a graphical representation of elements of X, generalising the usual partition notation. We will represent any $\lambda = (\lambda_1, \dots, \lambda_n) \in X$ by a sequence of n rows of boxes, where row i contains all boxes to the left of column λ_i inclusive, together with a vertical bar between columns 0 and 1. We set the content of a box ε in row i and column j to be $c(\varepsilon) = i - j$. (This is not the usual choice for partitions, for reasons which will become apparent later.) For example, when $n = 8$ the element $(6,2,4,-3,1,-2)$ (and the content of its boxes) is illustrated in Figure 3.

...	4	3	2	1	0	−1	−2	−3	−4	−5
...	5	4	3	2	1	0				
...	6	5	4	3	2	1	0	−1		
...	7									
...	8	7	6	5	4					
...	9	8								
...	10	9	8	7						
...	11	10	9	8						

FIGURE 3. *The element* $(6,2,4,-3,1,-2)$ *when* $n = 8$.

When λ is a partition we will usually omit the portion of the diagram to the left of the bar, and below the final non-zero row, thus recovering the usual Young diagram notation for partitions.

If $\lambda = (\lambda_1, \dots, \lambda_n)$ then the content $c(\lambda)_i$ of the last box in row i is $-\lambda_i + i$. Combining this with (3.2) and (3.3) we obtain

PROPOSITION 3.1. *For any two elements λ and μ in X there exists $w \in W$ with $\mu = w.\lambda$ if and only if there exists $\pi \in \Sigma_n$ and $\sigma : \mathbf{n} \to \{\pm 1\}$ with $d(\sigma)$ even such that for all $1 \le i \le n$ we have either*

$$\sigma(i) = 1 \quad \textit{and} \quad c(\mu)_i = c(\lambda)_{\pi(i)}$$

or

$$\sigma(i) = -1 \quad \textit{and} \quad c(\mu)_i + c(\lambda)_{\pi(i)} = 2 - \delta.$$

It is helpful when considering low rank examples in Lie theory to use a graphical representation of the action of a Weyl group. As our weight space is generally greater than two-dimensional, we can rarely use such an approach directly. However, we can still apply a limited version of this approach, by considering various two-dimensional projections of the weight lattice.

We can depict elements of the weight lattice X by projecting into the ij plane for various choices of $i < j$. Each weight λ is represented by the projected coordinate pair (λ_i, λ_j), and each such pair represents a fibre of weights, which may or may not include any dominant weights. For example, the point $(0,0)$ in the $1j$ plane represents precisely one dominant weight (the zero weight), while the $(0,0)$ point in the 23 plane represents the set of dominant weights $(\lambda_1, 0, 0, \dots, 0)$. Clearly a necessary condition for dominance is that $\lambda_i \ge \lambda_j \ge 0$.

We will represent such projections in the natural two-dimensional coordinate system, so that the set of points representing at least one dominant weight correspond to those shaded in Figure 4. (If $\delta = 2$ then the example shown is the case $i = 1$ and $j = 5$.)

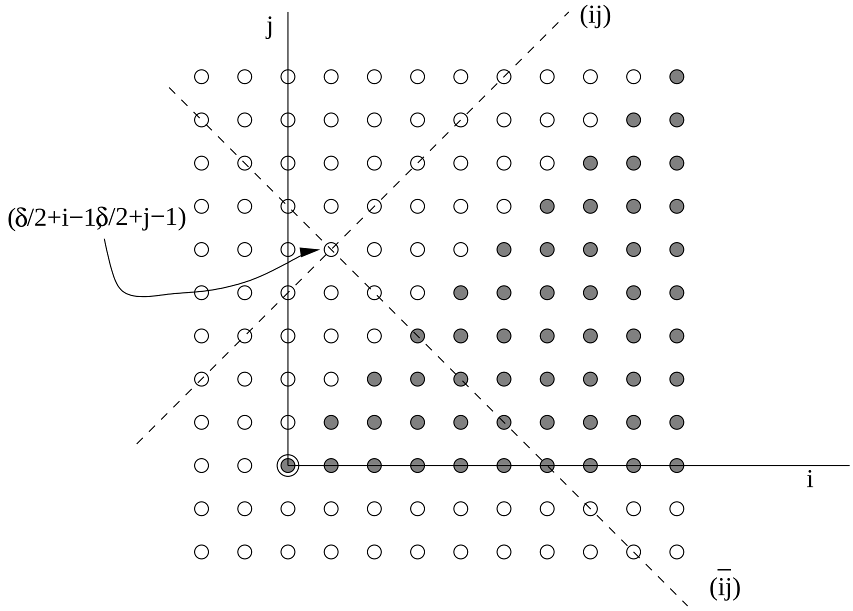

FIGURE 4. *A projection onto the ij plane.*

It will be convenient to give an explicit description of the action of $s_{\varepsilon_i-\varepsilon_j}$ and $s_{\varepsilon_i+\varepsilon_j}$ on a partition λ. We have that

$$s_{\varepsilon_i-\varepsilon_j}.\lambda = \lambda - (\lambda_i - \lambda_j - i + j)(\varepsilon_i - \varepsilon_j)$$

and hence if $r = \lambda_i - \lambda_j - i + j$ is positive (respectively negative) the effect of the dot action of $s_{\varepsilon_i-\varepsilon_j}$ on λ is to add r boxes to row j (respectively row i) and subtract r boxes from row i (respectively row j). Similarly,

$$s_{\varepsilon_i+\varepsilon_j}.\lambda = \lambda - (\lambda_i + \lambda_j - \delta + 2 - i - j)(\varepsilon_i + \varepsilon_j)$$

and hence if $r = \lambda_i + \lambda_j - \delta + 2 - i - j$ is positive (respectively negative) the effect of the dot action of $s_{\varepsilon_i+\varepsilon_j}$ on λ is to remove (respectively add) r boxes from each of rows i and j. In terms of our projection onto the ij plane these operations correspond to reflection about the dashed lines in Figure 4 labelled (ij) for $s_{\varepsilon_i-\varepsilon_j}$ and $(\overline{ij})$ for $s_{\varepsilon_i+\varepsilon_j}$. Note that the position of $(\overline{ij})$ depends on δ, but (ij) does not.

Various examples of reflections, and their effect on a dominant representative of each coordinate pair, are given in Figures 5, 6, and 7. For each reflection indicated, a dominant weight is illustrated, together with a shaded subcomposition corresponding to the image of that weight under the reflection. Where no shading is shown (as in Figure 5(a)) the image is the empty partition.

Note that some reflections may take a dominant weight to a non-dominant one, even if the associated fibres both contain dominant weights. For example the cases in Figure 7(a) and (b) correspond to the reflection of $(3,3,3)$ to $(1,3,1)$ and of $(4,3,3)$ to $(1,3,0)$. Also, some reflections may represent a family of reflections of dominant weights, as in Figure 7(e), where there are three possible weights in each fibre (corresponding to whether none, one or both of the boxes marked X are included).

4. *The blocks of the Brauer algebra in characteristic zero*

The main result in [**CDM05**] was the determination of the blocks of $B_n(\delta)$ when $k = \mathbb{C}$. In that paper, the blocks were described by a combinatorial condition on partitions. We would like to have a geometric formulation of this result. If $\delta \notin \mathbb{Z}$ then $B_n(\delta)$ is semisimple [**Wen88**], so we will assume that $\delta \in \mathbb{Z}$.

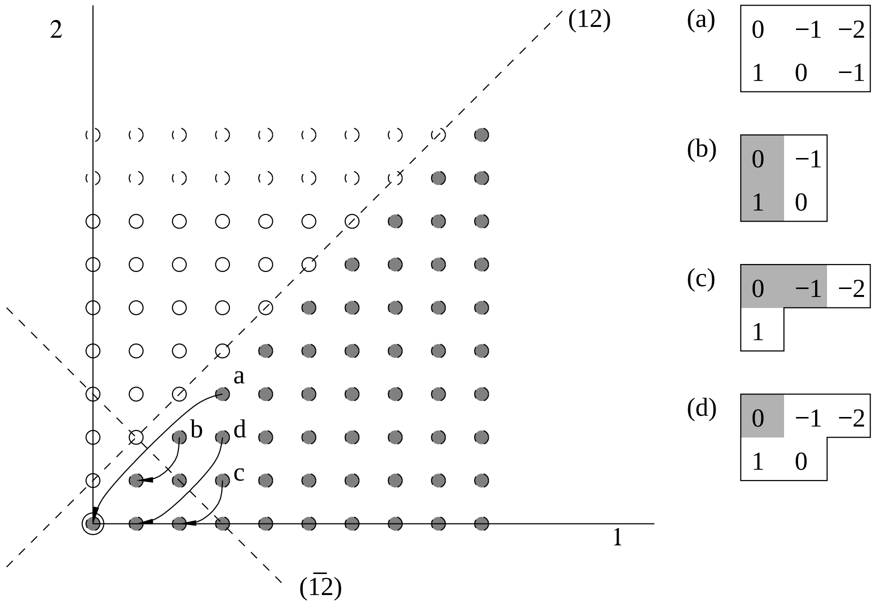

FIGURE 5. *Projections into the* 12 *plane with* $\delta = 2$.

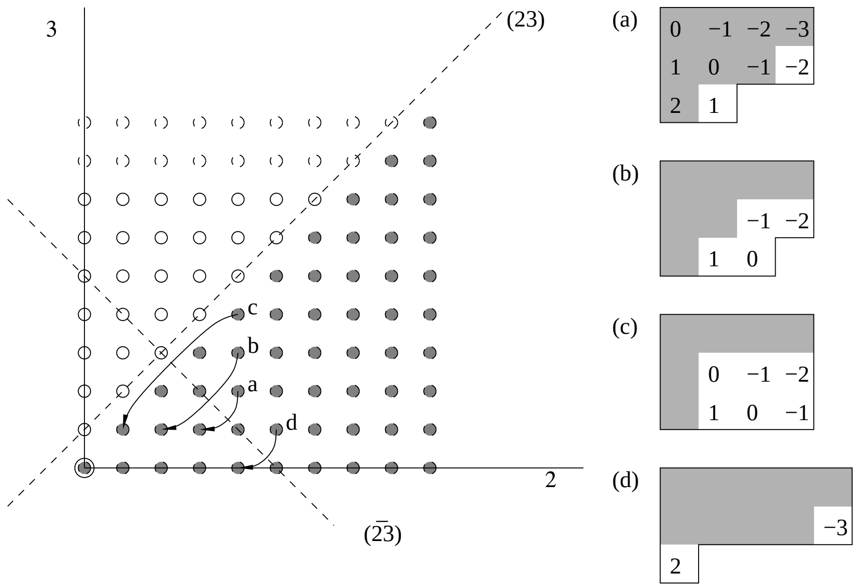

FIGURE 6. *Projections into the* 23 *plane with* $\delta = 2$.

We will identify the simple $B_n(\delta)$-modules with weights in X^+ using the correspondence

$$(\lambda \in X^+) \quad \longleftrightarrow \quad L(\lambda^T)$$

where λ^T denotes the conjugate partition of λ (i.e. the one obtained by reversing the roles of rows and columns in the usual Young diagram). Using this correspondence, we restate the main result of [**CDM05**] as follows. Given two partitions $\mu \subset \lambda$ we write λ/μ for the associated skew partition. We say that a pair of weights $\lambda, \mu \in X^+$ is a δ-*balanced pair* (or just *balanced pair* if δ is understood) if and only if the boxes of $\lambda/(\lambda\cap\mu)$ (respectively $\mu/(\lambda\cap\mu)$) can be paired up such that (i) the contents

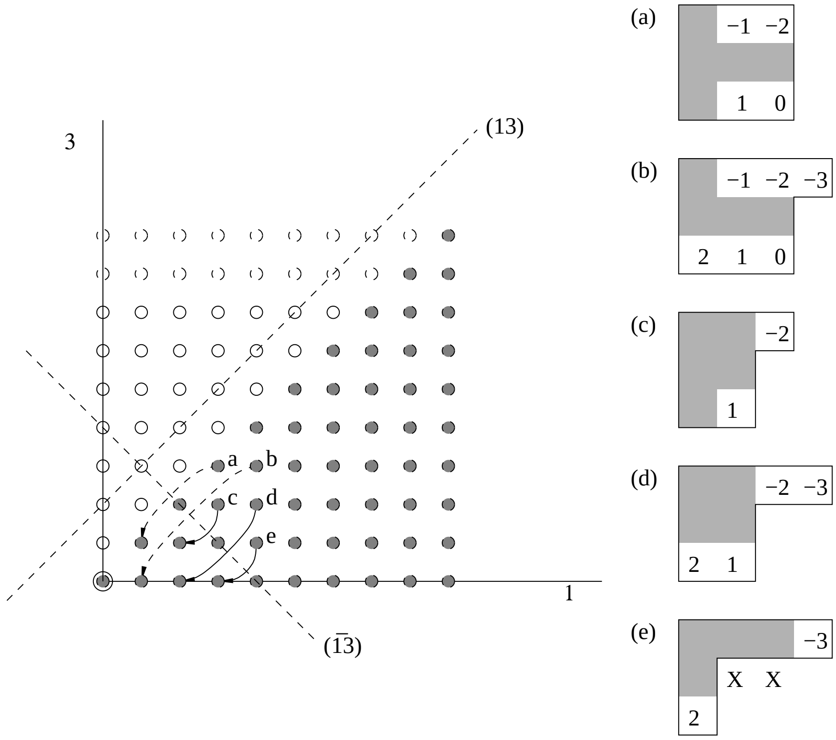

FIGURE 7. *Projections into the* 13 *plane with* $\delta = 2$.

of the boxes in each pair sum to $1-\delta$, and (ii) if δ is even and the boxes with content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ are configured as in Figure 8 then the number of rows in Figure 8 must be even.

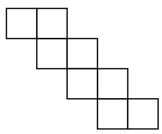

FIGURE 8. *A potentially unbalanced configuration.*

Noting that the definition of content given in Section 1 is the transpose of the one used in [**CDM05**], it is easy to see (simply by transposing everything) that [**CDM05**] Corollary 6.7 becomes

THEOREM 4.1. *Suppose that $k = \mathbb{C}$ and $\delta \in \mathbb{Z}$. Two simple $B_n(\delta)$-modules $L(\lambda^T)$ and $L(\mu^T)$ are in the same block if and only if λ and μ form a balanced pair.*

(Note that Theorem 4.1 includes as a special case the semisimplicity results over $\mathbb{C}$ in [**Rui05**].) We now give the desired geometric formulation of Theorem 4.1.

THEOREM 4.2. *Suppose that $k = \mathbb{C}$ and $\delta \in \mathbb{Z}$. Two simple $B_n(\delta)$-modules $L(\lambda^T)$ and $L(\mu^T)$ are in the same block if and only if $\mu \in W \cdot \lambda$*

Proof. We will show that this description is equivalent to that given in Theorem 4.1, by proceeding in two stages. First we will show, using the action of the generators of W on X, that two partitions in the same orbit form a balanced pair. This implies that the blocks are unions of W-orbits. Next we will show that two partitions which form a balanced pair lie in the same W-orbit.

Stage 1: The case $n=2$ is an easy calculation. For $n>2$, note that

$$s_{\varepsilon_i-\varepsilon_j} = s_{\varepsilon_j+\varepsilon_k} s_{\varepsilon_i+\varepsilon_k} s_{\varepsilon_j+\varepsilon_k}$$

where $i \neq k \neq j$, and so W is generated by reflections of the form $s_{\varepsilon_i+\varepsilon_j}$. Now consider the action of such a generator on a weight in X.

$$s_{\varepsilon_i+\varepsilon_j}.\lambda = \lambda - (\lambda_i+\lambda_j-\delta+2-i-j)(\varepsilon_i+\varepsilon_j).$$

If $\lambda_i+\lambda_j-\delta+2-i-j \geq 0$ then this involves the removal of two rows of boxes with respective contents

$$-\lambda_i+i+\lambda_i+\lambda_j-i-j-\delta+1,\ldots,-\lambda_i+i+1,-\lambda_i+i$$

and

$$-\lambda_j+j+\lambda_i+\lambda_j-i-j-\delta+1,\ldots,-\lambda_j+j+1,-\lambda_j+j$$

which simplify to

$$\lambda_j-j-\delta+1,\ldots,-\lambda_i+i+1,-\lambda_i+i$$

and

$$\lambda_i-i-\delta+1,\ldots,-\lambda_j+j+1,-\lambda_j+j.$$

If we pair these two rows in reverse order, each pair of contents sum to $1-\delta$. Note also that for δ even, the number of horizontal pairs of boxes of content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ is either unchanged or decreased by 2. The argument when $\lambda_i-\lambda_j-\delta+2-i-j<0$ is similar (here we add paired boxes instead of removing them).

Now take two partitions $\lambda,\mu \in X^+$ with $\mu = w\cdot\lambda$ for some $w\in W$. We need to show that they form a balanced pair, i.e. that the boxes of $\lambda/\lambda\cap\mu$ (respectively $\mu/\lambda\cap\mu$) can be paired up in the appropriate way. First observe that the set of contents of boxes in $\lambda/\lambda\cap\mu$ and in $\mu/\lambda\cap\mu$ are disjoint. To see this, suppose that there is a box ε in $\lambda/\lambda\cap\mu$ with the same content as a box η in $\mu/\lambda\cap\mu$. Then these two boxes must lie on the same diagonal. Assume, without loss of generality, that ε appears in an earlier row than η. As η belongs to μ and ε is above and to the left of η, we must have that ε is also in μ (as μ is a partition). But then ε belongs to $\lambda\cap\mu$ which is a contradiction.

Let us concentrate on the action of w on boxes either with a fixed content c say or with the paired content $1-\delta-c$. As w can be written as a product of the generators considered above, it will add and remove pairs of boxes of these content, say

$$(\tau_1+\tau_1')+(\tau_2+\tau_2')+\ldots+(\tau_m+\tau_m')$$

$$-(\sigma_1+\sigma_1')-(\sigma_2+\sigma_2')-\ldots-(\sigma_q+\sigma_q')$$

for some boxes τ_i, τ_i', σ_j and σ_j' with $c(\tau_i)=c=1-\delta-c(\tau_i')$ for $1\leq i\leq m$ and $c(\sigma_j)=c=1-\delta-c(\sigma_j')$ for $1\leq j\leq q$. Thus the number of boxes in $\mu=w\cdot\lambda$ of content c (resp. $1-\delta-c$) minus the number of boxes in λ of content c (resp. $1-\delta-c$) is equal to $m-q$. But this must be equal to the number of boxes in $\mu/(\lambda\cap\mu)$ of content c (resp. $1-\delta-c$) minus the number of boxes in $\lambda/(\lambda\cap\mu)$ of content c (resp. $1-\delta-c$). As we have just observed that the contents of boxes in $\lambda/(\lambda\cap\mu)$ and in $\mu/(\lambda\cap\mu)$ are disjoint, we either have $m-q\geq 0$ and

$$\begin{aligned} m-q &= |\{\text{boxes of content } c \text{ in } \mu/(\lambda\cap\mu)\}| \\ &= |\{\text{boxes of content } 1-\delta-c \text{ in } \mu/(\lambda\cap\mu)\}| \end{aligned}$$

or $m-q<0$ and

$$\begin{aligned} m-q &= -|\{\text{boxes of content } c \text{ in } \lambda/(\lambda\cap\mu)\}| \\ &= -|\{\text{boxes of content } 1-\delta-c \text{ in } \lambda/(\lambda\cap\mu)\}| \end{aligned}$$

Thus the boxes of $\lambda/(\lambda\cap\mu)$ (resp. $\mu/(\lambda\cap\mu)$) can be paired up such that the sum of the contents in each pair is equal to $1-\delta$. Moreover, for δ even, as each generator $s_{\varepsilon_i+\varepsilon_j}$ either adds or removes 2 (or no) horizontal pairs of boxes of contents $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$, we see that λ and μ are indeed a balanced pair.
Stage 2: We need to show that if λ and μ are a balanced pair of partitions then they are in the same W-orbit. Note that if λ and μ are a balanced pair then by definition so are λ and $\lambda\cap\mu$, and μ and $\lambda\cap\mu$. Thus it is enough to show that if $\mu\subset\lambda$ are a balanced pair then they are in the same W-orbit.

We will show that whenever we have a weight $\eta\in X$ with $\eta+\rho\in X^+$ and $\mu\subset\eta$ (i.e. $\mu_i\leq\eta_i$ for all i) such that μ,η form a balanced pair, we can construct $\eta^{(1)}\in W_p\cdot\eta$ such that either $\eta^{(1)}=\mu$ or $\eta^{(1)}\subset\eta$ having the same properties as η. Starting with $\eta=\lambda$ and applying induction will prove that $\mu\in W_p\cdot\lambda$.

Pick a box ε in η/μ such that
(i) it is the last box in a row of η,
(ii) $\frac{1-\delta}{2}-c(\varepsilon)$ is maximal.
If more than one such box exists, pick the southeastern-most one. Say that ε is in row i. Find a box ε' on the edge of η/μ (i.e. a box in η/μ such that there is no box to the northeast, east, or southeast of it in η/μ) with $c(\varepsilon)+c(\varepsilon')=1-\delta$. Say that ε' is in row j.

Note that $i\neq j$ as if i were equal to j then there would either be a box of content $\frac{1-\delta}{2}$ (for δ odd) or a pair of boxes of content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ (for δ even) in between ε and ε'. Now, as η,μ is a balanced pair and $\eta_{i-1}-\eta_i\geq -1$, it must contain another such box or pair of boxes of the same content(s) in row $i-1$, as illustrated in Figure 9 (where the shaded area is part of μ). As $\eta_{i-1}-\eta_i\geq -1$ we see that η/μ contains at least two boxes of content $c(\varepsilon)$. But as ε was chosen with maximal content and μ is a partition, η/μ can only have one box of content $c(\varepsilon')$, as otherwise the box χ would be in η/μ. This contradicts the fact that η,μ is a balanced pair.

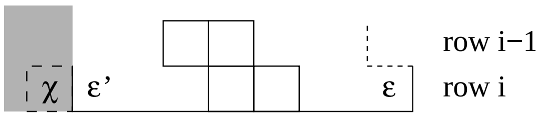

FIGURE 9. *The (impossible) configuration occurring if $i=j$.*

Now let α be the last box in row j and let α' be the southeastern-most box on the edge of η/μ having content $c(\alpha')=1-\delta-c(\alpha)$. Say that α' is in row k.
Case 1: $k=j$.
In this case there must either be a box of content $\frac{1-\delta}{2}$ (for δ odd) or a pair of boxes of content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ (for δ even) in between α' and α. Now, as η,μ is a balanced pair and $\eta_{j-1}-\eta_j\geq -1$, it must contain another such box or pair of boxes of the same content(s) in row $j-1$, as illustrated in Figure 10.

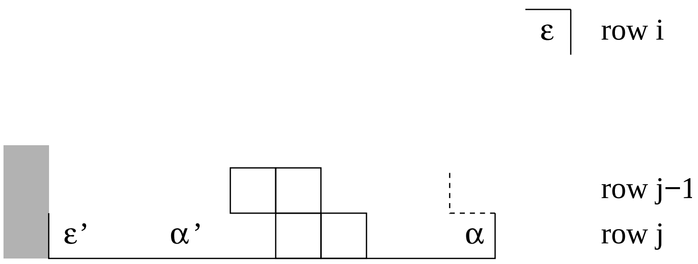

FIGURE 10. *The case $j=k$.*

For each $c(\varepsilon) \leq c < c(\alpha)$, define i_c by saying that the southeastern-most box of content c on the edge of η/μ is in row i_c. For $c = c(\alpha)$, define $i_{c(\alpha)} = j-1$. Note that the i_c's are not necessarily all distinct. Consider all distinct values of i_c and order them

$$i = i_{c_0} < i_{c_1} < \ldots < i_{c_l} = j-1.$$

Now consider

$$\eta^{(1)} = (s_{\varepsilon_i - \varepsilon_{i_{c_1}}} \ldots s_{\varepsilon_i - \varepsilon_{i_{c_{l-1}}}} s_{\varepsilon_i - \varepsilon_{j-1}} s_{\varepsilon_i + \varepsilon_j}) \cdot \eta.$$

This is illustrated schematically in Figure 11, where curved lines indicate boundaries whose precise configuration does not concern us. Then $\eta^{(1)} \subset \eta$ with $\mu, \eta^{(1)}$ a balanced pair and $\eta^{(1)} + \rho \in X^+$ as required.

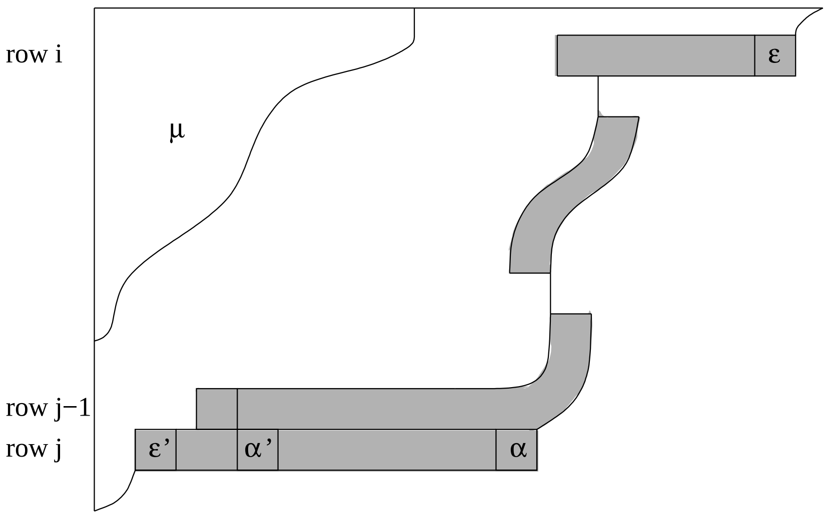

FIGURE 11. *The elements* $\mu \subset \eta^{(1)} \subset \eta$.

Case 2: $k \neq j$.
If $i = k$ then consider

$$\eta^{(1)} = s_{\varepsilon_i + \varepsilon_j} \cdot \eta$$

then $\eta^{(1)} \subset \eta$ with $\mu, \eta^{(1)}$ a balanced pair and $\eta^{(1)} + \rho \in X^+$.

If $k \neq i$, then as in Case 1, for each $c(\varepsilon) \leq c \leq c(\alpha')$ we define i_c by saying that the southeastern most box in η/μ is in row i_c. As before, there are not necessarily all distinct but we can pick a set of representatives

$$i = i_{c_0} < i_{c_1} < \ldots < i_{c_l} = k.$$

Now consider

$$\eta^{(1)} = (s_{\varepsilon_i - \varepsilon_{i_{c_1}}} \ldots s_{\varepsilon_i - \varepsilon_{i_{c_{l-1}}}} s_{\varepsilon_i - \varepsilon_k} s_{\varepsilon_i + \varepsilon_j}) \cdot \eta.$$

Again, this is illustrated schematically in Figure 12, where curved lines indicate boundaries whose precise configuration does not concern us. As before we have $\eta^{(1)} \subset \eta$ with $\mu, \eta^{(1)}$ a balanced pair and $\eta^{(1)} + \rho \in X^+$. $\square$

EXAMPLE 4.3. We illustrate Stage 2 of the proof above by an example. Take $\lambda = (8,8,8,7,3,3,2)$ and $\mu = (6,5,1,1)$ and $\delta = 2$. Then it is easy to see that μ, λ form a balanced pair. We will construct $w \in W_p$ such that $\mu = w \cdot \lambda$.

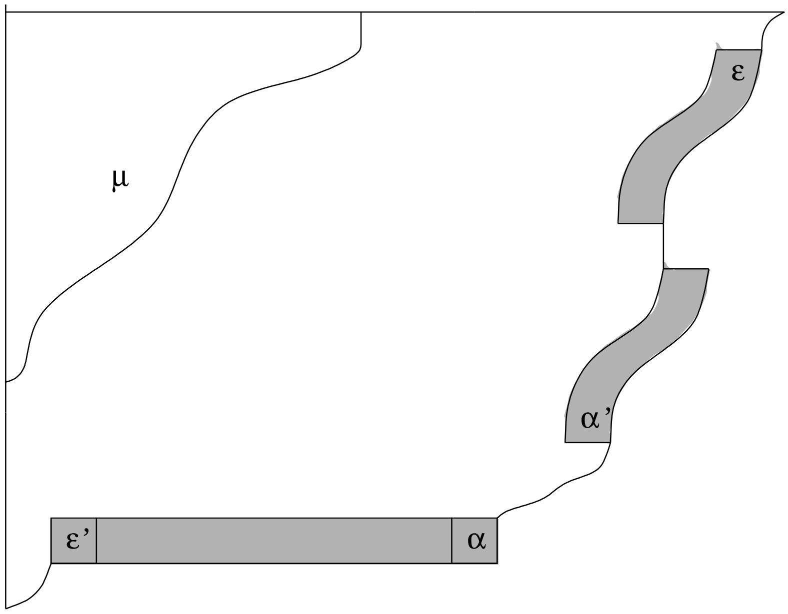

FIGURE 12. *The elements* $\mu \subset \eta^{(1)} \subset \eta$.

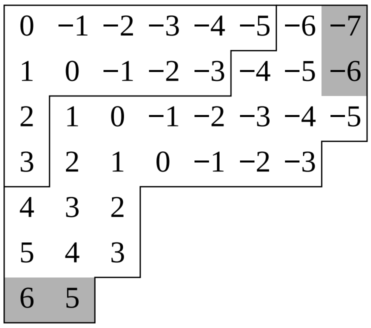

FIGURE 13. *The elements* $\mu \subset \lambda^{(1)} \subset \lambda$.

First consider $\lambda^{(1)} = s_{\varepsilon_1-\varepsilon_2}s_{\varepsilon_1+\varepsilon_7}\cdot\lambda$. The elements λ and μ are illustrated in outline in Figure 13, with the boxes removed to form $\lambda^{(1)}$ shaded.

Repeating the process we next consider $\lambda^{(2)} = s_{\varepsilon_1-\varepsilon_3}s_{\varepsilon_1+\varepsilon_6}\cdot\lambda^{(1)}$, as in Figure 14, followed by $\lambda^{(3)} = s_{\varepsilon_2-\varepsilon_4}s_{\varepsilon_2+\varepsilon_5}\cdot\lambda^{(2)}$ as in Figure 15. Finally consider $\lambda^{(4)} = s_{\varepsilon_3+\varepsilon_4}\cdot\lambda^{(3)}$ as shown in Figure 16.

5. *Orbits of the affine Weyl group of type D*

We would like to have a block result in characteristic $p > 0$ similar in spirit to Theorem 4.2. For this we first need a candidate to play the role of W. To motivate our choice of such, we begin by considering a possible approach to modular representation theory via reduction from characteristic 0.

The verification that the Brauer algebra is cellular is a characteristic-free calculation over $\mathbb{Z}[\delta]$. Thus all of our algebras and cell modules have a $\mathbb{Z}[\delta]$-form, from which the corresponding objects over k can be obtained by specialisation. If the maps between cell modules that have been constructed in characteristic zero in [**DWH99**, **CDM05**] also had a corresponding integral form, then they would also specialise to maps in characteristic p. As these maps were not constructed explicitly, we are unable to

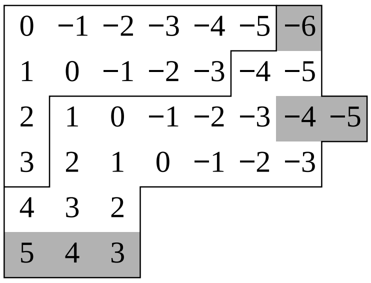

FIGURE 14. *The elements* $\mu \subset \lambda^{(2)} \subset \lambda^{(1)}$.

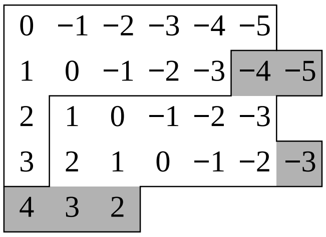

FIGURE 15. *The elements* $\mu \subset \lambda^{(3)} \subset \lambda^{(2)}$.

verify this except in very small examples. However, if we assume for the moment that it holds, this will suggest a candidate for our new reflection group.

We will wish to consider the dot action of W for different values of shift parameter ρ. In such cases we will write $w._{\delta}\lambda$ for the element

$$w(\lambda+\rho(\delta))-\rho(\delta).$$

When we wish to emphasise the choice of dot action we will also write W^{δ} for W.

Fix $\delta \in \mathbb{Z}$, and suppose that maps between cell modules in characteristic 0 do reduce mod p. Then we would expect weights to be in the same block in characteristic p if they are linked by the action of W^{δ} in characteristic zero. However, all elements of the form $\delta + rp$ in characteristic zero reduce to the same element δ mod p, and so weights should be in the same block if they are linked by the action of $W^{\delta+rp}$ for some $r \in \mathbb{Z}$. Thus our candidate for a suitable reflection group will be $\mathbf{W} = \langle W^{\delta+rp} : r \in \mathbb{Z}\rangle$.

Note however that a block result does not follow *automatically* from the integrality assumption, as: (i) the chain of reflections from $\mathbf{W}$ linking two weights might leave the set of weights for $B_n(\delta)$; (ii) in characteristic p there may be new connections between weights not coming from connections in characteristic zero. We shall see that the former is indeed a problem, but that the latter does not occur.

Now fix a prime number $p > 2$ and consider the affine Weyl group W_p associated to W. This is defined to be

$$W_p = \langle s_{\beta,rp} : \beta \in \Phi, r \in \mathbb{Z}\rangle$$

where

$$s_{\beta,rp}(\lambda) = \lambda - ((\lambda,\beta) - rp)\beta.$$

As before, we consider the dot action of W_p on X (or E) given by

$$w.\lambda = w(\lambda+\rho)-\rho.$$

It is an easy exercise to show

0	−1	−2	−3	−4	−5
1	0	−1	−2	−3	
2	1	0	−1	−2	−3
3	2	1	0	−1	−2

FIGURE 16. *The elements* $\mu = \lambda^{(4)} \subset \lambda^{(3)}$.

LEMMA 5.1. *For all* $r \in \mathbb{Z}$ *and* $1 \leq i \neq j \neq k \leq n$*, we have*

$$s_{\varepsilon_i+\varepsilon_j} \cdot_{\delta+rp} \lambda = s_{\varepsilon_i+\varepsilon_j, rp} \cdot_\delta \lambda.$$
$$s_{\varepsilon_i-\varepsilon_j} \cdot_{\delta+rp} \lambda = s_{\varepsilon_i-\varepsilon_j} \cdot_\delta \lambda.$$
$$s_{\varepsilon_i-\varepsilon_j, rp} = s_{\varepsilon_j+\varepsilon_k} s_{\varepsilon_i+\varepsilon_k, rp} s_{\varepsilon_j+\varepsilon_k}.$$

and

$$s_{\varepsilon_i+\varepsilon_j, rp} s_{\varepsilon_i+\varepsilon_j} \text{ is translation by } rp(\varepsilon_i+\varepsilon_j).$$

In particular, for $n > 2$ *we have*

$$W_p = \langle s_{\varepsilon_i+\varepsilon_j, rp} : 1 \leq i < j \leq n \text{ and } r \in \mathbb{Z} \rangle.$$

It follows from the first two parts of the Lemma that the group

$$W^{[r]} = \langle s_{\varepsilon_i+\varepsilon_j, rp}, s_{\varepsilon_i-\varepsilon_j} : 1 \leq i < j \leq n \rangle$$

is isomorphic to the original group $W^{\delta+rp}$, and its δ-dot action on X is the same as the $(\delta+rp)$-dot action of $W^{\delta+rp}$ on X. Further, the usual dot action of W_p on X is generated by all the $W^{[r]}$ with $r \in \mathbb{Z}$. Thus we have

COROLLARY 5.2. *For* $p > 2$ *we have* $\mathbf{W} \cong W_p$*, and the isomorphism is compatible with their respective dot actions on* X*.*

The above considerations suggest that the affine Weyl group is a potential candidate for the reflection group needed for a positive characteristic block result. It will be convenient to have a combinatorial description of the orbits of this group on X.

PROPOSITION 5.3. *Suppose that* λ *and* μ *are in* X *with* $|\lambda| - |\mu|$ *even. Then* $\mu \in W_p \cdot \lambda$ *if and only if there exists* $\pi \in \Sigma_n$ *and* $\sigma : \mathbf{n} \to \{\pm 1\}$ *with* $d(\sigma)$ *even such that for all* $1 \leq i \leq n$ *we have either*

$$\sigma(i) = 1 \quad \text{and} \quad c(\mu)_i \equiv c(\lambda)_{\pi(i)} \mod p$$

or

$$\sigma(i) = -1 \quad \text{and} \quad c(\mu)_i + c(\lambda)_{\pi(i)} \equiv 2 - \delta \mod p$$

Proof. We have $\mu \in W_p \cdot \lambda$ if and only if

$$\mu + \rho = w(\lambda + \rho) + p\nu$$

for some $w \in W$ and $\nu \in \mathbb{Z}\Phi$. Note that for any $\nu \in X$ we have $\nu \in \mathbb{Z}\Phi$ if and only if $|\nu| = \sum \nu_i$ is even, as

$$2\varepsilon_i = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_i + \varepsilon_{i+1})$$

and

$$2\varepsilon_{i+1} = (\varepsilon_i + \varepsilon_{i+1}) - (\varepsilon_i - \varepsilon_{i+1})$$

for all $1 \leq i \leq n-1$. Thus, if $|\lambda| - |\mu|$ is even, then $\mu \in W_p \cdot \lambda$ if and only if $\mu + \rho = w(\lambda + \rho) + p\nu$ for some $w \in W$ and some $\nu \in X$. Combining this with Proposition 3.1 gives the result. □

As in the non-affine case, we may represent reflections graphically via projection into the plane. In this case each projection will contain two families of reflections; those parallel to $s_{\varepsilon_i - \varepsilon_j}$ and those parallel to $s_{\varepsilon_i + \varepsilon_j}$. This is illustrated for $p = 5$ in Figure 17. An example of the effect of various such reflections on partitions will be given in Figure 23, after we have introduced a third, abacus, notation.

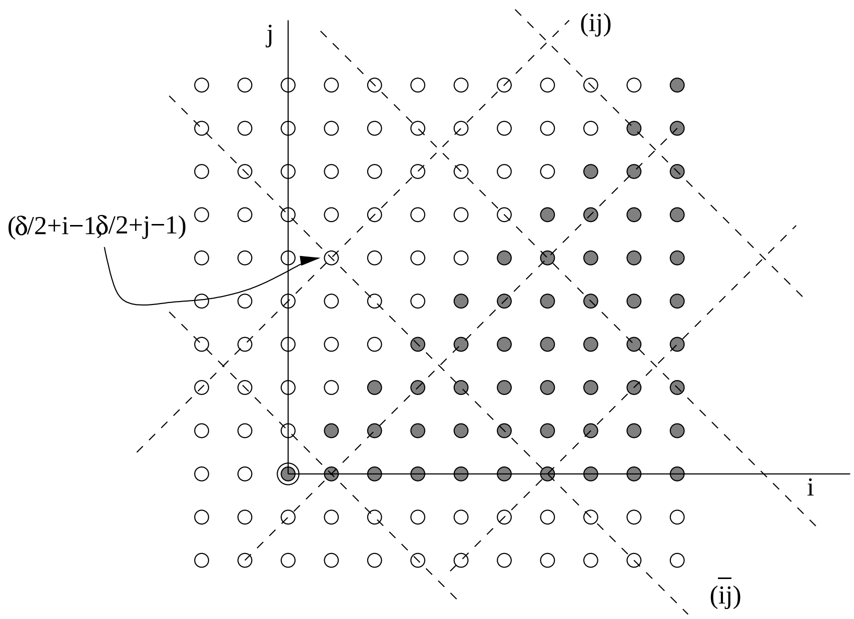

FIGURE 17. *A projection onto the ij plane with $p = 5$.*

6. *On the blocks of the Brauer algebra in characteristic p*

We have already seen that the blocks of the Brauer algebra in characteristic 0 are given by the restriction of orbits of W to the set of partitions. We would like a corresponding result in characteristic $p > 0$ involving the orbits of W_p. One does not expect the blocks of the Brauer algebra to be given by W_p in *exactly* the same manner as in characteristic 0. Instead, we can ask if the orbits of W_p are unions of blocks. We will show that this is the case, and give examples in Section 7 to show that indeed these orbits are not in general single blocks. (A similar result for the symplectic Schur algebra has been given by the second author [**DeV08**].) Throughout the next two sections we will assume that we are working over a field of characteristic $p > 0$.

We will need a positive characteristic analogue of [**CDM05**, Proposition 4.2]. Denote by $[\lambda]$ the set of boxes in λ, and recall that we denote the cell/standard modules by $\Delta_n(\lambda)$. In the next Proposition note that δ can be an arbitrary element of k. To make sense of formulas involving contents $c(d)$ and δ we will regard $c(d)$ as an element of $\mathbb{Z}_p \subset k$ and work mod p.

PROPOSITION 6.1. *Let $\lambda, \mu \in X^+$ with $|\lambda| - |\mu| = 2t \geq 0$. If there exists $M \leq \Delta_n(\mu^T)$ with*

$$\mathrm{Hom}_{B_n(\delta)}(\Delta_n(\lambda^T), \Delta_n(\mu^T)/M) \neq 0$$

then

$$t(\delta-1)+\sum_{d\in[\lambda]}c(d)-\sum_{d\in[\mu]}c(d)\equiv 0 \mod p.$$

Proof. The main steps in the proof of this result in characteristic 0 in [**CDM05**, Proposition 4.2] are (i) to give an explicit description of the action of a certain element in the Brauer algebra, and (ii) to use a symmetric group result to show that this action is as a scalar (and to determine its value). The proof of step (i) in [**DWH99**, Lemma 3.2] does not depend on the field, and hence also holds in characteristic p.

For step (ii), we may assume (by exactness of localisation) that λ is a partition of n. Hence as a module for Σ_n we have $\Delta_n(\lambda^T)\cong S^{\lambda^T}$, the Specht module labelled by λ^T. In characteristic 0 this is irreducible, so any central element in $\mathbb{C}\Sigma_n$ acts by a scalar. In particular a certain central element $x\in\mathbb{Z}\Sigma_n\subset\mathbb{C}\Sigma_n$, as described in the proof of [**CDM05**, Proposition 4.2], must act by a scalar s.

However, for our fixed choice of $\delta\in\mathbb{Z}$ the basis for $\Delta_n(\lambda)$ given in [**CDM05**, Lemma 2.4] is defined over $\mathbb{Z}$ (assuming we start with an integral basis for the Specht module), and is a $\mathbb{Z}$-form for the module. As x and s are also defined over $\mathbb{Z}$ the calculation over $\mathbb{C}$ restricts to this $\mathbb{Z}$-form. Tensoring over $\mathbb{Z}$ with our field k we deduce that the element x acts by the same scalar (mod p) in our case. This is what is required to complete the proof in positive characteristic. □

By the last result, if δ is not in the prime subfield $\mathbb{Z}_p\subset k$ then the only composition factors of a cell module $\Delta_n(\lambda^T)$ that can occur are those labelled by weights μ^T with $|\lambda|=|\mu|$. Thus to determine the blocks it is enough to consider homomorphisms between cell modules labelled by partitions of the same degree. As localisation is an exact functor we may assume that λ and μ are both partitions of n, in which case both cell modules are the lifts of Specht modules. Thus we have

THEOREM 6.2. *Suppose that $\delta\notin\mathbb{Z}_p$. Then two simple $B_n(\delta)$-modules $L(\lambda^T)$ and $L(\mu^T)$ are in the same block if and only if $|\lambda|=|\mu|$ and the corresponding simple $k\Sigma_{|\lambda|}$-modules are in the same block.*

Thus we can restrict our attention to the case where $\delta\in\mathbb{Z}_p$. We wish to replace the role played by the combinatorics of partitions by the action of our affine reflection group W_p.

THEOREM 6.3. *Suppose that $\delta\in\mathbb{Z}_p$ and $\lambda,\mu\in X^+$. If there exists $M\leq\Delta_n(\mu^T)$ with*

$$\mathrm{Hom}_{B_n(\delta)}(\Delta_n(\lambda^T),\Delta_n(\mu^T)/M)\neq 0$$

then $\mu\in W_p.\lambda$.

Proof. First note that $\mathrm{Hom}_{B_n(\delta)}(\Delta_n(\lambda^T),\Delta_n(\mu^T)/M)\neq 0$ implies that $|\lambda|-|\mu|=2t\geq 0$. As if we had $|\lambda|<|\mu|$ then using the fact that the localisation function F is exact, we can assume that $\mu\vdash n$, so $\Delta_n(\mu^T)\cong S^{\mu^T}$. However, this module only contains composition factors of the form $L_n(\eta)$ where $\eta\vdash n$, which gives a contradiction.

We now use induction on n. If $n=1$ then $\lambda=\mu=(1)$ and so there is nothing to prove. Assume $n>1$. If $\lambda=\emptyset$ then by the above remark we have $\mu=\emptyset$ and we are done. Now suppose that λ has a removable box in row i say. Then we have

$$\mathrm{Ind}\,\Delta_{n-1}((\lambda-\varepsilon_i)^T)\twoheadrightarrow\Delta_n(\lambda^T)$$

and so, using our assumption we have

$$\mathrm{Hom}_{B_n(\delta)}(\mathrm{Ind}\,\Delta_{n-1}((\lambda-\varepsilon_i)^T),\Delta_n(\mu^T)/M)$$

$$=\mathrm{Hom}_{B_{n-1}(\delta)}(\Delta_{n-1}((\lambda-\varepsilon_i)^T),\mathrm{Res}\,(\Delta_n(\mu^T)/M))\neq 0.$$

Thus either (Case 1) we have

$$\mathrm{Hom}_{B_{n-1}(\delta)}(\Delta_{n-1}((\lambda-\varepsilon_i)^T),\Delta_{n-1}((\mu-\varepsilon_j)^T)/N)\neq 0$$

for some positive integer j with $\mu-\varepsilon_j\in X^+$ and some $N\leq\Delta_{n-1}((\mu-\varepsilon_j)^T)$,
or (Case 2) we have

$$\mathrm{Hom}_{B_{n-1}(\delta)}(\Delta_{n-1}((\lambda-\varepsilon_i)^T),\Delta_{n-1}((\mu+\varepsilon_j)^T)/N)\neq 0$$

for some positive integer j with $\mu+\varepsilon_j\in X^+$ and some $N\leq\Delta_{n-1}((\mu+\varepsilon_j)^T)$.
Case 1: Using Proposition 6.1 for λ and μ and for $\lambda-\varepsilon_i$ and $\mu-\varepsilon_j$, we see that $c(\lambda)_i\equiv c(\mu)_j$ mod p. Now, using induction on n we have that $\mu-\varepsilon_j\in W_p\cdot(\lambda-\varepsilon_i)$. By Proposition 5.3, we can find $\pi\in\Sigma_n$ and $\sigma:\mathbf{n}\to\{\pm1\}$ such that $d(\sigma)$ is even and for all $1\leq m\leq n$, if $\sigma(m)=1$ we have

$$c(\mu-\varepsilon_j)_m\equiv c(\lambda-\varepsilon_i)_{\pi(m)} \bmod p$$

and if $\sigma(m)=-1$ we have

$$c(\mu-\varepsilon_j)_m+c(\lambda-\varepsilon_i)_{\pi(m)}\equiv 2-\delta \bmod p.$$

We will now construct $\pi'\in\Sigma_n$ and $\sigma':\mathbf{n}\to\{\pm1\}$ to show that $\mu\in W_p\cdot\lambda$. Suppose $\pi(j)=k$ and $\pi(l)=i$ for some $k,l\geq 1$. Define π' by $\pi'(j)=i$, $\pi'(l)=k$ and $\pi'(m)=\pi(m)$ for all $m\neq j,l$. Now if $\sigma(j)=\sigma(l)$ then define σ' by $\sigma'(j)=\sigma'(l)=1$ and $\sigma'(m)=\sigma(m)$ for all $m\neq j,l$. And if $\sigma(j)=-\sigma(l)$ the define σ' by $\sigma'(j)=1$, $\sigma'(l)=-1$ and $\sigma'(m)=\sigma(m)$ for all $m\neq j,l$. Now it's easy to check, using the fact that $c(\mu)_j\equiv c(\lambda)_i$ mod p, that π' and σ' satisfy the conditions in Proposition 5.3 for λ and μ, and so $\mu\in W_p\cdot\lambda$.
Case 2: This case is similar to Case 1. Using Proposition 6.1 we see that $c(\lambda)_i+c(\mu)_j\equiv 2-\delta$ mod p. Now using induction on n we have $\pi\in\Sigma_n$ and $\sigma:\mathbf{n}\to\{\pm1\}$ satisfying the conditions in Proposition 5.3 for $\lambda-\varepsilon_i$ and $\mu+\varepsilon_j$. Suppose $\pi(j)=k$ and $\pi(l)=i$ for some $k,l\geq 1$. Define π' by $\pi'(j)=i$, $\pi'(l)=k$ and $\pi'(m)=\pi(m)$ for all $m\neq j,l$. Now if $\sigma(j)=\sigma(l)$ the define σ' by $\sigma'(j)=-1$, $\sigma'(l)=-1$ and $\sigma'(m)=\sigma(m)$ for all $m\neq j,l$. And if $\sigma(j)=-\sigma(l)$ then we define $\sigma'(j)=-1$, $\sigma'(l)=1$ and $\sigma'(m)=\sigma(m)$ for all $m\neq j,l$. □

Note that by the cellularity of $B_n(\delta)$ this immediately implies

THEOREM 6.4. *Two simple $B_n(\delta)$-modules $L(\lambda^T)$ and $L(\mu^T)$ are in the same block only if $\mu\in W_p.\lambda$.*

Thus we have the desired necessary condition in terms of the affine Weyl group for two weights to lie in the same block.

7. *Abacus notation and orbits of the affine Weyl group*

In this section we will show that, even if n is arbitrarily large, being in the same orbit under the affine Weyl group is not sufficient to ensure that two weights lie in the same block. This is most conveniently demonstrated using the abacus notation [**JK81**], and so we first explain how this can be applied in the Brauer algebra case. We begin by recalling the standard procedure for constructing an abacus from a partition, and then show how this is compatible with the earlier orbit results for W_p. As in the preceding section, we assume that our algebra is defined over some field of characteristic $p>2$, and that $\delta\in\mathbb{Z}$.

To each partition we shall associate a certain configuration of beads on an abacus in the following manner. An *abacus with p runners* will consist of p columns (called runners) together with some number of beads distributed amongst these runners. Such beads will lie at a fixed height on the abacus, and there may be spaces between beads on the same runner. We will number the possible bead positions from left to right in each row, starting from the top row and working down, as illustrated in Figure 18.

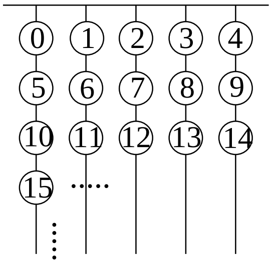

FIGURE 18. *The possible bead positions with* $p = 5$.

For a fixed value of n, we will associate to each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of m, with $m \leq n$ and $n - m$ even, a configuration of beads on the abacus. Let b be a positive integer such that $b \geq n$. We then represent λ on the abacus using b beads by placing a bead in position numbered

$$\lambda_i + b - i$$

for each $1 \leq i \leq b$, where we take $\lambda_i = 0$ for $i > t$. In representing such a configuration we will denote the beads for $i \leq n$ by black circles, for $n < i \leq b$ by grey beads, and the spaces by white circles (or blanks if this is unambiguous). Runners will be numbered left to right from 0 to $p - 1$. For example, the abacus corresponding to the partition $(5,3,3,2,1,1,0^{10})$ when $p = 5$, $n = 16$, and $b = 20$ is given in Figure 19. Note that the abacus uniquely determines the partition λ.

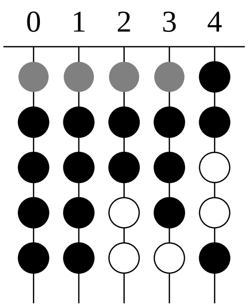

FIGURE 19. *The abacus for* $(5,3,3,2,1,1,0^{10})$ *when* $p = 5$, $n = 16$, *and* $b = 20$.

We would like a way of identifying whether two partitions λ and μ are in the same W_p orbit directly from their abacus representation. First let us rephrase the content condition which we had earlier.

Recall from Proposition 5.3 and the definition of $c(\lambda)$ that λ and μ are in the same W_p-orbit if and only if there exists $\pi \in \Sigma_n$ such that for each $1 \leq i \leq n$ either

$$\mu_i - i \equiv \lambda_{\pi(i)} - \pi(i) \mod p$$

or

$$\mu_i - i \equiv \delta - 2 - (\lambda_{\pi(i)} - \pi(i)) \mod p$$

and the second case occurs an even number of times.

Choose $b \in \mathbb{N}$ with $2b \equiv 2 - \delta \mod p$ (such a b always exists as $p > 2$). Then λ and μ are in the same W_p-orbit if and only if there exists $\pi \in \Sigma_n$ such that for each $1 \leq i \leq n$ either

$$\mu_i + b - i \equiv \lambda_{\pi(i)} + b - \pi(i) \mod p \tag{7.1}$$

or

$$\mu_i + b - i \equiv p - (\lambda_{\pi(i)} + b - \pi(i)) \mod p \tag{7.2}$$

and the second case occurs an even number of times. Thus if we also choose b large enough such that λ and μ can be represented on an abacus with b beads then (7.1) says that the bead corresponding to μ_i is on the same runner as the bead corresponding to $\lambda_{\pi(i)}$, and (7.2) says that the bead corresponding to μ_i is on runner l only if the bead corresponding to $\lambda_{\pi(i)}$ is on runner $p-l$. Note that for corresponding black beads on runner 0 both (7.1) and (7.2) hold, and so we can use this pair of beads to modify $d(\sigma)$ to ensure that it is even. Obviously if there are no such black beads then the number of beads changing runners between λ and μ must be even. Further, the grey beads (for $i > n$) are the same on each abacus. Summarising, we have

PROPOSITION 7.1. *Choose $b \geq n$ with $2b \equiv 2-\delta \mod p$, and λ and μ in Λ_n. Then λ and μ are in the same W_p-orbit if and only if*

(i) the number of beads on runner 0 is the same for λ and μ, and

(ii) for each $1 \leq l \leq p-1$, the total *number of beads on runners l and $p-l$ is the same for λ and μ, and*

(iii) if there are no black beads on runner 0, then the number of beads changing runners between λ and μ must be even.

Note that condition (iii) plays no role when n is large (compared to p) as in such cases every partition will have a black bead on runner 0.

To illustrate this result, consider the case $n = 16$ and the partitions

$$\lambda = (5,3,3,2,1,1), \quad \mu = (2,2,2,1,1,1), \quad \eta = (5,3,3,2,1,1,1). \tag{7.3}$$

Take $p = 5$ and $\delta = 2$, then $b = 20$ satisfies $2b \equiv 2-\delta \mod p$, and is large enough for all three partitions to be represented using b beads. The respective abacuses are illustrated in Figure 20, with the matching rows for condition (ii) in Proposition 7.1 indicated.

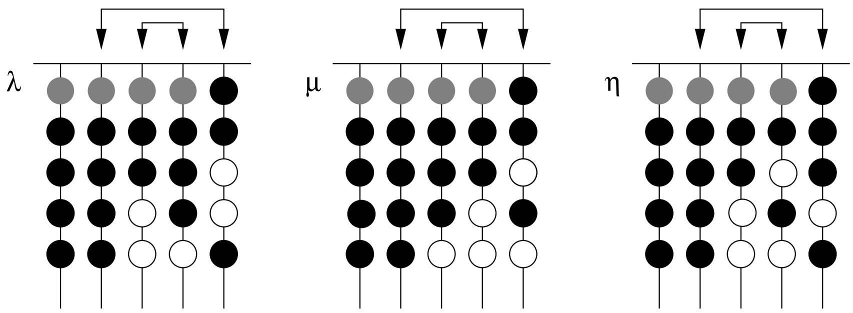

FIGURE 20. *Abacuses representing the elements λ, μ and η in (7.3) with $b = 20$.*

We see that $\mu \in W_p.\lambda$, as the number of beads on runner 0, and on runners 1/4 and 2/3 are the same for both λ and μ (respectively 5, 8, and 7) and there is a black bead on runner 0. (The number of beads moving from runner l to a distinct runner $p-l$ is 1, which is odd. However, as discussed above, we can chose σ such that one of the two black beads on runner 0 is regarded as moving (to the same runner), to obtain the required even number of such moves. If there were no black beads on runner 0 then this would not be possible.) However, $\eta \notin W_p.\lambda$ as columns 1/4 and 2/3 have 9 and 6 entries respectively.

Having reinterpreted the orbit condition in terms of the abacus, we will now show that the orbits of W_p can be *non-trivial* unions of blocks for $B_n(\delta)$.

THEOREM 7.2. *Suppose that k is of characteristic $p > 2$. Then for arbitrarily large n there exist $\lambda \vdash n$ and $\mu \vdash n-2$ (corresponding to the partial abacuses in Figure 21 or Figure 22) which are in the same W_p-orbit but not in the same $B_n(\delta)$-block.*

Proof. Let $b \in \mathbb{N}$ be such that

$$2b \equiv 2 - \delta \mod p.$$

If b is even (respectively odd), consider the partial abacuses illustrated in Figure 21 (respectively Figure 22).

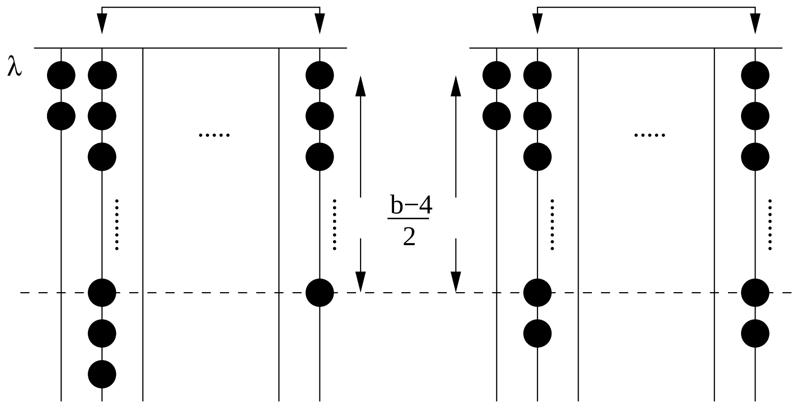

FIGURE 21. *The partial abacuses for λ and μ when b is even.*

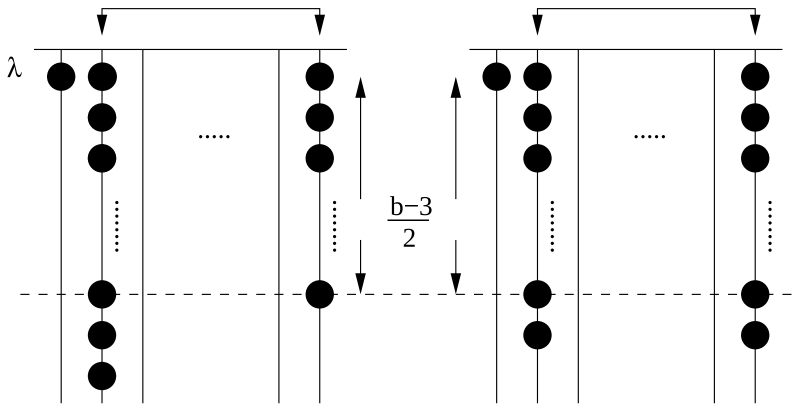

FIGURE 22. *The partial abacuses for λ and μ when b is odd.*

These will not correspond directly to partitions λ and μ, as the degree of each partition will be much larger than b. However, by completing each in the same way (by adding the same number of black beads in rows from right to left above each partition, followed by a suitable number of grey beads), they can be adapted to form abacuses of partitions $\lambda \vdash n$ and $\mu \vdash n-2$ for some $n >> 0$ and for some $b' \equiv b \mod p$. (This corresponds to adding sufficiently many zeros to the end of each partition such that each has $|\lambda|$ parts.)

It is clear from Proposition 7.1 that in each case λ and μ are in the same W_p-orbit. Note that for both λ and μ, all beads are as high as they can be on their given runner. If we move any bead to a higher numbered position then this corresponds to increasing the degree of the associated partition. Thus λ and μ are the only partitions with degree at most $|\lambda|$ in their W_p-orbit.Also it is easy to check that μ is obtained from λ by removing two boxes from the same row. Clearly by increasing b we can make n arbitrarily large.

To complete the proof, it is enough to show that $L_n(\lambda^T)$ and $L_n(\mu^T)$ are not in the same $B_n(\delta)$-block. We will reduce this to a calculation for the symmetric group, and use the corresponding (known) block result in that case. To state this we need to recall the notion of a p-core.

A partition is a *p-core* if the associated abacus has no gap between any pair of beads on the same runner. We associate a unique p-core to a given partition τ by sliding all beads in some abacus representation of τ as far up each runner as they can go, and taking the corresponding partition. By the Nakayama conjecture (see [**MT76**] for a survey of its various proofs), two partitions τ and η are in the same block for $k\Sigma_n$ if and only if they have the same p-core. It is also easy to show (using the definition of p-cores involving the removal of p-hooks [**Mat99**]) that if τ is a p-core then so is τ^T.

Returning to our proof, as $\lambda \vdash n$ we have that the cell module $\Delta_n(\lambda^T)$ is isomorphic to the Specht module S^{λ^T} as a $k\Sigma_n$-module (by [**DWH99**, Section 2]). As λ is a p-core so is λ^T, and hence $\Delta_n(\lambda^T)$ is in a $k\Sigma_n$-block on its own (by the Nakayama conjecture) so is simple as a $k\Sigma_n$-module, isomorphic to D^{λ^T}, and hence equal to $L_n(\lambda^T)$ as a $B_n(\delta)$-module.

If
$$[\Delta_n(\mu^T) : L_n(\lambda^T)] \neq 0$$
then we must have
$$[\mathrm{res}_{k\Sigma_n} \Delta_n(\mu^T) : D^{\lambda^T}] \neq 0.$$
However, $\mathrm{res}_{k\Sigma_n} \Delta_n(\mu^T)$ has a Specht filtration where the multiplicity of S^{η^T} in this filtration is given by the Littlewood-Richardson coefficient $c^{\eta^T}_{\mu^T(2)}$. (This is proved over $\mathbb{C}$ in [**HW90**, Theorem 4.1] and in arbitrary characteristic in [**Pag07**, Proposition 8].) In particular, as λ^T is obtained from μ^T by adding two boxes in the same column we see that $S^{\lambda^T} = D^{\lambda^T}$ does not appear as a Specht subquotient in this filtration [**DWH99**, remarks after Theorem 3.1]. However, we still have to prove that it cannot appear as a composition factor of some other S^{η^T}. But this is clear, as if it did then η^T would have to have the same p-core as λ, but λ is already a p-core and hence this is impossible.

This proves that $\Delta_n(\mu^T) = L_n(\mu^T)$, and so λ and μ are in different blocks for $B_n(\delta)$. ☐

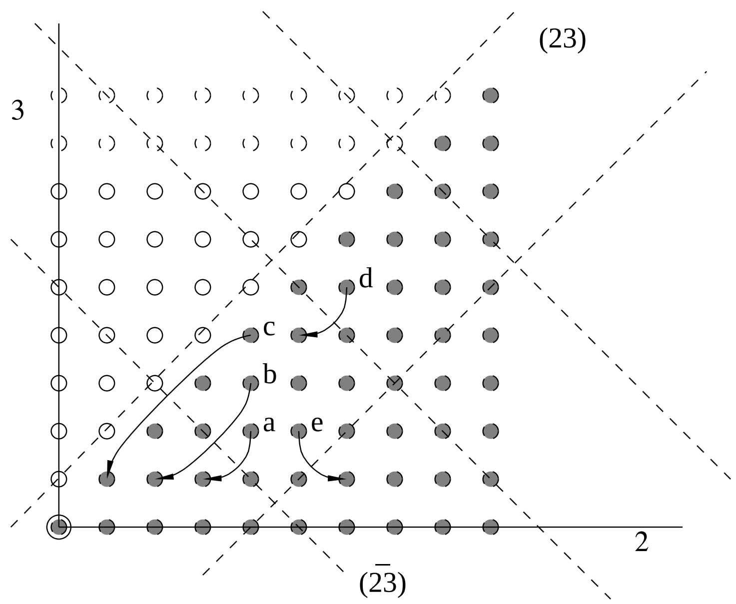

FIGURE 23. *Assorted examples with $p = 5$ and $\delta = 2$.*

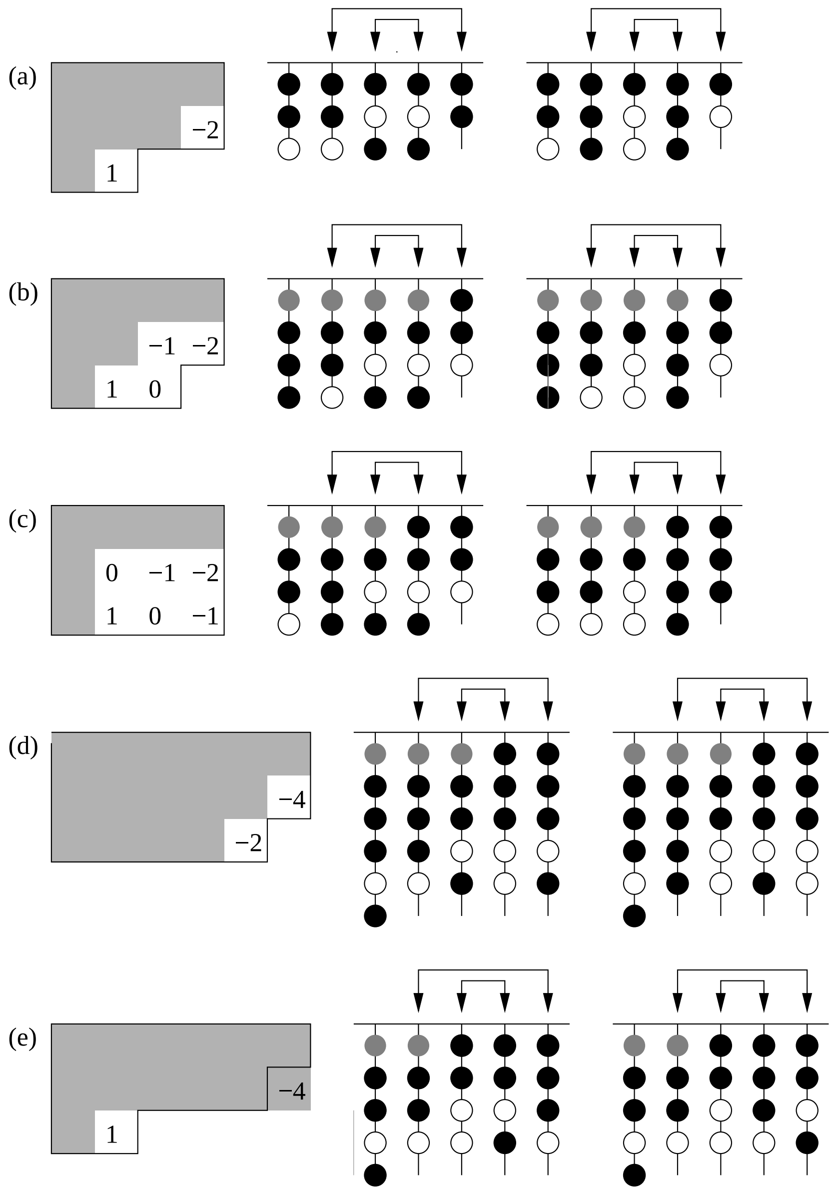

FIGURE 24. *Assorted examples with $p = 5$ and $\delta = 2$.*

REMARK 7.3. Theorems 6.4 and 7.2 imply that the orbits of the affine Weyl group provide a necessary, but not sufficient, condition for two weights to be in the same block as $B_n(\delta)$. In the Lie theoretic context, Theorem 6.4 corresponds to the linkage principle [**Jan03**, II, 6.17]. For practical purposes this is the key condition that we need.

To conclude, we illustrate some examples of various affine reflections together with the corresponding partitions and abacuses, when $p = 5$ and $\delta = 2$. Our condition on b implies that it must be chosen to be a multiple of 5. Reflections are labelled (a)–(e) in Figure 23, with the corresponding partitions and abacuses in Figure 24. Case (a) corresponds to the reflection from $(4,4,2)$ to $(4,3,1)$, with $n = b = 10$. Case (b) corresponds to the reflection from $(4,4,3)$ to $(4,2,1)$, with $n = 11$ and $b = 15$. Case (c) corresponds to the reflection from $(4,4,4)$ to $(4,1,1)$ with $n = 12$ and $b = 15$. These three cases only use elements from W, and so would be reflections in any characteristic. Hence the condition on matched contents in these cases are equalities, not merely equivalences mod p. Case (d) corresponds to the reflection from $(6,6,5)$ to $(6,5,4)$ with $n = 17$ and $b = 20$. This is a strictly affine phenomenon, and so the paired boxes only sum to $1-\delta$ mod p. Finally, case (e) corresponds to the reflection from $(6,5,2)$ to $(6,6,1)$ with $n = 13$ and $b = 15$. This is our only example of reflection about an affine (ij) line, and so is the only case illustrated where the number of boxes is left unchanged.

References

Bou68 N. Bourbaki, *Groupes et algèbres de Lie (Chapitres 4–6)*, Hermann, 1968.

Bra37 R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.

Bro55 W. Brown, *An algebra related to the orthogonal group*, Michigan Math. J. **3** (1955), 1–22.

CDM05 A. G. Cox, M. De Visscher, and P. P. Martin, *The blocks of the Brauer algebra in characteristic zero*, preprint, (2005) `http://xxx.lanl.gov/abs/math.RT/0601387`.

CMPX06 A. G. Cox, P. P. Martin, A. E. Parker, and C. Xi, *Representation theory of towers of recollement: theory, notes, and examples*, J. Algebra **302** (2006), 340–360.

DDH08 R. Dipper, S. Doty, and J. Hu, *Brauer's centralizer algebras, symplectic Schur algebras and Schur-Weyl duality*, Trans. AMS **360** (2008), 189–213.

DeV08 M. De Visscher, *On the blocks for semisimple algebraic groups and associated generalized Schur algebras*, J. Algebra **319** (2008), 952–965.

Don87 S. Donkin, *Good filtrations of rational modules for reductive groups*, Arcata Conference on Representations of Finite Groups, Proceedings of Symp. in Pure Math., vol. 47, 1987, pp. 69–80.

Dot98 S. Doty, *Polynomial representations, algebraic monoids, and Schur algebras of classical type*, J. Pure Appl. Algebra **123** (1998), 165–199.

DWH99 W. F. Doran, D. B. Wales, and P. J. Hanlon, *On the semisimplicity of the Brauer centralizer algebras*, J. Algebra **211** (1999), 647–685.

GL96 J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

Gre80 J. A. Green, *Polynomial representations of GL_n*, Lecture Notes in Mathematics 830, Springer, 1980.

HP06 R. Hartmann and R. Paget, *Young modules and filtration multiplicities for Brauer algebras*, Math. Z. **254** (2006), 333–357.

HW90 P. J. Hanlon and D. B. Wales, *Computing the discriminants of Brauer's centralizer algebras*, Math. Comp. **54** (1990), 771–796.

Jan03 J. C. Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, AMS, 2003.

JK81 G. D. James and A. Kerber, *The representation theory of the Symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, 1981.

Mat99 A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric groups*, University lecture series, vol. 15, American Mathematical Society, 1999.

MT76 N. Meier and J. Tappe, *Ein neuer Beweis der Nakayama-Vermutung über die Blockstruktur Symmetrischen Gruppen*, Bull. London Math. Soc. **8** (1976), 34–37.

MW98 P. P. Martin and D. Woodcock, *The Partition algebra and a new deformation of the Schur algebras*, J. Algebra **203** (1998), 91–124.

Oeh01 S. Oehms, *Centralizer coalgebras, FRT-construction and symplectic monoids*, J. Algebra **244** (2001), 19–44.

Pag07 R. Paget, *A family of modules with Specht and dual Specht filtrations*, J. Algebra **312** (2007), 880–890.

Rui05 H. Rui, *A criterion on the semisimple Brauer algebras*, J. Comb. Theory Ser. A **111** (2005), 78–88.

Wen88 H. Wenzl, *On the structure of Brauer's centralizer algebra*, Ann. Math. **128** (1988), 173–193.

Anton Cox and Maud De Visscher,
Centre for Mathematical Science,
City University,
Northampton Square,
London,
EC1V 0HB,
England.

A.G.Cox@city.ac.uk
M.Devisscher@city.ac.uk

Paul Martin,
Department of Pure Mathematics,
University of Leeds,
Leeds,
LS29JT.

ppmartin@maths.leeds.ac.uk