

A geometric construction of elliptic conic bundles in \mathbf{P}^4

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Dedicated to the memory of Nobuo Sasakura

In the fall of 1996 Abo, Decker and Sasakura [ADS] discovered smooth elliptic conic bundles in \mathbf{P}^4 . They show that these surfaces all arise as degeneracy loci of four sections of a fixed rank-5 vector bundle on \mathbf{P}^4 . In particular the family of elliptic conic bundles in \mathbf{P}^4 is irreducible.

An elliptic conic bundle in \mathbf{P}^4 is a smooth surface $S \subset \mathbf{P}^4$ with a map $S \rightarrow C$ onto an elliptic curve C , whose general fibers are embedded as smooth conic sections in \mathbf{P}^4 . In this note I give a construction of the general elliptic conic bundle in \mathbf{P}^4 via Cremona transformations. More precisely I construct the general elliptic conic bundle which has a minimal model isomorphic to a smooth quintic elliptic scroll in \mathbf{P}^4 , i.e. a scroll with minimal selfintersection of a section equal to 1. I do not prove that every conic bundle have a minimal model like that, but that the general one does.

For any smooth elliptic quintic scroll S_5 in \mathbf{P}^4 we find reducible curves $G \cup L$ consisting of a rational normal quartic curve G which meet the scroll in 10 points and a line L which is secant to G and a ruling in the scroll. The quadric hypersurfaces passing through $G \cup L$ define a Cremona transformation of \mathbf{P}^4 , and the image under this transformation of the elliptic scroll is a smooth conic bundle in \mathbf{P}^4 . Conversely, any elliptic conic bundle in \mathbf{P}^4 which have a minimal model isomorphic to a scroll S_5 , contains reducible curves $C_2 \cup C_3$ consisting of a curve C_2 of degree 6 and genus 4 and a curve C_3 of degree 9 and genus 6 meeting C_2 in 6 points three on each of a pair of skew lines. The curve C_2 lies on a quadric surface, while the curve C_3 lies on a rational cubic scroll which meet the quadric surface along the two lines. The cubic hypersurfaces through the union of the quadric and the cubic surface defines a Cremona transformation inverse to the one above, and the image of the elliptic conic bundle is a smooth elliptic scroll of degree 5.

A conic bundle S is not minimal, and the points blown up on a minimal model S_5 to obtain a surface S may not be chosen in general position. The Cremona transformation gives an extrinsic description in the \mathbf{P}^4 of S_5 of how the points chosen as part of the intersection of S_5 and a rational normal quartic curve gives rise to a surface S . An intrinsic characterization of the 8 points blown up is given in section 3 and compared with the Cremona construction.

1. Background and minimal models.

1.1. Let H denote the hyperplane section of S , and K the canonical divisor and let $g(H)$ denote the sectional genus, i.e. by adjunction $2g(H) - 2 = H^2 + H \cdot K$.

Recall from [ES, BR]:

1.2 Proposition. *Any elliptic conic bundle S in \mathbf{P}^4 has degree 8, sectional genus 5 has 8 singular conic fibers. The elliptic threefold scroll V_S of planes generated by the conic fibers is a cone, has degree 4 and contains one or two planes which meet every plane in the ruling along a line and the conic bundle in plane quartic curves.*

Proof. An elliptic conic bundle is a smooth surface fibered over an elliptic curve in conic sections. By adjunction $|H + K|$ is composed of this fibered structure so $(H + K)^2 = 0$. The Euler characteristic of S is $\chi(\mathcal{O}_S) = 0$. The double point formula for surfaces in \mathbf{P}^4 , cf. [H], yields the relation:

$$(H^2)^2 - 10H^2 - 5H \cdot K - 2K^2 = 0.$$

Combined with $(H + K)^2 = 0$ and the adjunction formula, the sectional genus is

$$g(H) = \frac{1}{2}((H^2)^2 - 7H^2 + 2).$$

The Castelnuovo bound for space curves, cf. [H], says

$$g(H) \leq \frac{1}{4}((H^2)^2 - 1) - H^2 + 1,$$

so $H^2 \leq 10$. On the other hand eliminating $H \cdot K$ from the double point formula yields $K^2 = 15H^2 - 2(H^2)^2$. Since $K^2 \leq 0$ we get the lower bound $H^2 \geq 8$, so $8 \leq H^2 \leq 10$. For $H^2 = 10$ the sectional genus is 16, which means that the surface is a complete intersection of a quadric and a quintic hypersurface, while $H^2 = 9$ yields the sectional genus 10, which means that the surface is a complete intersection of two cubic hypersurfaces. Either case is clearly absurd, so $H^2 = 8$ is the only possibility. By adjunction $g(H) = 5$, and $K^2 = -8$ so there are 8 singular conic fibers. The threefold scroll V_S of planes generated by the conic sections, have a natural desingularization V in the incidence variety I of points and planes in $\mathbf{P}^4 \times \text{Grass}(2, \mathbf{P}^4)$:

$$I = \{(x, P) | x \in P \subset \mathbf{P}^4, P \in \text{Grass}(2, \mathbf{P}^4)\} \subset \mathbf{P}^4 \times \text{Grass}(2, \mathbf{P}^4)$$

The preimage of S in V is a smooth surface, in fact isomorphic to S , and the projection to the Grassmannian defines the conic bundle structure via this isomorphism. Thus the elliptic base curve C has a natural embedding in $\text{Grass}(2, \mathbf{P}^4)$. We denote the natural projections, and their restrictions to S by $p : V \rightarrow V_S$ and $q : V \rightarrow C$. If h is the pullback on V of the hyperplane in \mathbf{P}^4 and f is the class of a plane, and d is the degree of V_S , then $S \cong 2h + (8 - 2d)f$, numerically. The canonical divisor on V is $K_V \cong -3h + df$, so by adjunction, $K_S \cong (-h + (8 - d)f)|_S$. But $K_S^2 = -8$ so $(-h + (8 - d)f)^2 \cdot (2h + (8 - 2d)f) = 2h^3 + (8 - 2d - 32 + 4d)h^2f = 4d - 24 = -8$, i.e. $d = 4$. Clearly the degree of V_S is also the Plücker degree of the C in the Grassmannian. So C is an elliptic quartic curve. It spans a \mathbf{P}^3 . The intersection of this \mathbf{P}^3 with the Grassmannian is not proper, in fact it is a quadric surface: The Grassmannian of planes in \mathbf{P}^4 is isomorphic to the Grassmannian of lines in \mathbf{P}^4 . Via this isomorphism the curve C correspond to an elliptic quartic surface

scroll S_C in \mathbf{P}^4 . This scroll spans \mathbf{P}^4 only if it is a cone. Via the above isomorphism again S_C is a cone if and only if V_S is degenerate, while S_C is degenerate if and only if V_S is a cone. Since V_S is not degenerate, it must therefore be a cone. In this case C lies in a Schubert variety corresponding to all planes through a point. This Schubert variety is a quadric 4-fold in a \mathbf{P}^5 , and may be identified with a Grassmannian of lines in a \mathbf{P}^3 . The \mathbf{P}^3 of C is contained in this \mathbf{P}^5 and intersect the quadric hypersurface in a quadric surface. The pencil of \mathbf{P}^4 s defining the \mathbf{P}^3 of C is generated by two tangent hyperplanes to the quadric 4-fold. These tangent hyperplanes correspond to Schubert varieties of planes through the vertex which meet a given plane through the vertex in a line. If the quadric surface is singular, then the tangent hyperplanes coincide, and all planes in V_S meet one plane in a line (through the vertex). If the quadric surface is smooth, then every plane in V_S meet two distinct planes in a line. In either case the special planes are contained in V_S . Since C is the intersection of the quadric surface with another quadric it meets every ruling of either in 2 points. For the rulings belonging to the Grassmannian, this means that the corresponding planes meet along a line. These planes together with one of the special planes, which meet every plane in V_S in line, span a hyperplane. Therefore the intersection of the special plane with S is a curve of degree at most 4. It is clearly a bisection, i.e. it meets every ruling in two points. On the other hand through every line through the vertex there are two planes of the ruling, since C is of type $(2, 2)$. Therefore the special plane meet S in a curve of degree 4. \square

1.3 Minimal models. The singular conic fibers of the map $q : S \rightarrow C$, which defines the conic bundle structure, consists of pairs of (-1) -lines on the surface. Since there are 8 singular fibers, there are therefore 2^8 different minimal models for S , and each minimal model is a \mathbf{P}^1 -bundle over an elliptic curve. On a \mathbf{P}^1 -bundle the parity of the selfintersection of all sections coincide. The choice of (-1) -line to blow down in a singular conic on S corresponds precisely to this choice of parity downstairs. There are therefore 2^7 minimal models of each parity, which we call even or odd accordingly. Any section of the map q will together with 8 exceptional curves and the pullback of the Picard group of the elliptic base curve C generate the Picard group of S . Given a section C_0 with selfintersection $C_0^2 = a$ on S , one may always blow down all the (-1) -lines on S which do not meet C_0 to get a minimal model. Thus the selfintersection of C_0 on the minimal model is unchanged. Numerically $H \cong 2C_0 + (4 - a)F - E_1 - \dots - E_8$, so $H \cdot C_0 = 2a + 4 - a = a + 4 = C_0^2 + 4$. Since C_0 is elliptic the degree is at least 3 so the selfintersection is at least -1 . In fact every section C_0 with selfintersection $C_0^2 = -1$ is a plane cubic curve on S . On the other hand any plane cubic curve on S is contained in a pencil of hyperplanes, so it is at most a bisection on S . If it was a bisection, the residual moving part of degree 5 of the pencil of hyperplane sections is supported on a finite number of fibers. But this is impossible since the residual moving part would have to include an exceptional line and these do not move. Therefore every plane cubic curve on S is a section with selfintersection -1 . Thus

1.4 Lemma. *S has 2^7 odd minimal models, and the minimal selfintersection of a section is in each case 1 or -1 . The sections with selfintersection -1 correspond precisely to the plane cubic curves on S .*

1.5 Remark. With minimal selfintersection of a section equal to 1, the minimal model is isomorphic to a smooth elliptic quintic scroll in \mathbf{P}^4 . Since the family of elliptic conic bundles is irreducible, and conic bundles with plane cubic curves cannot degenerate to conic bundles without plane cubic curves, it is clear that the conic bundles with a minimal model isomorphic to a smooth quintic scroll are the general ones on the family. On the other hand it is not clear whether every conic bundle S has a minimal model isomorphic to a smooth quintic scroll.

2. The Cremona construction

2.1. Consider an elliptic quintic scroll S_5 and a general complete intersection of two cubics containing it. The residual to S_5 is a smooth Veronese surface V_4 , the smooth projection of a Veronese surface in \mathbf{P}^5 . The curve $D = S_5 \cap V_4$ is linearly equivalent to $H - K$ on each of the two surfaces. Therefore it is a plane quintic curve on the plane (i.e. of degree 10 on the Veronese surface), while it is a 3-section on the scroll. Now, pick a general member of the ruling of S_5 . It is a line L , which meet D in 3 distinct points. Pick 2 of the three points, and consider the rational quartic curves on V_4 which pass through these two points. These curves form a web, while the hyperplane sections of V_4 through the two points form a net, so the general member of the web spans \mathbf{P}^4 . Pick a smooth member G which spans \mathbf{P}^4 . The union $B = L \cup G$ has arithmetic genus 1, and has ideal generated by the 5 quadric Pfaffians of a skewsymmetric 5×5 matrix with linear entries. B is a degeneration of an elliptic normal quintic curve. The 5 quadrics which cut out B define a Cremona transformation ϕ_B of \mathbf{P}^4 , cf. [S]. This Cremona transformation collapses the secant lines to B . More precisely, let $U \rightarrow \mathbf{P}^4$ be the blowup of \mathbf{P}^4 in the ideal of B , and let Z_U be a scheme of length 2 on U whose image Z in \mathbf{P}^4 also has length 2. Then Z_U is mapped to a point by $\phi_B : U \rightarrow \mathbf{P}^4$, if and only if the line L_Z generated by Z intersect B in a scheme of length 2. From various projections of B to a plane one easily derives that the secants of B form a reducible hypersurface. The secants to G is a cubic hypersurface, while the lines joining L and G form a quadric hypersurface, singular along L .

2.2. We restrict ϕ_B to S_5 . Since G meets D in 10 points, and 2 of them lie on L , we see that B intersect S_5 in L and in 8 additional points. Therefore ϕ_B restrict to S_5 as a map defined by a linear system $|2H - L|$ with assigned basepoint at these 8 points. Let S denote the blow up of S_5 in these 8 points, and let E, \dots, E_8 be the exceptional curves. The linear system on S becomes

$$|h| = |2H - L - E_1 - \dots - E_8|,$$

where H and L are the pullback of the corresponding divisors on S_5 . For very ampleness of $|h|$, we investigate the intersection of secants to B with S_5 . Consider first a subscheme Z of length 2 on S_5 which does not intersect B and for which the line L_Z generated by Z is a secant line to B . Since L_Z has a scheme of length 4 of intersection with $S_5 \cup V_4$, and this union is a complete intersection of two cubics, it must be contained in the union of the two surfaces. But V_4 contains no lines, so L_Z must be a ruling of S_5 . Thus Z is collapsed by ϕ_B only if L_Z is a ruling of S_5 , which is a secant line to G .

Next, consider a subscheme Z_U of length 2 on S whose image Z on S_5 has length 2 and intersect B in a point. If Z_U is collapsed to a point, the line L_Z generated by Z must be

a secant to B . There are two possibilities. If Z does not intersect L , then L_Z must be a trisecant to S_5 , tangent to S_5 at a point of intersection with G . In addition it must be a secant line to G , so it intersects the union $S_5 \cup V_4$ in a scheme of length at least 4. As above it must be a member of the ruling of S_5 , which is at the same time a secant to G . If Z intersect L , the line L_Z must again be a trisecant to S_5 , tangent to S_5 at a point on L . It is also a secant to B so it must intersect G at some point disjoint from L . Again L_Z is a 4-secant to the union of the two surfaces, so it must be a member of the ruling, i.e. it must coincide with L , absurd. Finally we consider a subscheme Z_U of length 2 of S whose image Z on S_5 is contained in B . If Z has length 2 then L_Z is a 4-secant to S_5 tangent at two points, but this is impossible, since S_5 is cut out by cubics, unless L_Z is a member of the ruling. If Z is a point, it is one of the 8 points of intersection of G and S_5 outside L . But, the Cremona transformation has no basepoints on the blowup, so every exceptional curve of S is embedded as a line, and has therefore no double points.

As a conclusion

2.3 Proposition. *The Cremona transformation defines an embedding of S as soon as G meets the scroll S_5 in 10 distinct points, and no secant of G except L coincide with a ruling of S_5 .*

2.4 The converse Cremona transformation. The secants of B have two components, the secant variety of G , a cubic hypersurface V_3 , and the join of L and G which is a rank 3 quadric V_2 , singular along L . Both of these are collapsed to surfaces by the Cremona transformation. The map defined by all quadrics through G maps V_3 to a Veronese surface in \mathbf{P}^5 , the 5 quadrics of the Cremona transformation projects this Veronese surface from a point on the surface, the image of the secant line L , to a cubic scroll S_3 in \mathbf{P}^4 . The quadric V_2 is one of the quadrics which define the Cremona transformation, so its image is contained in a hyperplane. The pencil of lines joining L and a fixed point on G sweep out a plane. This plane and G lie in 3 quadrics so the image of the pencil is a line. Clearly distinct points on G generate disjoint pencils with disjoint image lines by the Cremona, so the image of V_2 is a smooth scroll, i.e. a quadric surface S_2 . The opposite ruling on S is formed by the pencils of lines joining G and a fixed point on L . The intersection of the surfaces S_2 and S_3 is formed by the two pencils of secants to G from the two points of intersection between G and L . Thus $S_2 \cap S_3$ form two skew lines. The cubic scroll S_3 is smooth and the directrix lie in the hyperplane of S_2 and intersect S_2 properly in two points.

The converse Cremona transformation is defined by the cubic hypersurfaces through the quadric surface S_2 and the cubic scroll S_3 . The union $T = S_2 \cup S_3$ is a degenerate quintic elliptic scroll, its ideal is generated by 5 cubics. The converse Cremona collapses subschemes of length 2 precisely when the line generated by the subscheme intersect T in a scheme of length 3, i.e. is a trisecant to T . In the case of a smooth elliptic quintic scroll, the trisecants are trisecants of an elliptic pencil of plane cubic curves on the surface. In our degenerate case, the plane cubic curves belong to two families: the pencil of planes through the directrix of S_3 inside the span of S_2 , this family of planes form a line in the Grassmannian of planes and is mapped to the line L by the Cremona transformation. The other family of plane cubic curves on T , are defined by the by planes through the rulings

of S_2 which are secants to S_3 . Each secant line on S_3 is a secant to a unique conic section on S_3 . This conic section spans a plane which therefore intersect T in a cubic curve. The family of planes defined this way form a rational curve in the Grassmannian of planes, and it has two planes in common with the linear pencil of planes. Each of these planes are collapsed to a point by the Cremona transformation, and the family is mapped to the rational normal quartic curve G .

2.5. The surface S intersect S_2 and S_3 in curves C_2 and C_3 . C_2 is part of a hyperplane section, it is the section defined V_2 , since this quadric is singular along L , C_2 is equivalent to $h - L$, i.e. a curve of degree 6 and genus 4. Since a curve of degree 6 and genus 4 is a complete intersection $(2, 3)$ in a \mathbf{P}^3 , we see that S_2 is determined by C_2 . The curve C_3 is defined by the cubic V_3 , since this hypersurface is the secant variety of G , it contains L and is singular in all the eight points which are blown up. Thus

$$C_3 \equiv 3H - L - 2E_1 - \dots - 2E_8 \equiv 3h - H - C_2 \equiv 2h - H + L.$$

We see that on S , the curve C_3 is residual to $H - L$ in a quadric. But $H - L$ moves in a net and the general member is a smooth elliptic curve of degree 7 on S . An elliptic curve of degree 7 lies on at least one quadric, so C_3 is the baselocus on S of the quadrics containing the curves in the net $|H - L|$. Thus the quadrics form a net and define a cubic scroll in \mathbf{P}^4 . This scroll of course coincides with S_3 . C_3 has degree 9 and genus 6, so it must be of type $5l - e$ on the scroll, when we think of the scroll as the blowup of the plane in a point, l is the class of a line and e the class of the exceptional curve, the directrix on the scroll. Since $C_2 \cdot C_3 = 6$ the curve C_3 intersect C_2 in 3 points on each of the common rulings of the scrolls S_2 and S_3 .

2.6. We may recover this converse Cremona directly from S for the general conic bundle S with minimal model S_5 isomorphic to a smooth elliptic quintic scroll. For this we start by considering sections C_0 of S_5 with selfintersection 3. By 1.4 such curves always exist and moves in a net without basepoints on S_5 . We denote the strict transform of C_0 on S also by C_0 . It has degree 7 in the embedding in \mathbf{P}^4 and the general member of the net $|C_0|$ on S is smooth. Each curve in $|C_0|$ lie on at least one quadric. Since S does not lie in any quadric hypersurface, any curve C_3 in $|2h - C_0|$ must lie in a net of quadrics. C_3 has degree 9, so the net must define a cubic scroll: Since the general curve C_0 is smooth, the general quadric which contain C_3 is irreducible. Therefore the net of quadrics define at most a surface. Since the intersection of the quadrics contains a curve of degree 9 the intersection is at least a surface, so the intersection must be a surface. It lies on three quadrics so it is a possibly degenerate cubic scroll S_3 .

For the general surface S and linear system $|C_0|$ we may assume that the cubic scroll is smooth, in fact that is the case in the construction above. In the smooth scroll, C_3 must be equivalent to $5l - e$, since it has degree 9 and arithmetic genus 6. So it contains the directrix as a component or meets it in a unique point x . By genericity we assume that C_3 meets the directrix properly in x . Since $|C_0|$ is a net without basepoints, the intersection of S with the cubic scroll is precisely C_3 . Therefore the directrix and L intersect only in x , and they span a unique hyperplane. Residual to L in this hyperplane is a curve C_2 on S which has degree 6 and arithmetic genus 4. This is a complete intersection $(2, 3)$. By

genericity again we may assume that the quadric surface S_2 containing C_2 is smooth, in fact that is the case in the above example. The hyperplane of S_2 intersect the cubic scroll S_3 along the directrix and two rulings. Since the directrix does not lie on the quadric surface, and the intersection $C_2 \cdot C_3 = 6$ the two rulings of the cubic scroll must also be rulings of the quadric surface S_2 . The union $S_3 \cup S_2$ form a degenerate elliptic quintic scroll. The cubics through $S_3 \cup S_2$ define a Cremona transformation of \mathbf{P}^4 , cf. [S]. This transformation maps S onto a smooth quintic elliptic scroll: In fact the restriction of the transformation to the surface S is a linear system

$$|H| = |3h - C_2 - C_3| = |3h - (h - L) - (2h - C_0)| = |C_0 + L|$$

of sections with selfintersection 5 coming from a minimal model isomorphic to a smooth quintic elliptic scroll. Such a linear system is very ample on the minimal model obtained by blowing down all exceptional lines that does not intersect the general curve in the linear system.

We may conclude:

2.7 Proposition. *If S is a general elliptic conic bundle with minimal model isomorphic to a smooth quintic elliptic scroll, then S is obtained by a Cremona transformation from such a scroll.*

2.8 Remark. The construction of elliptic conic bundles in \mathbf{P}^4 described in [ADS] shows in particular that the family of all these conic bundles is irreducible. In the construction of conic bundles from elliptic quintic scrolls in 2.2, it is clear that the genericity conditions of 2.6 are satisfied. Since these conditions are open it follows from 2.7 and remark 1.5 that the general elliptic conic bundle appear via a Cremona transformation.

3. Construction via linear systems and plane quartic curves

3.1. Again we restrict the attention to conic bundles S with minimal model isomorphic to a smooth quintic scroll S_5 . In this case as we saw in 2.2 that the linear system of hyperplane sections on S can be written as

$$|h| = |2H - L - E_1 - \dots - E_8|,$$

where H is the pullback of a hyperplane section on S_5 , the divisor L is the pullback of a ruling and the E_i are exceptional curves. This linear system is special, in fact the 8 points blown up to obtain S from S_5 may not be chosen generally. By Riemann Roch one computes that $h^0(\mathcal{O}_S(h)) = 5$ precisely when $h^1(\mathcal{O}_S(h)) = 1$. The open condition of very ampleness for $|h|$ requires that among the 8 points on S_5 no 2 of the points are infinitely close or lie on the same ruling of S_5 . The more important closed condition that is required for $h^0(\mathcal{O}_S(h)) = 5$ can be understood by looking at a plane quartic curve on S . Recall the projection map $q : S \rightarrow C$ which defines the conic bundle structure on S . The blowup map $\pi : S \rightarrow S_5$ is a factor of the projection q , therefore the curve C now have a natural embedding in the Grassmannian of lines in the \mathbf{P}^4 of S_5 . We denote Plücker divisor on C of this embedding by γ .

Now, on the conic bundle S we know from 1.2 that there is a plane quartic curve linearly equivalent to $A \equiv h - q^*\alpha$ for some divisor α of degree 2 on C . Since D is a plane quartic

curve in the embedding by h this linear system restricts as the canonical linear series on S . The canonical series on S is $K \equiv -2H + q^*\gamma + E_1 + \dots + E_8$. The adjoint series to A is $q^*(\gamma - \alpha) - L \equiv q^*\beta$ the pullback of a divisor of degree 2 on C , and the restriction of h to A coincides with this only if $h - q^*\beta$ restricts trivially to A . When $q^*\alpha$ and $q^*\beta$ are distinct then $h^1(\mathcal{O}_S(h - q^*\beta - A)) = 0$, so $h - q^*\beta$ restricts trivially to A if and only if there is a curve $A_1 \equiv h - q^*\beta$ on S . On S_5 this means that the 8 points blown up are precisely the complete intersection $\pi(A) \cap \pi(A_1)$ on S_5 . The argument now applies also to A_1 , so when α and β are not equivalent as divisors on C , then S have the two plane quartic curves A and A_1 and these determine the exceptional curves completely. If $\alpha \equiv \beta$ then A and A_1 may coincide. In fact in this case the 8 points on $\pi(A)$ belong to the linear series $|\pi(A)|_{\pi(A)}$, which are not the intersection of a curve $\pi(A_1)$ linearly equivalent to $\pi(A)$. This is possible: Since S_5 is irregular, the restriction of $|\pi(A)|$ to $\pi(A)$ is not complete.

3.2 This intrinsic description of the linear system $|h|$ on S , may also be related to the Cremona construction in section 2.

In the Cremona construction a plane quartic curve A on S is defined via the transformation by a pencil of quadrics which contain B and 2 additional rulings of S_5 . Following the construction of the linear system $|h|$ we start with a ruling L and a smooth curve $A_0 \equiv 2H - L - q^*\alpha$ on S_5 for some divisor α of degree 2 on the elliptic base curve. The union $A_0 \cup L$ has degree 8 and arithmetic genus 4 and lies in a pencil of quadrics. By proving that the three quadrics through A_0 define a complete intersection, one can show that the pencil through $A_0 \cup L$ defines a smooth complete intersection. Furthermore, $A_0 \cup L$ lies on many cubic hypersurfaces which does not contain S_5 . A general cubic through $A_0 \cup L$ define together with the pencil of quadrics a complete intersection $(2, 2, 3)$. In this complete intersection $A_0 \cup L$ is linked to a smooth rational normal quartic curve G . In fact any curve in $|G|$ on the complete intersection of the quadrics, are linked to $A_0 \cup L$ in a complete intersection $(2, 2, 3)$. The line L is a secant to G , so the quadrics through $G \cup L$ define a Cremona transformation as above. G intersect A_0 in 8 points, which are blown up on S_5 by the Cremona transformation.

Consider the set of cubics through S_5 and G . There are precisely a pencil containing both. This pencil defines a complete intersection containing S_5 and a quartic surface S_4 . The quartic surface S_4 is a Veronese surface or a degeneration of a Veronese surface. Any such degeneration is reducible and does not contain rational normal quartic curves, so since G is smooth S_4 must be a smooth Veronese surface. Since there is precisely a pencil of cubics through $S_5 \cup G$, the Veronese surface S_4 is completely determined by G . By construction precisely a pencil of the quadrics through $G \cup L$ contain A_0 , while the Cremona transformation has degree 4 on A_0 , so A_0 is canonically embedded as a plane quartic curve by the Cremona transformation. Therefore the above argument applies to give the intrinsic description of the 8 points blown up on S_5 . In particular it follows that A_0 may or may not be the only plane quartic curve on S depending on the choice of L and the divisor α .

3.4 Open problems. The constructions in section 2 and 3 give the general elliptic conic bundle in \mathbf{P}^4 . They leave open whether every conic bundle with minimal model an elliptic quintic scroll can be constructed this way. In particular, are there conic bundles with only

one plane quartic curve? The argument also leaves open the problem whether there are conic bundles S which do not admit S_5 as a minimal model, i.e. for which every minimal model have a -1 -section.

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