A GEOMETRIC CHARACTERIZATION OF THE PERFECT SUZUKI-TITS OVOIDS

Hendrik Van Maldeghem[1]

We characterize in a geometrical way those Suzuki-Tits ovoids which are defined over a perfect field $\mathbb{K}$ (or equivalently living inside a self polar symplectic quadrangle). We simplify our axioms in the particular cases that (1) the associated Suzuki group has exactly two orbits in the set of lines of $PG(3, \mathbb{K})$, and (2) the ovoid is finite.

1 INTRODUCTION

The perfect symplectic generalized quadrangle $W(\mathbb{K})$, where $\mathbb{K}$ is a (commutative) field of characteristic 2 admitting an automorphism σ whose square is the Frobenius automorphism of $\mathbb{K}$, always has an ovoid, namely the *Suzuki-Tits ovoid* obtained by considering the absolute points of a polarity, see Tits [7]. In case that $\mathbb{K}$ has a quadratic extension, there is also a *classical ovoid*, which is in fact an elliptic quadric in projective 3-space over $\mathbb{K}$. If $\mathbb{K}$ is a finite field, then these are the only ovoids known to exist. They are also ovoids of the projective 3-space over $\mathbb{K}$ in which $W(\mathbb{K})$ is embedded.

In general, an ovoid of $PG(3, \mathbb{K})$ can be structured as a *Möbius circle plane* or *inversive plane* by considering the plane sections. This Möbius plane is in fact a one-point extension of the affine plane over $\mathbb{K}$. There are some charaterizations of the (finite) Möbius plane corresponding with the classical ovoid (which is called the *Miquelian inversive plane*) in characteristic 2, see for instance Thas [5, 6] and Glynn [2]. In this paper we present a geometric characterization of the perfect Suzuki-Tits ovoids as Möbius planes. This characterization — which I already announced in [8] as a remark — arose naturally from an attempt to characterize geometrically the Ree-Tits generalized octagons over a perfect field, where the pencils of lines have the structure of the Möbius plane defined by a perfect Suzuki-Tits ovoid (this can also be seen through the action of a Suzuki group on the pencils), see [9].

[1]Senior Research Associate at the Belgian National Fund for Scientific Research

2 MAIN RESULT

Let $\Gamma = (\mathcal{P}, \mathcal{C}, \in)$ be a geometry consisting of a set $\mathcal{P}$ of at least 4 points; a set $\mathcal{C}$ of at least 2 circles and an incidence relation $\in \subseteq \mathcal{P} \times \mathcal{C}$. Then Γ is a *(Möbius) inversive plane* if it satisfies conditions (MP1) and (MP2) stated below. A Möbius inversive plane is a *characteristic* 2 *inversive plane* if it moreover satisfies (CH1) and (CH2). It is called *perfect* if it satisfies (P). Finally, a perfect characteristic 2 inversive plane is a *Suzuki-Tits inversive plane*, or briefly an *STi-plane*, if it satisfies condition (ST). As the terminology already shows, the only axiom distinguishing the Suzuki-Tits ovoid from the classical ovoid (if it exists) in the perfect case is Axiom (ST).

We now state all axioms mentioned above. We view circles as sets of points incident with them. Two circles *touch* if they intersect in exactly one point.

(MP1) *Three points of* Γ *are contained in exactly one circle.*

(MP2) *For every circle* $C \in \mathcal{C}$, *and every two points* $x, y \in \mathcal{P}$ *such that* $x \in C$ *and* $y \notin C$, *there exists a unique circle* $D \in \mathcal{C}$ *touching* C *in* x *and containing* y.

(CH1) *For every circle* C *and every pair of points* $x, y \notin C$, *either there is a unique circle containing both* x *and* y *and touching* C, *or all circles through* x *and* y *touch* C.

(CH2) *There do not exist three circles two by two touching each other in distinct points.*

(P) *For every two triplets* $\{C_i, D_i, E_i\}$, $i = 1, 2$, *of pairwise disjoint circles, we have that* E_1 *touches* E_2 *whenever both* C_i *and* D_i *touch* C_j, D_j *and* E_j, $\{i, j\} = \{1, 2\}$.

(ST) *The inversive plane* Γ *is furnished with a map* $\partial : \mathcal{C} \to \mathcal{P} : C \mapsto \partial C \in C$ *such that:*
(ST1) *For every pair of points* $x, y \in \mathcal{P}$ *there is a unique circle* $C \in \mathcal{C}$ *containing* x *and* y *and such that* $\partial C = x$.
(ST2) *For every circle* C *and every point* $x \notin C$, *there is at most one circle* D *containing* x *and* ∂C *and such that* $\partial D \in C$.

In fact an STi-plane should be denoted by its geometry Γ and the map ∂. Since we will always use the same notation ∂, we will by abuse of language call Γ an STi-plane when in fact the pair (Γ, ∂) is. The map ∂ is by the way not necessarily unique. For the inversive plane arising from the elliptic quadric over the field of two elements (which is isomorphic to the inversive plane arising from the Suzuki-Tits ovoid related to $Sz(2)$, see TITS [7]), there are essentially three choices since $Sz(2)$ has index 3 in $L_2(4)$ and since the latter acts transitively on the incident point-circle pairs. For a general given (perfect) Suzuki-Tits ovoid $\mathcal{O}$ defined over some field $\mathbb{K}$, a map δ is constructed as follows. Let $W(\mathbb{K})$ be a symplectic quadrangle such that $\mathcal{O}$ is the set of absolute points of a polarity in $W(\mathbb{K})$; let $PG(3, \mathbb{K})$ be the ambient projective space. Every circle C of the inversive plane corresponding with $\mathcal{O}$ is the intersection of $\mathcal{O}$ with a plane π of $PG(3, \mathbb{K})$. This plane π is the image under the symplectic polarity defining $W(\mathbb{K})$ of a unique point x. All points of C are collinear in $W(\mathbb{K})$ with x, and conversely, every line in $W(\mathbb{K})$ through x contains a unique point of C. The set of lines obtained from $\mathcal{O}$ by applying the polarity of $W(\mathbb{K})$ defining $\mathcal{O}$ is a spread of $W(\mathbb{K})$ and so there is a unique line L of this set through x. The point of $\mathcal{O}$ on L is by definition δC. We remark that δC is called "le nœud" by TITS [7]. We will call it the *corner* of C.

In the next section we will prove some lemmas which will allow us to construct a symplectic quadrangle in which a given STi-plane lives. More exactly we will show:

MAIN RESULT. *The point set $\mathcal{P}$ of any* STi-*plane* Γ *can be embedded in a projective* 3-*space* $PG(3,\mathbb{K})$*, where* $\mathbb{K}$ *is a field of characteristic* 2 *in which there exists an automorphism* σ *whose square is the Frobenius map* $x \mapsto x^2$*, in such a way that the circles of* Γ *correspond to plane sections of* $\mathcal{P}$ *(now viewed as a set of points of* $PG(3,\mathbb{K})$*) and such that* $\mathcal{P}$ *is projectively equivalent with a Suzuki-Tits ovoid. Conversely, every inversive plane obtained from a Suzuki-Tits ovoid over a perfect field by considering plane sections is an* STi-*plane.*

In the finite case, the main result can be stated in a more elegant way as follows.

MAIN RESULT — FINITE CASE. *An inversive plane of even order arises naturally from a finite Suzuki-Tits ovoid* **if and only if** *it satisfies Axiom* (ST) *stated above.*

Another particular case is the case where the Suzuki group has only two orbits on the set of lines of $PG(3,\mathbb{K})$ not belonging to the symplectic quadrangle in its natural action on $PG(3,\mathbb{K})$. In this case, we call the Suzuki group and the associated Suzuki-Tits ovoid *special*. In particular, this is true if $\mathbb{K}$ is finite. In the special case the situation described in Axiom (P) does not occur. So we might state an alternative axiom:

(P′) *If for two pairs* $\{C_i, D_i\}$*,* $i = 1, 2$*, of disjoint circles we have that* C_i *touches both* C_j *and* D_j*,* $\{i, j\} = \{1, 2\}$*, then* D_1 *does not touch* D_2*.*

A characteristic 2 inversive plane which satisfies (P′) and (ST) will be called *a special* STi-plane. Note that if (P′) holds, then the conditions of (P) can never be satisfied, and so (P) is always valid. Indeed, the circles C_1, C_2, D_1, D_2 of Axiom (P) can never satisfy Axiom (P′). Hence every special STi-plane is an STi-plane, but the converse is not true, as we will see below.

We now have:

MAIN RESULT — SPECIAL CASE. *The point set $\mathcal{P}$ of any special* STi-*plane* Γ *can be embedded in a projective* 3-*space* $PG(3,\mathbb{K})$*, where* $\mathbb{K}$ *is a field of characteristic* 2 *in which there exists an automorphism* σ *whose square is the Frobenius map* $x \mapsto x^2$*, in such a way that the circles of* Γ *correspond to plane sections of* $\mathcal{P}$ *(now viewed as a set of points of* $PG(3,\mathbb{K})$*) and such that* $\mathcal{P}$ *is projectively equivalent with a special Suzuki-Tits ovoid. Conversely, every inversive plane obtained from a special Suzuki-Tits ovoid over a perfect field by considering plane sections is a special* STi-*plane.*

For the rest of the paper, we denote by Γ a given STi-plane.

3 PRELIMINARY RESULTS.

LEMMA 1 *Let* $C, D, E \in \mathcal{C}$ *all contain the point* x*. If* C *touches* D *in* x *and* D *touches* E *in* x*, then* C *touches* E *in* x *or* $C = E$*.*

PROOF. If C would meet E in a second point $y \neq x$, then there exist two circles C and E containing y and touching D in x. This would contradict Axiom (MP2).

LEMMA 2 *For every circle C and every point $x \notin C$, there is a unique circle D containing x and ∂C and such that $\partial D \in C$.*

PROOF. Let $y \in C$ be arbitrary but such that $y \neq \partial C$. Let E be the unique circle containing ∂C such that $\partial E = y$ (Axiom (ST1)). If $x \in E$, then the result follows from Axiom (ST2). So suppose that $x \notin E$. Let D be the unique circle containing x and touching E in ∂C. Note that $y \in E \cap C$ and E touches D in ∂C, hence by Axiom (MP2) C does not touch D in ∂C. That means that C and D share another point z. Let F be the circle with corner z containing ∂C. Since $z \in C$, it follows from Axiom (ST2) that E and F touch in ∂C (because $y \notin F$ otherwise $F = C$ by Axiom (MP1)). But $z \in F \cap D$ and both D and F touch E in ∂C. This implies by Axiom (MP2) that $F = D$ and hence $z = \partial D \in C$. The lemma is proved. □

LEMMA 3 *If a circle C touches D in ∂D, then $\partial C = \partial D$.*

PROOF. Choose $x \in C$, $x \neq \partial D$, and let C' be the circle with corner ∂D and containing x. Axiom (ST1) implies that C' touches D in ∂D and so by Axiom (MP2) we must have $C' = C$ (because they both contain x). □

LEMMA 4 *If three circles containing a point x touch another circle which does not contain x, then these three circles have two points in common.*

PROOF. Let C, D, E touch F with $x \in C \cap D \cap E$ and $x \notin F$. By Axiom (CH2), C and D meet in a second point y, and C and E also meet in a second point z. We have to show $y = z$. Suppose by way of contradiction $y \neq z$. Then $z \notin D$. By Axiom (MP2), there exists a circle G touching D in x and containing z. By Axiom (CH1) G touches F. Since $x \notin F$, this contradicts Axiom (CH2). Hence the result. □

LEMMA 5 *If three circles touch two disjoint circles, then they either all have two points in common, or they are pairwise disjoint.*

PROOF. Suppose C, D and E all touch the disjoint circles F and G. Suppose that C, D, E are not pairwise disjoint. By Axiom (CH2), we may assume that C and D meet in two distinct points x and y. Note that $x \in F$ would imply by Axiom (MP2) that $C = D$. Suppose E touches F in z. By Axiom (CH1) the circle H containing x, y and z touches both F and G. From Axiom (MP2) follows readily that either H coincides with E, or H touches E. But the latter violates Axiom (CH2). Whence the result. □

LEMMA 6 *There do not exist four circles C_i, $i = 1, 2, 3, 4$, such that C_i touches C_{i+1} and C_i meets C_{i+2} in two distinct points, for all subscripts modulo 4.*

PROOF. Let $C_1 \cap C_3 = \{x_1, x_3\}$ and $C_2 \cap C_4 = \{x_2, x_4\}$. Axiom (MP2) implies that $x_1, x_3 \neq x_2, x_4$. Hence there is a unique circle D containing x_1, x_2 and x_3. By Axiom (CH1), D touches both C_2 and C_4. But this contradicts Axiom (MP2) since both C_2 and C_4 contain x_4 and touch D. □

LEMMA 7 *If C and D are two circles, then every point x not in $C \cup D$ is contained in at least one circle D_x touching both C and D. If C and D meet in two points, then D_x is unique and the claim holds for every $x \notin C \cap D$.*

PROOF. If C and D touch, then the result follows readily from Axiom (MP2) and Lemma 1. So we may assume that C and D do not touch.

Let $x \in \mathcal{P} \setminus (C \cup D)$. Let C_1 and C_2 be two circles touching C and containing x. By Axiom (CH2), C_1 and C_2 meet in another point $x_C \neq x$. By Axiom (CH1), all circles through X and x_C touch C. Similarly, there is a point x_D such that all circles through x and x_D touch D. Hence a circle through x, x_C and x_D touches both C and D. If C and D meet, then $x_C \neq x_D$ by Lemma 6. Moreover every circle through x touching C must contain x_C by Lemma 4 and similarly for D. This proves the uniqueness in the case that C and D are non-disjoint. If $x \in C \setminus D$, then by considering a third circle C' containing $C \cap D$, $C \neq C' \neq D$, the result follows from Axiom (CH1). $\square$

LEMMA 8 *For every point x outside any circle C, there exists a unique circle D with corner x touching C.*

PROOF. There is at most one such circle by combining Axioms (CH2) and (ST1). Let D_1 and D_2 be two circles containing x and touching C (these exist since C contains at least three elements and for each element of C we can construct such a circle by Axiom (MP2)). By Axiom (CH2) the circles D_1 and D_2 meet in a second point $y \neq x$. Let D be the circle with corner x containing y, then the result follows from Axiom (CH1). $\square$

4 THE SYMPLECTIC QUADRANGLE $W(\mathbb{K})$

We introduce the following geometry $\Gamma_\Box = (\mathcal{P}_\Box, \mathcal{L}_\Box, I)$. Both the point set $\mathcal{P}_\Box$ and the line set $\mathcal{L}_\Box$ of $\Gamma_\Box$ are the union of $\mathcal{P}$ and $\mathcal{C}$. Hence if $x \in \mathcal{P}$, then x can be viewed as a point or as a line of $\Gamma_\Box$. To make that difference, we write x_p respectively x_ℓ if we view x as an element of $\mathcal{P}_\Box$ respectively $\mathcal{L}_\Box$. Similarly for circles. We now define incidence in $\Gamma_\Box$. A point x_p, $x \in \mathcal{P}$, is incident with a line y_ℓ, $y \in \mathcal{P}$, if and only if $x = y$. A point x_p (respectively line x_ℓ), $x \in \mathcal{P}$, is incident with a line C_ℓ (respectively point C_p), $C \in \mathcal{C}$, if and only if $\partial C = x$. A point C_p, $C \in \mathcal{C}$, is incident with a line D_ℓ, $D \in \mathcal{C}$, if and only if $\partial C \in D$, $\partial D \in C$ and $\partial C \neq \partial D$

PROPOSITION 9 *The geometry $\Gamma_\Box$ as defined above is a symplectic generalized quadrangle.*

PROOF. First we claim that, if $C, D \in \mathcal{C}$, then C_p and D_p are collinear in $\Gamma_\Box$ if and only if C touches D. If $C_p \,\mathbf{I}\, x_\ell \,\mathbf{I}\, D_p$ with $x \in \mathcal{P}$, then the claim follows from Axiom (ST1). Suppose now $C_p \,\mathbf{I}\, E_\ell \,\mathbf{I}\, D_p$ with $E \in \mathcal{C}$. Then $\partial E \in C \cap D$ and since $D \neq C$, we have $\partial D \neq \partial C$. Clearly also $\partial D \neq \partial E \neq \partial C$. Since $\partial C, \partial D \in E$, the result follows from Axiom (ST2). Conversely, suppose that C and D are touching circles. If they touch in ∂C, then by Lemma 3 $\partial C = \partial D$ and $C_p \mathbf{I} (\partial C)_\ell \mathbf{I} D_p$. So we may assume that $\{x\} = C \cap D$ with $\partial C \neq x \neq \partial D$. By Lemma 2, there exists a unique circle E containing ∂C and ∂D and such that $\partial E = x$. We now have $D_p \mathbf{I} E_\ell \mathbf{I} D_p$.

Next, we show that $\Gamma_\Box$ is a thick generalized quadrangle. So we must show that, whenever X is a point of $\Gamma_\Box$ and L is a line of $\Gamma_\Box$ not incident with X, then there exists a unique point-line pair (Y, M) such that $X \mathbf{I} M \mathbf{I} Y \mathbf{I} L$. There are essentially three cases to distinguish.

Case 1. $X = x_p$ and $L = y_\ell$ with $x, y \in \mathcal{P}$. Clearly $x \neq y$ and neither Y nor M can be elements of $\mathcal{P}$. So M must be equal to C_ℓ with $C \in \mathcal{C}$ and $\partial C = x$. Similarly $Y = D_p$ with $D \in \mathcal{C}$ and $\partial D = y$. Since C_ℓ and D_p are incident in $\Gamma_\Box$ we must have that $x \in D$ and $y \in C$. But that defines C and D uniquely by Axiom (ST1).

Case 2. $X = x_p$ and $L = C_\ell$ with $x \in \mathcal{P}$ and $C \in \mathcal{C}$. By assumption $\partial C \neq x$. Suppose first that $x \in C$. Let $D \in \mathcal{C}$ be such that $\partial D = x$ and $\partial C \in D$ (D is uniquely defined by Axiom (ST1)). Then $x_p \mathbf{I} x_\ell \mathbf{I} D_p \mathbf{I} C_\ell$. If E is any circle such that $E_\ell \mathbf{I} x_p$, then $\partial E = x$. If E_ℓ is concurrent with C_ℓ, then similarly as for collinear points in $\Gamma_\Box$, C and E touch. But Lemma 8 implies that $\partial C = \partial E = x$, contradictory to our assumptions. We have shown the uniqueness of Y and M if $x \in C$.

Suppose now $x \notin C$. If F is a circle such that $F_p \mathbf{I} C_\ell$ and F_p is collinear with x_p in $\Gamma_\Box$, then as above, $\partial F \in C$ and $x, \partial C \in F$. By Lemma 2, F exists and is unique. If E is the unique circle with corner x and containing ∂F, then we have $x_p \mathbf{I} E_\ell \mathbf{I} F_p \mathbf{I} C_\ell$ and this chain is unique.

The case where X is a circle and L is a point of Γ is completely similar.

Case 3. $X = C_p$ and $L = D_\ell$ with $C, D \in \mathcal{C}$. First suppose that $\partial C = \partial D$. Clearly $X \mathbf{I} (\partial C)_\ell \mathbf{I} (\partial D)_p \mathbf{I} L$. Suppose E and F are circles such that $X \mathbf{I} E_\ell \mathbf{I} F_p \mathbf{I} D_\ell$. Then E contains ∂C, ∂F and ∂E. Since $\partial E \neq \partial F$ (by definition of incidence in $\Gamma_\Box$), we also have $E \neq F$. But also F contains $\partial C = \partial D$, ∂E and ∂F, contradicting the fact that, by definition of incidence in $\Gamma_\Box$, there holds $\partial C \neq \partial F$ and $\partial E \neq \partial C$. Clearly $(\partial C)_p$ cannot be collinear with C_p in $\Gamma_\Box$.

Suppose that $\partial C \in D$. Since X and L are not incident we have $\partial D \notin C$. Clearly $X \mathbf{I} (\partial C)_\ell \mathbf{I} E_p \mathbf{I} L$, where $E \in \mathcal{C}$ with $\partial E = \partial C$ and $\partial D \in E$. Clearly this chain is unique with the property that it contains a point of Γ. Suppose now $C_p \mathbf{I} F_\ell \mathbf{I} G_p \mathbf{I} D_\ell$ with $F, G \in \mathcal{C}$. Then G contains ∂D, D contains ∂G and G touches C in, say, x. By Axiom (MP1), $x \notin D$ (because otherwise either $G = D$ or $G = E$ and $x = \partial C = \partial G$). The circle E containing ∂D with corner ∂C touches C by Axiom (ST2), and it touches G by Axiom (ST2). Hence by Axiom (CH2) either $C \cap E \cap G$ is non-empty, or the set $\{C, E, G\}$ has two elements. In either case, one obtains $E = G$. This concludes the case $\partial C \in D$. Similarly the case $\partial D \in C$ is proved.

So we may assume that $\partial C \notin D$ and $\partial D \notin C$. Let C_1 and C_2 be two circles containing ∂D and touching C (these exist by Axiom (MP2) since a circle contains at least three points). By Axiom (CH2) C_1 and C_2 meet in a further point $x \neq \partial D$. By Axiom (CH1) all circles through ∂D and x touch C, hence so does the unique circle E with $\partial D, x \in E$ and $\partial E \in D$ (which exists by Lemma 2). Note that E is unique with respect to the properties that

$\partial E \in D$, $\partial D \in E$ and E touches C. Indeed, if E' is any circle meeting these conditions, then E and E' touch by Axiom (ST2) and so $E = E'$ by Axiom (MP2). Hence $E_p \, \mathbf{I} \, D_\ell$ and E_p is collinear with C_p. By the uniqueness of E, the proof of the fact that $\Gamma_\Box$ is a generalized quadrangle is complete.

The quadrangle $\Gamma_\Box$ is clearly thick since every circle contains at least three elements (and the number of points of $\Gamma_\Box$ on a line C_p equals the number of points of Γ on C; similarly dually).

Now we show that every pair of non-collinear points $\{X, Y\}$ of $\Gamma_\Box$ is regular, i.e. whenever U, V, W are collinear with X and Y, and Z is collinear with U and V, then Z is collinear with W. Again, there are three cases.

Case 1. Suppose first that $X = x_p$ and $Y = y_p$ with $x, y \in \mathcal{P}$. It is easily seen that U, V and W must be circles of Γ containing x and y (but not as their corners). So we put $U = C_\ell$, $V = D_\ell$ and $W = E_\ell$ with $C, D, E \in \mathcal{C}$. Note that $C \cap D \cap E = \{x, y\}$, which implies that, if Z is collinear with U and V, then $Z = F_p$ with $F \in \mathcal{C}$. Also we know (see above) that F touches both C and D. Axiom (CH1) implies that F touches also E, hence Z and W are collinear.

Case 2. Suppose now that $X = x_p$ and $Y = G_p$ with $x \in \mathcal{P}$ and $G \in \mathcal{C}$. Again U, V and W must be circles and we again put $U = C_\ell$, $V = D_\ell$ and $W = E_\ell$, with $C, D, E \in \mathcal{C}$ and $x \in C \cap D \cap E$. We know that G touches C, D and E, hence by Lemma 4, C, D and E meet in a further point y. We are back at the situation of the preceding paragraph and so the result follows.

Case 3. By switching the roles of X, Y, Z and U, V, W in the preceding paragraphs, we may now assume that they are all circles of Γ. So we put $X = H_p$, $Y = G_p$, $Z = F_p$, $U = C_\ell$, $V = D_\ell$ and $W = E_\ell$, with $C, D, E, F, G, H \in \mathcal{C}$. Now C touches H; H touches D; D touches G and G touches C, while C and D respectively G and H do not touch each other. Hence by Lemma 6, we may assume that G and H are disjoint. By Lemma 5, either C, D and E share two points x and y and hence E touches F as before, or C, D, E are pairwise disjoint and the result follows directly from Axiom (P).

Hence we have shown that all points (and dually lines) of $\Gamma_\Box$ are regular. We now show that each triad of points has at least one center, i.e. if X, Y, Z are three pairwise non-collinear points, then there exists at least one point W collinear with all of them.

case 1. If X, Y and Z are all points of Γ, say $X = x_p$, $Y = y_p$ and $Z = z_p$, then the point C_p, with C the unique circle containing x, y and z, is collinear with X, Y and Z in $\Gamma_\Box$.

case 2. If X and Y are points of Γ, say $X = x_p$ and $Y = y_p$, and if Z is a circle, say $Z = C_p$, then by Axiom (CH1), there is at least one circle F containing both x and y and touching C (note that indeed $x, y \notin C$ otherwise X respectively Y is collinear with Z). The point F_p of $\Gamma_\Box$ is collinear with X, with Y and with Z.

Case 3. Let $X = x_p$, $x \in \mathcal{P}$, let $Y = D_p$ and $Z = C_p$, $C, D \in \mathcal{C}$. We have by assumption $x \notin C \cup D$ and C and D do not touch each other. By Lemma 7, there is at least one circle F containing x and touching both C and D. The point F_p of $\Gamma_\Box$ is again collinear with all three X, Y, Z.

case 4. Finally let $X = E_p, Y = D_p, Z = C_p$ with $C, D, E \in \mathcal{C}$. If $C \cap D \cap E$ contains a point $x \in \mathcal{P}$, then x_p is collinear with all three X, Y, Z. So suppose that $C \cap D \cap E$ is empty. First suppose that C, D, E are pairwise disjoint. Let F be a circle containing ∂E and touching both C and D. If F touches E, then we're done for then F_p is collinear with X, Y and Z. So we may assume that f does not touch E, hence equivalently, we may assume that $\partial F \notin E$ (by Lemma 3). Let F', $F' \neq F$, be any circle touching both C and D. We have already shown above that there exists a unique circle C' containing ∂E, touching F' and having its corner in F. By Axiom (ST2), F touches C'. If F and F' are disjoint, then by Axiom (P), C' touches every circle which touches both C and D, which means that the set of circles touching C, D and E coincides with the set of circles touching C', D and E. If F and F' are non-disjoint, then the same conclusion is derived from Lemma 5 and Axiom (CH1). Hence we may assume that C and D are not disjoint. Let $\{x, y\} = C \cap D$ and let E' be a circle containing both x and y and touching E (E' exists by Axiom (CH1)) in, say, z. By Lemma 7 there exists a unique circle F touching both C and D and containing z. By Axiom (CH1), F also touches E' (in z). Since E' touches E in z, we conclude with Lemma 1 that F touches E (since clearly $F \neq E$). Hence F_p is collinear with X, Y and Z.

So we have shown that for every three pairwise non-collinear points of $\Gamma_\square$ there exists a point collinear with all three of them. In fact this means that, by the regularity of all points of $\Gamma_\square$, for every point X of $\Gamma_\square$ the geometry with point set $\{Y \in \mathcal{P}_\square : Y = X \text{ or } Y \text{ is collinear with } X\}$, line set the set of all sets of points collinear with X and some other point Y of $\Gamma_\square$ and natural incidence relation, is a projective plane. It then follows from SCHROTH [3] that $\Gamma_\square$ is a symplectic quadrangle over some commutative field $\mathbb{K}$. $\square$

As an immediate corollary, we see that in an STi-plane, for every three pairwise non-touching circles, either there exists a unique circle touching all three given circles, or every circle touching two of the three given circles also touches the third. This follows from the regularity of points in $\Gamma_\square$, but it is not so obvious to prove it directly, especially when the three given circles are pairwise disjoint.

5 PROOF OF THE MAIN RESULTS

We can now finish the proof of our Main Results. First note that the map $x_p \mapsto x_\ell$ and $C_p \mapsto C_\ell$, $x \in \mathcal{P}$ and $C \in \mathcal{C}$, induces a polarity in the symplectic quadrangle $\Gamma_\square$. By the definition of incidence in $\Gamma_\square$, the set of absolute points of this polarity is exactly the set of points of Γ. An arbitrary plane section is a set of points collinear with some non-absolute point X of the quadrangle (use the symplectic polarity). Hence $X = C_p$ for some $C \in \mathcal{C}$. But the set of absolute points collinear with C_p is exactly C viewed as the set of elements of $\mathcal{P}$. By TITS [7],Théorème 3.6, Γ is the inversive plane arising from a Suzuki-Tits ovoid.

Suppose now that Γ is a special STi-plane. Let L be any line of $PG(3, \mathbb{K})$ not belonging to $\Gamma_\square$. Then L is a set of points collinear with two non-collinear points X and Y of $\Gamma_\square$. If L contains a point of Γ, then it contains exactly two points of Γ (since in that case X and Y must be circles which intersect in two points). All such lines lie in the same orbit under the action of the associated Suzuki group by the doubly-transitivity of these groups (see TITS [7],Théoreème 6.1) on the points of the ovoid. If L does not contain a point of Γ, then, if X

and Y are circles, they are disjoint. By Axiom (P′), all points of L are circles of Γ meeting in two points x and y (by Lemma 5). Clearly x and y belong to the line L^θ, where θ is the symplectic polarity of $PG(3,\mathbb{K})$ associated with $\Gamma_\Box$. Hence all such lines are also in the same orbit (since L and L^θ are determined by each other). So the corresponding Suzuki group is special.

If Γ is finite and of even order q, i.e. every circle contains $q+1$ points and there are in total q^2+1 points, then every circle C not containing a certain point x is an oval in the *internal* affine plane Γ_x of order q in x (which may be defined as follows: the points of the plane are the points of Γ distinct from x; the lines are the circles through x; incidence is the natural one). Hence this oval has a unique nucleus (see for instance Barlotti [1]) which lies in the affine plane (and not at infinity because otherwise the line at infinity would be a tangent line and the oval would only contain q affine points). This observation already shows that Axioms (CH1) and (CH2) are satisfied. We now show Axiom (P′). Suppose C_1 and C_2 are disjoint circles both touching the disjoint circles D_1 and D_2. For every point x of D_1, there exists a unique circle C_x containing x and touching both D_1 and D_2 (indeed, the circle C_x is the unique line of Γ_x parallel to D_1 and containing the nucleus of D_2 in Γ_x). The circle C_x is disjoint form C_1. For otherwise they meet in two points u and v and all circles through u and v touch both D_1 and D_2. Hence also the circle C' containing u and v and $D_1 \cap C_2$. Since C_2 and C' touch D_1 in the same points, they touch each other. But since they both touch D_2, they should be equal by Axiom (CH2), which we already proved, a contradiction. Similarly, one shows that for distinct $x, y \in D_1$ the circles C_x and C_y are disjoint. Hence we obtain a set of $q+1$ (the number of points of D_1) disjoint circles. This is a contradiction since there are only $q^2+1 < (q+1)^2$ points in total.

There remains to show that the inversive plane arising from the Suzuki-Tits ovoid satisfies the given axioms. Axioms (CH1), (CH2) and (ST) follow immediately from Van Maldeghem [8], Lemmas 4.1, 4.2 and 4.3. There remains to show Axiom (P) (in the general case) and Axiom (P′) (in the special case).

Axiom (P) is a direct consequence of the following observations. Let the Suzuki-Tits ovoid be defined as the set of absolute points of a polarity in the symplectic quadrangle $W(\mathbb{K})$ naturally embedded in $PG(3,\mathbb{K})$. Each circle of a perfect Suzuki-Tits ovoid defines a plane of $PG(3,\mathbb{K})$; two circles touch if and only if these planes meet in a line of the generalized quadrangle, hence they are conjugated with respect to the symplectic polarity θ defining $W(\mathbb{K})$. So disjoint circles have planes meeting in a non-singular line L and every circle meeting both circles has a plane which contains L^θ. Axiom (P) now follows from the fact that every plane through L is conjugated with every plane through L^θ.

In the special case, either L or L^θ contains two points of the Suzuki-Tits ovoid and hence all circles arising from intersections with planes through one of these lines meet (in exactly two points). This shows Axiom (P′) in the special case.

The Main Results are now completely proved.

REMARK 1. By choosing coordinates in $PG(3,\mathbb{K})$, it is not difficult to show that the Suzuki group defined over $\mathbb{K}$ is special if and only if the equation

$$(x^{\sigma+2}+x+a)(x^{\sigma+2}+x+a+1) \quad = \quad 0 \tag{1}$$

has at least one solution, for every $a \in \mathbb{K}$ (where σ is the corresponding automorphism whose square is the Frobenius automorphism). In this case, one factor has no solution and the other one exactly two.

Raising the first factor of Equation 1 to the power σ respectively 2, we obtain

$$x^{2\sigma+2} + x^{\sigma} = a^{\sigma} \tag{2}$$
$$x^{2\sigma+4} + x^{2} = a^{2} \tag{3}$$

Multiplying Equation 2 by x^2 and adding the result to the sum of Equation 3 and the first factor of Equation 1, one obtains

$$(a^{\sigma} + 1)x^2 + x = a^2 + a,$$

which is equivalent to (by putting $y = (a^{\sigma} + 1)x$)

$$y^2 + y = (a^2 + a)(a^{\sigma} + 1). \tag{4}$$

Putting $z = y + a$ in Equation 4, we obtain

$$z^2 + z = (a^2 + a)(a^{\sigma}), \tag{5}$$

which is the same equation as Equation 4 with a substituted by $a + 1$. Hence if Equation 1 has a solution, then so does Equation 5. Now consider the quotient field $\mathbb{K}$ of the ring of (finite) "polynomials" over $\mathbb{Z}/2\mathbb{Z}$ in the indeterminate t with powers in $\mathbb{Z}[\frac{\sqrt{2}}{2}]$. Put $a = t^{\sqrt{2}-1}$ in Equation 5, then it is easily seen that

$$z = \sum_{i=0}^{+\infty} t^{2^i} + \sum_{j=0}^{+\infty} (t^{\sqrt{2}})^{2^j}$$

is a solution in the completion of $\mathbb{K}$ with respect to the natural valuation. But this solution does not lie in $\mathbb{K}$ since it is non-repeting. Hence we have established a field $\mathbb{K}$ of characteristic 2 in which there exists a map σ (raising to the power $\sqrt{2}$) whose square is raising to the power 2 over which the corresponding Suzuki group is not special, as we promised in the introduction.

Suzuki-Tits ovoids which are not special have some properties which seem unusual compared to the finite case. For example, there are two kinds of linear *flocks* (cp. THAS [5]). One kind partitions the ovoid minus two points into ovals using planes through one fixed line; the other kind partitions the whole ovoid into ovals using planes through a fixed line. As pointed out to me by the referee, also some elliptic quadrics have these two kinds of linear flocks, namely, those defined over a field $\mathbb{K}$ which admits two non-isomorphic quadratic extensions.

REMARK 2. If $|\mathbb{K}| = 8$, then the circles of a Suzuki-Tits ovoid over $\mathbb{K}$ are *pointed conics* in the sense of SEGRE [4] (cp. BARLOTTI [1]), and their nuclei are exactly the corners in our sense.

References

[1] A. Barlotti, Some topics in finite geometrical structures, *Institute of Statistics Mimeo Series* **439** (1965).

[2] D. G. Glynn, The Hering classification for inversive planes of even order, *Simon Stevin* **58** (1984), 319 – 353.

[3] A. E. Schroth, Characterising symplectic quadrangles by their derivations, *Arch. Math.* **58** (1992), 98 – 104.

[4] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, *Acta Arith.* **5** (1959), 315 – 332.

[5] J. A. Thas, Flocks of finite egglike inversive planes, **in** *Finite Geometric Structures and their Applications* (Bressanone, 1972), Cremonese, Roma (1973), 189 – 191.

[6] J. A. Thas, Flocks of non-singular ruled quadrics in $PG(3,q)$, *Atti Accad. Naz. Lincei Rend.* **59** (1975), 83 – 85.

[7] J. Tits, Ovoïdes et groupes de Suzuki, *Arch. Math.* **13** (1962), 187 – 198.

[8] H. Van Maldeghem, The geometry of traces in Ree octagons, **in** *Groups of Lie Type and their Geometries* (ed. W. M. Kantor & L. Di Martino), Cambridge University Press, *London Math. Soc. Lect. Notes Ser.* **207** (1995), 157 – 171.

[9] H. Van Maldeghem, A geometric characterization of the perfect Ree-Tits generalized octagons, I. Octagons satisfying certain geometric conditions, *preprint*.

Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra
University of Ghent
Galglaan 2,
9000 Gent
BELGIUM