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A Geometric Characterization of Homogeneous Production Models in Economics

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Abstract. In this paper, we give a simple geometric characterization of homogeneous production functions, by studying geometric properties of their associated graph hypersurfaces. For a homogeneous production function, we prove that its corresponding hypersurface with constant sectional curvature must be flat. Therefore, by combining this with Chen and Vîlcu's recent results, we obtain a new geometric characterization of homogeneous production functions having constant return to scale.

1. Introduction

In microeconomics and macroeconomics, the production functions are defined as non-constant positive functions which specify the output of a firm, an industry, or an entire economy for all combinations of inputs. Hence, almost all economic theories presuppose a production function, either on the firm level or the aggregate level.

Economists always using a production function in economic analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process. C. W. Cobb and P. H. Douglas [12] in 1928 introduced a two inputs production function, nowadays called Cobb-Douglas production function, which is defined by

$$Y = bL^kC^{1-k}, \tag{1.1}$$

where L denotes the labor input, C is the capital input, b is the total factor productivity and Y is the total production.

We could define the generalized form of the Cobb-Douglas production function with an arbitrary number of inputs as

$$F(x_1, \dots, x_n) = Ax_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.2}$$

where $x_i > 0$ $(i = 1, \dots, n)$, A is a positive constant and $\alpha_1, \dots, \alpha_n$ are nonzero constants.

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [1] introduced another two inputs production function written as

$$Q = b(aK^r + (1-a)L^r)^{\frac{1}{r}}, \tag{1.3}$$

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where Q is the output, b the factor productivity, a the share parameter, K and L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution. Hence it is also called constant elasticity of substitution (CES) production function [13, 14]. Also, the generalized form of CES production function with an arbitrary number of inputs is given by

$$F(x_1, \dots, x_n) = A\Big(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho}\Big)^{\frac{\gamma}{\rho}}, \tag{1.4}$$

where a_i, γ, A, ρ are nonzero constants, $A, a_i > 0$ and $\rho < 1$. Note that the generalized CES production function is a generalization of the generalized Cobb-Douglas production function, and both of them belong to a much larger class of production functions-homogeneous production functions.

It is well known that a production function $Q = F(x_1, \dots, x_n)$ is said to be γ-homogeneous or homogeneous of degree γ, if given any positive constant t,

$$F(tx_1, \dots, tx_n) = t^{\gamma} F(x_1, \dots, x_n). \tag{1.5}$$

for some nonzero constant γ. If $\gamma > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $\gamma < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale.

Recently, Vîlcu et al. showed in [15, 16] that the generalized Cobb-Douglas production functions and generalized CES production functions have constant return to scale if and only if the corresponding hypersurfaces have vanishing Gauss-Kronecker curvature. These results establish an interesting link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces in Euclidean spaces.

Motivated by the work mentioned above, a natural question in economic analysis is to study important production functions via geometric properties of their associated graph hypersurfaces.

Very recently, Chen and Vîlcu et al.'s made a series of nice work on this topic, see [2-8, 10, 11, 17] and references therein.

In particular, Chen and Vîlcu proved in [10] that a homogeneous production function with an arbitrary number of inputs defines a flat hypersurface if and only if either it has constant return to scale or it is a multinomial production function.

In the theory of differential geometry, the study of hypersurfaces with constant sectional curvature in a Riemannian space is very important and receives extensive attention by geometers [9].

In this paper, we generalize Chen-Vîlcu's results [10] to the case of homogeneous hypersurfaces with constant sectional curvature. We show that homogeneous hypersurface with constant sectional curvature must be flat. Hence, Chen and Vîlcu's result can be generalized to: a homogeneous production function with an arbitrary number of inputs defines a hypersurface with constant sectional curvature if and only if either it has constant return to scale or it is a multinomial production function. In particular, we prove that, in the two inputs case, a homogeneous production function with two inputs defines a production surface with constant Gauss curvature if and only if either it has constant return to scale or it is a binomial production function.

2. Basic Theory of Hypersurfaces in Differential Geometry

It is well known that each production function $F(x_1, \dots, x_n)$ can be identified with a graph of a nonparametric hypersurface of an Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ given by

$$f(x_1, \dots, x_n) = (x_1, \dots, x_n, F(x_1, \dots, x_n)). \tag{2.1}$$

We call this hypersurface as *a production hypersurface*.

Let us recall some basic concepts in the theory of hypersurfaces in Euclidean spaces.

Let M^n be a orientable hypersurface in an $(n+1)$-dimension Euclidean space. Since M^n is orientable, the Gauss map v can be defined globally given by

$$v : M^n \to S^n \subset \mathbb{E}^{n+1},$$

which maps M^n to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $v(p)$ is a unit normal vector $\xi(p)$ of M^n at point p.

The differential dv of the Gauss map v can be used to define an extrinsic curvature, known as the shape operator. It is well known that the shape operator S_p and dv can be related by:

$$g(S_p u, w) = g(dv(u), w),$$

where $u, w \in T_pM$ and g is the metric tensor on M^n.

Moreover, the second fundamental form h is related with the shape operator S_p by

$$g(S_p u, w) = g(h(u, w), \xi(p))$$

for $u, w \in T_pM$.

Denote the partial derivatives $\frac{\partial F}{\partial x_i}, \frac{\partial^2 F}{\partial x_i x_j}, \ldots$, etc. by $F_i, F_{ij}, \ldots$, etc. Put

$$W = \sqrt{1 + \sum_{i=1}^{n} F_i^2}. \tag{2.2}$$

Recall some well-known results for a graph of hypersurface (2.1) in $\mathbb{E}^{n+1}$ from [9, 2].

Proposition 2.1. *For a production hypersurface of $\mathbb{E}^{n+1}$ defined by*

$$h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, , x_n)) \tag{2.3}$$

we have:

1. *The unit normal ξ is given by:*
$$\xi = \frac{1}{W}(-F_1, \ldots, -F_n, 1). \tag{2.4}$$
2. *The coefficient $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ of the metric tensor is given by:*
$$g_{ij} = \delta_{ij} + F_i F_j, \tag{2.5}$$
where $\delta_{ij} = 1$ if $i = j$, otherwise 0.
3. *The volume element is*
$$dV = \sqrt{g_{ij}} dx_1 \wedge \cdots \wedge dx_n. \tag{2.6}$$
4. *The inverse matrix (g^{ij}) of (g_{ij}) is*
$$g^{ij} = \delta_{ij} - \frac{F_i F_j}{W^2}. \tag{2.7}$$
5. *The matrix of the second fundamental form h is*
$$h_{ij} = \frac{F_{ij}}{W}. \tag{2.8}$$
6. *The matrix of the shape operator S_p is*
$$a_i^j = \sum_k h_{ik} g^{kj} = \frac{1}{W}(F_{ij} - \sum_k \frac{F_{ik} F_k F_j}{W^2}). \tag{2.9}$$
7. *The Gauss-Kronecker curvature G is*
$$G = \frac{\det h_{ij}}{\det g_{ij}} = \frac{\det F_{ij}}{W^{n+2}}. \tag{2.10}$$
8. *The sectional curvature K_{ij} with respect to the plane section spanned by $\{\frac{\partial}{\partial_i}, \frac{\partial}{\partial_j}\}$ is given by*
$$K_{ij} = \frac{F_{ii} F_{jj} - F_{ij}^2}{W^2 (1 + F_i^2 + F_j^2)}. \tag{2.11}$$
In particular, when $n = 2$, the sectional curvature K_{12} is the Gauss-Kronecker curvature G or also known as the Gauss curvature.

3. The Homogeneous Production Hypersurfaces

In [10], Chen and Vîlcu studied flat homogeneous production hypersurfaces and give a complete classification. The aim of this section is to extend their results to the hypersurfaces (arbitrary dimension) with constant sectional curvature.

3.1. The case of dimension $n = 2$.

We first deal with the case of homogeneous production surfaces with constant Gauss curvature in $\mathbb{E}^3$.

Lemma 3.1. *The homogeneous production surfaces with constant Gauss curvature in $\mathbb{E}^3$ must be flat.*

Proof. Suppose that M is homogeneous production surface in $\mathbb{E}^3$ defined by a graph

$$h(x_1, x_2) = (x_1, x_2, F(x_1, x_2)), \tag{3.1}$$

where F is the homogeneous production function of degree γ with the form

$$F(x_1, x_2) = x_1^{\gamma} f(\frac{x_2}{x_1}). \tag{3.2}$$

Hence, it follows from (3.2) that

$$F_1 = \gamma x_1^{\gamma-1} f(\frac{x_2}{x_1}) - x_1^{\gamma-2} x_2 f'(\frac{x_2}{x_1}), \tag{3.3}$$

$$F_2 = x_1^{\gamma-1} f'(\frac{x_2}{x_1}), \tag{3.4}$$

$$F_{11} = \gamma(\gamma-1)x_1^{\gamma-2} f(\frac{x_2}{x_1}) - 2(\gamma-1)x_1^{\gamma-3} x_2 f'(\frac{x_2}{x_1}) + x_1^{\gamma-4} x_2^2 f''(\frac{x_2}{x_1}), \tag{3.5}$$

$$F_{12} = (\gamma-1)x_1^{\gamma-2} f'(\frac{x_2}{x_1}) - x_1^{\gamma-3} x_2 f''(\frac{x_2}{x_1}), \tag{3.6}$$

$$F_{22} = x_1^{\gamma-2} f''(\frac{x_2}{x_1}), \tag{3.7}$$

where we denote $\frac{x_2}{x_1}$ by a new variable u and "$\,'\,$" denotes the derivative with respect to u.

By (3.3) and (3.4), we compute

$$\begin{aligned} W^2 &= 1 + F_1^2 + F_2^2 \\ &= 1 + \gamma^2 x_1^{2\gamma-2} f^2 - 2\gamma x_1^{2\gamma-2} u f f' + x_1^{2\gamma-2}(1+u^2) f'^2. \end{aligned} \tag{3.8}$$

Moreover, from (3.5-3.7) we have

$$F_{11}F_{22} - F_{12}^2 = -(\gamma-1)^2 x_1^{2\gamma-4} f'^2 + \gamma(\gamma-1) x_1^{2\gamma-4} f f''. \tag{3.9}$$

By assumption, the Gauss curvature K is constant. Hence, substituting (3.8) and (3.9) into (2.11), we have

$$-(\gamma-1)^2 x_1^{2\gamma-4} f'^2 + \gamma(\gamma-1) x_1^{2\gamma-4} f f'' = G\Big(1 + \gamma^2 x_1^{2\gamma-2} f^2 - 2\gamma x_1^{2\gamma-2} u f f' + x_1^{2\gamma-2}(1+u^2) f'^2\Big)^2. \tag{3.10}$$

Note that if $\gamma = 1$, then $G = 0$. We will assume $\gamma \neq 1$ in the following.

Suppose that the Gauss-Kronecker curvature G is a nonzero constant. We will derive a contradiction.

Equation (3.10) reduces to

$$x_1^{\gamma-2}\Big(\frac{\gamma(\gamma-1) f f'' - (\gamma-1)^2 f'^2}{G}\Big)^{\frac{1}{2}} = 1 + x_1^{2\gamma-2}\Big(\gamma^2 f^2 - 2\gamma u f f' + (1+u^2) f'^2\Big). \tag{3.11}$$

If $\gamma = 0$, (3.11) reduces to

$$(\frac{-f'^2}{G})^{\frac{1}{2}} - (1+u^2) f'^2 = x_1^2,$$

which is impossible since f is a function with respect to u. Thus, we have $\gamma \neq 0$.

Differential both sides of equation (3.11) with respect to x_1, we obtain

$$(\gamma - 2)\left(\frac{\gamma(\gamma-1)ff'' - (\gamma-1)^2 f'^2}{G}\right)^{\frac{1}{2}} = 2(\gamma-1)x_1^{\gamma}\Big(\gamma^2 f^2 - 2\gamma u f f' + (1+u^2)f'^2\Big). \tag{3.12}$$

Since $\gamma \neq 1, 0$ and f is a function of u, it follows from (3.11) and (3.12) that

$$\begin{aligned} &\gamma - 2 = 0, \\ &\gamma^2 f^2 - 2\gamma u f f' + (1+u^2)f'^2 = 0, \end{aligned}$$

Hence

$$4f^2 - 4uff' + (1+u^2)f'^2 = 0,$$

which is equivalent to

$$(2f - uf')^2 + f'^2 = 0. \tag{3.13}$$

Equation (3.13) means that $f = 0$, whihc is a contradiction.

Therefore, we conclude that the Gauss curvature for the homogeneous production surfaces in Euclidean 3-space $\mathbb{E}^3$ must be vanishing. □

Therefore, combining Chen and Vîlcu's result (Theorem 4.1 in [10]) with Lemma 3.1 gives

Theorem 3.2. *A homogeneous production function with two inputs defines a production surface with constant Gauss curvature if and only if either it has constant return to scale or it is a binomial production function.*

Note that a binomial production function F (see, for details in [10]) is given by

$$F(x_1, x_2) = (c_1 x_1 + c_2 x_2)^{\gamma}, \quad \gamma \neq 1 \tag{3.14}$$

for two constants c_1 and c_2.

3.2. The case of dimension $n > 2$.

In the following, we focus on the general case of homogeneous production function with n inputs for $n > 2$.

Theorem 3.3. *A homogeneous production function with n inputs defines a production hypersurface with constant sectional curvature if and only if either it has constant return to scale or it is a multinomial production function.*

Proof. Let F be an n inputs homogeneous production function with degree of γ defined in (1.5). Then F corresponds a graph in Euclidean space $\mathbb{E}^{n+1}$,

$$h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)). \tag{3.15}$$

Now suppose that F has the following form

$$F(x_1, \ldots, x_n) = x_1^{\gamma} f(u_2, u_3, \ldots, u_n), \tag{3.16}$$

where

$$u_i = \frac{x_i}{x_1}, \quad i = 2, \ldots, n.$$

Since in the case $\gamma = 1$ the conclusion follows easily, we assume that $\gamma \neq 1$. By (3.6), a direct computation shows that

$$F_1 = x_1^{\gamma-1}\Big(\gamma f - \sum_{k=2}^{n} u_k f_{u_k}\Big), \tag{3.17}$$

$$F_i = x_1^{\gamma-1} f_{u_i}, \quad i = 2, \ldots, n, \tag{3.18}$$

$$F_{11} = x_1^{\gamma-2}\Big(\gamma(\gamma-1)f - 2(\gamma-1)\sum_{k=2}^{n} u_k f_{u_k} + \sum_{k,l=2}^{n} u_k u_l f_{u_k u_l}\Big), \tag{3.19}$$

$$F_{1i} = x_1^{\gamma-2}\Big((\gamma-1)f_{u_i} - \sum_{k=2}^{n} u_k f_{u_i u_k}\Big), \quad i = 2, \ldots, n, \tag{3.20}$$

$$F_{ii} = x_1^{\gamma-2} f_{u_i u_i}, \quad i = 2, \ldots, n, \tag{3.21}$$

$$F_{ij} = x_1^{\gamma-2} f_{u_i u_j}, \quad i, j = 2, \ldots, n, \text{ and } i \neq j, \tag{3.22}$$

and

$$W^2 = 1 + \sum_{k=1}^{n} F_k^2 = 1 + x_1^{2(\gamma-1)} A, \tag{3.23}$$

where

$$A = \Big((\gamma f - \sum_{k=2}^{n} u_k f_{u_k})^2 + \sum_{k=2}^{n} f_{u_k}^2\Big). \tag{3.24}$$

According to the assumption that the sectional curvature K_{ij} is a constant c, we assume that $c \neq 0$ and distinguish the following two cases:

Case A. $i = 1, j = 2, \ldots, n$. In this case, from (3.17-3.21) we have

$$1 + F_1^2 + F_j^2 = 1 + x_1^{2(\gamma-1)} B, \tag{3.25}$$

$$F_{11}F_{jj} - F_{1j}^2 = x_1^{2(\gamma-2)} C, \tag{3.26}$$

where

$$B = \Big((\gamma f - \sum_{k=2}^{n} u_k f_{u_k})^2 + f_{u_j}^2\Big), \tag{3.27}$$

$$C = \left[f_{u_j u_j}\Big(\gamma(\gamma-1)f - 2(\gamma-1)\sum_{k=2}^{n} u_k f_{u_k} + \sum_{k,l=2}^{n} u_k u_l f_{u_k u_l}\Big) - \left((\gamma-1)f_{u_j} - \sum_{k=2}^{n} u_k f_{u_j u_k}\right)^2 \right]. \tag{3.28}$$

Notice that all of A, B, C are functions with respect to variables $u_2, \ldots, u_n$ and not vanishing.

Hence, by (2.11), (3.23-3.26) we obtain

$$c(1 + x_1^{2(\gamma-1)} A)(1 + x_1^{2(\gamma-1)} B) = x_1^{2(\gamma-2)} C. \tag{3.29}$$

Differentiating (3.30) with respect to x_1 gives

$$4c(\gamma-1)x_1^{2\gamma} AB + 2c(\gamma-1)x_1^2(A+B) - 2(\gamma-2)C = 0. \tag{3.30}$$

Differentiating (3.31) with respect to x_1, one has

$$2\gamma x_1^{2\gamma-2} AB + A + B = 0, \tag{3.31}$$

which is a contradiction since $\gamma \neq 1$ and both of A and B are nonzero functions of $u_2, \ldots, u_n$.

Case B. $i, j = 2, \ldots, n$ and $i \neq j$. From (3.18), (3.21) and (3.22), we compute

$$1 + F_i^2 + F_j^2 = 1 + x_1^{2(\gamma-1)}(f_{u_i}^2 + f_{u_j}^2), \tag{3.32}$$

$$F_{ii}F_{jj} - F_{ij}^2 = x_1^{2(\gamma-2)}(f_{u_iu_i}f_{u_ju_j} - f_{u_iu_j}^2) \tag{3.33}$$

for $i \neq j$ and $i, j = 2, \ldots, n$.

It follows from (2.11), (3.33) and (3.34) that

$$c(1 + x_1^{2(\gamma-1)}A)[1 + x_1^{2(\gamma-1)}(f_{u_i}^2 + f_{u_j}^2)] = x_1^{2(\gamma-2)}(f_{u_iu_i}f_{u_ju_j} - f_{u_iu_j}^2). \tag{3.34}$$

After differentiating (3.35) twice with respect to x_1, a similar discussion as the above case yields a contradiction as well.

Thus, we conclude that the sectional curvature for a homogeneous production hypersurface must be vanishing. So, by the results in [10] we complete the proof of Theorem 3.3. □

Remark that a multinomial production function F (see details in [10]) is given by

$$F(x_1, x_2, \ldots, x_n) = (c_1x_1 + c_2x_2 + \cdots + c_nx_n)^\gamma, \quad \gamma \neq 1$$

for constants c_i, $i = 1, \ldots, n$.

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