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A geometric environment for building-up loops

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Abstract We start from the embedding of the Klein model of a hyperbolic plane $\mathbf{H}$ over a Euclidean field $\mathbf{K}$ in its direct motion group $\mathcal{G} := \mathrm{PSL}_2(\mathbf{K})$ and of both in $\mathrm{PG}(3, \mathbf{K})$. We present a geometric procedure to obtain loops which are related to suitable regular subsets of direct motions as transversals of the coset space $\mathcal{G}/\mathcal{D}$, where $\mathcal{D}$ is the subgroup of hyperbolic rotations fixing a given point $o \in \mathbf{H}$. We investigate some properties of such loops and we determine their automorphism groups.

Keywords loop · transversal · section · hyperbolic plane · direct motion group · automorphism

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1 Introduction

It is well known that the point-set of a hyperbolic plane over a Euclidean field $\mathbf{K}$ can be equipped with the algebraic structure of a *Bruck loop* (or *K-loop*, see [6]) which is obtained in a standard way (called *loop derivation*) by means of the regular involution set of the point reflections. The recent papers [4] and

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[12] introduce another class of loops associated to a general hyperbolic plane, generalizing a topological loop firstly constructed by Nagy and Strambach in [10]: in [4] a construction of a loop arising from a suitable selected subset of limit rotations is proposed. In the subsequent paper [12], exploiting a nice geometric representation of the hyperbolic plane motion group $\mathrm{PGL}_2(\mathbf{K})$ as the point-set of $\mathrm{PG}(3, \mathbf{K})$ deprived of a ruled quadric $\mathcal{Q}$, the authors characterize the left multiplications of the limit rotation loop as one of the two sheets $\mathbf{\Lambda}^+$ of the tangent cone $\mathbf{\Lambda}$ to $\mathcal{Q}$ having vertex the point $\mathbf{1}$ (the identity of $\mathrm{PGL}_2(\mathbf{K})$). Moreover the half-cone $\mathbf{\Lambda}^+$ itself turns out to be a *transversal* of a suitable coset space of the group $\mathrm{PSL}_2(\mathbf{K})$, thus it is equipped with a left loop operation, which is indeed a loop operation isomorphic to that of the limit rotation loop via a well known algebraic technique (see e.g. [7]) which, in this particular case, produces a left conjugacy closed loop as an *invariant section* of the group $\mathrm{PSL}_2(\mathbf{K})$ in the sense of [9]. In the present paper we carry on the research begun in [4] and [12] and, starting from the same environment and employing the related geometric insight, we characterize the planar transversals of the same group, which give rise to loops or groups. Moreover this representation makes the determination of the automorphism groups of these loops quite easy.

The paper is organized as follows. After reviewing in Section 2 the algebraic background involving loops, coset spaces, sections and transversals, loop derivations, in Section 3 the geometric setting we are working with is presented. In Section 4 we prove some algebraic properties of the loops arising as transversals by means of the geometric properties of $\mathrm{PG}(3, \mathbf{K}) \setminus \mathcal{Q}$.

In Sections 5 and 6 we obtain our main results. Section 5 deals with the case of *planar transversals*, namely transversals arising from planes of $\mathrm{PG}(3, \mathbf{K})$, and we prove that such transversals are given only by those planes through the point $\mathbf{1}$ which are tangent or external to the cone $\mathbf{\Lambda}$. They are loops, indeed groups for tangent planes. Finally in Section 6 we prove some general results on the automorphism group of a loop transversal and we fully determine it in the cases of planar transversals and of the limit rotation loop.

2 Algebraic background

A non-empty set L with a binary operation " $\cdot$ " is called *left (right) loop* if there is a *neutral element* $1 \in L$, such that $\forall\, a \in L:\ a \cdot 1 = 1 \cdot a = a$ and there is a unique solution $x \in L$ of the equation $a \cdot x = b$ $(x \cdot a = b)$ for all $a, b \in L$. If $(L, \cdot)$ is a left loop, consider for all $a \in L$ the *left translation*, namely the permutation $\lambda_a : L \to L;\ x \mapsto a \cdot x$, the set $\lambda(L) = \{\lambda_a \in \mathrm{Sym}\, L \mid a \in L\}$ and the *left translation group* of L, namely the group $\mathcal{M}_\ell := \langle \lambda(L) \rangle$. For all $a, b \in L$ denote by $\delta_{a,b} := \lambda_{a\cdot b}^{-1} \circ \lambda_a \circ \lambda_b \in \mathcal{M}_\ell$ the *precession map* and let $\Delta := \langle \{\delta_{a,b} \mid a, b \in L\} \rangle$ be the *left inner mapping group*. It is known (see e.g. [7, (2.6)]) that $\Delta = \{\alpha \in \mathcal{M}_\ell \mid \alpha(1) = 1\}$.

A *loop* $(L, \cdot)$ is both a left and right loop. In a loop $(L, \cdot)$ for all $a \in L$ we can consider the *right (left) inverse* $a_r^{-1} \in L$ $(a_\ell^{-1} \in L)$ such that $a \cdot a_r^{-1} = 1$

($a_\ell^{-1} \cdot a = 1$). When $a_r^{-1} = a_\ell^{-1} =: a^{-1}$ we shall say that a^{-1} is the *inverse* of a.

A loop $(L, \cdot)$ is called:

- a *loop with inverses* if every element of L possesses an inverse;
- a loop with the *left inverse property* if $\forall\, x, y \in L :\ x_\ell^{-1}(xy) = y$;
- a *left conjugacy closed loop* if $\forall\, x, y \in L :\ \lambda_x^{-1}\lambda_y\lambda_x \in \lambda(L)$;
- a *Bol loop* if $\forall\, x, y \in L :\ \lambda_x\lambda_y\lambda_x = \lambda_{(x(yx))}$;
- a *K-loop* (or *Bruck loop*, see [8]) if L is a Bol loop (hence a loop with inverses) and moreover the following *automorphic inverse property* holds true: $\forall\, x, y \in L :\ (xy)^{-1} = x^{-1}y^{-1}$.

There are many techniques to obtain loops. We now describe two of them we will use in the following sections.

Consider a *regular permutation set* (L, Γ), where L is a non-empty set and Γ is a set of permutations acting regularly on L, and fix an element $o \in L$ (we shall write (L, Γ, o)). For all $a, b \in L$ the unique element of Γ mapping a to b is denoted by $\widetilde{a\,b}$ (when $a = b$ we simply write $\widetilde{a}$).

Starting from (L, Γ, o) we can define the following binary operation on L (see e.g. [2]): $\forall\, a, b \in L :\ a \cdot b := \widetilde{oa} \circ \widetilde{o}^{-1}(b)$. Then $(L, \cdot)$ is a loop with neutral element o called the *loop derivation* of (L, Γ, o) and denoted by $\mathcal{L}(L, \Gamma, o)$.

For the second construction let us consider a group $\mathcal{G}$ and a subgroup $\mathcal{D}$ and let $\mathcal{G}/\mathcal{D}$ be the *coset space*, namely the set of all left cosets of $\mathcal{D}$ in $\mathcal{G}$, and $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{D}$ the quotient map, thus $\pi(g) = g\mathcal{D}$. A *section* of the coset space $\mathcal{G}/\mathcal{D}$ is a map $\sigma : \mathcal{G}/\mathcal{D} \to G$ such that $\pi \circ \sigma = \mathrm{id}$ and $\sigma(\mathcal{D}) = 1$. Then $L := \sigma(\mathcal{G}/\mathcal{D})$ is a complete set of representatives of the left cosets of $\mathcal{D}$ in $\mathcal{G}$ with $1 \in L$, i.e.:

$$\forall\, g \in \mathcal{G} : \quad |g\mathcal{D} \cap L| = 1 \quad \text{and} \quad 1 \in L.$$

Such a set of representatives is called a *transversal* of $\mathcal{D}$ in $\mathcal{G}$.

In a transversal L of a coset space $\mathcal{G}/\mathcal{D}$ for all $a, b \in L$ there are unique $a * b \in L$ and $d_{a,b} \in \mathcal{D}$ such that $ab = (a * b)d_{a,b}$, thus, following [7, § 2.B], the set L can be equipped with a left loop operation $a * b := (ab)d_{a,b}^{-1}$ and the binary operation "$*$" can be characterized also in set-theoretic terms in the following way:

$$\forall\, a, b \in L : \quad a * b = \sigma(ab\mathcal{D}) = ab\mathcal{D} \cap L.$$

Moreover the left loop $(L, *)$ is in fact a loop if and only if for every $g \in \mathcal{G}$ it holds that L is a transversal of $g\mathcal{D}g^{-1}$ (see [7, (2.7)])

Conversely if $(L, \cdot)$ is a left loop then it is known that $\lambda(L)$ is a transversal of $\mathcal{M}_\ell/\Delta$ and the left loop $(\lambda(L), *)$ is isomorphic to $(L, \cdot)$ by [7, (2.11)].

3 The geometric setting

Let $(\mathbf{K}, +, \cdot)$ be a Euclidean[1] field and identify the 3-dimensional projective space $\mathrm{PG}(\mathrm{Mat}_2(\mathbf{K}))$ with $\mathrm{PG}(3, \mathbf{K})$ in the usual way, so that the singular matrices correspond to the points of the ruled quadric $\mathcal{Q}$ of equation $x_1x_4 - x_2x_3 = 0$ and, with abuse of notation, we can write $\mathrm{PGL}_2(\mathbf{K}) = \mathrm{PG}(3, \mathbf{K}) \setminus \mathcal{Q}$. Note in particular that $\mathrm{PSL}_2(\mathbf{K})$, identified with the subgroup $\{\mathbf{K}^*A \mid A \in \mathrm{Mat}_2(\mathbf{K}),\ \det A > 0\} \leqslant \mathrm{PGL}_2(\mathbf{K})$, consists exactly of one of the two disjoint parts in which $\mathrm{PG}(3, \mathbf{K})$ is divided by the quadric $\mathcal{Q}$, namely the one that contains the identity point $\mathbf{1} = \mathbf{K}^*(1, 0, 0, 1)$. In the following we shall denote by ω the polarity induced by $\mathcal{Q}$ in $\mathrm{PG}(3, \mathbf{K})$.

If we consider the Klein model of the hyperbolic plane[2] over $\mathbf{K}$ it is well known that the group of motions of such plane is isomorphic to the group $\mathrm{PGL}_2(\mathbf{K})$ of projectivities of the conic bordering the model. In particular the elements of $\mathrm{PSL}_2(\mathbf{K})$ can be identified with the direct motions. Moreover the Klein model of the hyperbolic plane itself can be embedded in $\mathrm{PG}(3, \mathbf{K}) \setminus \mathcal{Q}$ as:

$$\mathbf{H} := \{\mathbf{K}^*A \in \mathrm{PSL}_2(\mathbf{K}) \mid \mathrm{tr}\, A = 0\}$$

namely as the internal points of the conic $\mathcal{C}$ obtained as intersection of $\mathcal{Q}$ with the polar plane π_∞ of the point $\mathbf{1}$, of equation $x_1 + x_4 = 0$. The points of $\mathcal{C}$ are in fact of the form:

$$\mathcal{C} := \{\mathbf{K}^*A \mid A \in \mathrm{Mat}_2(\mathbf{K}),\ \det A = 0 \ \wedge\ \mathrm{tr}\, A = 0\}\,.$$

Note that in this representation each point a of the hyperbolic plane is identified with the element of $\mathbf{H}$ corresponding to the central symmetry fixing a, hence the action of $\mathrm{PGL}_2(\mathbf{K})$ on $\mathbf{H}$ is by conjugation, i.e. for all $\mathbf{K}^*A \in \mathrm{PGL}_2(\mathbf{K})$ we consider:

$$\widehat{\mathbf{K}^*A} : \begin{cases} \mathbf{H} & \longrightarrow \mathbf{H} \\ \mathbf{K}^*X & \longmapsto \mathbf{K}^*(A \cdot X \cdot A^{-1}). \end{cases}$$

In the following we will consider also the action of an element $\mathbf{K}^*A \in \mathrm{PGL}_2(\mathbf{K})$ on the points of the whole projective space $\mathrm{PG}(3, \mathbf{K})$, extending the action by conjugation described above and denoting it again by $\widehat{\mathbf{K}^*A}$.

From now on we shall denote by $\mathcal{G} := \mathrm{PSL}_2(\mathbf{K})$ and by

$$\mathcal{D} := \left\{ \mathbf{K}^* \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{K}) \mid (a, b) \in \mathbf{K}^2 \setminus \{(0, 0)\} \right\}$$

the subgroup of rotations fixing the point $o := \mathbf{K}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbf{H}$. Note that the subgroup $\mathcal{D}$ is abelian and it is a line in $\mathrm{PG}(3, \mathbf{K})$ with no intersection with $\mathcal{Q}$.

[1] We recall that a Euclidean field $(\mathbf{K}, +, \cdot)$ is an ordered field where the positive elements are exactly the non-zero squares.

[2] For all the properties of the hyperbolic plane over a Euclidean field $\mathbf{K}$ we refer to [1] and [5].

Remark 3.1 If we equip $\mathbf{H}$ with a loop operation "$\cdot$" such that $\lambda(\mathbf{H}) \subseteq \mathcal{G}$, then by [7, (2.11)] $\lambda(\mathbf{H})$ is a transversal of $\mathcal{G}/\mathcal{D}$ and the two loops $(\lambda(\mathbf{H}), *)$ and $(\mathbf{H}, \cdot)$ are isomorphic. The stated isomorphism can be provided explicitly by the map

$$\varphi : \begin{cases} \lambda(\mathbf{H}) \to \mathbf{H} \\ \mathbf{K}^* A \mapsto \widehat{\mathbf{K}^* A}(o) = \mathbf{K}^* A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^{-1}. \end{cases}$$

Let us recall two relevant loops that can be defined on $\mathbf{H}$ obtained by loop derivation $\mathcal{L}(\mathbf{H}, \varGamma, o)$ from suitable regular subsets $\varGamma$ of the direct motion group $\mathcal{G}$.

1. The so called *K-loop of the hyperbolic plane* $(\mathbf{H}, \oplus, o)$ where $\varGamma$ is given by the central symmetries (cf [6]).
2. The *limit rotation loop*, where $\varGamma$ is a selected regular subset of the set of limit rotations of the hyperbolic plane (cf [4]).

A geometric interpretation of the latter loop in the setting here presented is performed in [12], in particular recall that the set $\mathbf{\Lambda}$ of limit rotations of the hyperbolic plane $\mathbf{H}$ corresponds to the tangent lines through $\mathbf{1}$ to the quadric $\mathcal{Q}$ and has equation $(x_1 - x_4)^2 + 4x_2x_3 = 0$, or equivalently

$$\mathbf{\Lambda} = \{\mathbf{K}^* L \in \mathrm{PSL}_2(\mathbf{K}) \mid \mathrm{tr}\, L = \pm 2 \text{ and } \det L = 1\}.$$

Each limit rotation in $\mathbf{\Lambda}$ fixes precisely a point $a \in \mathcal{C}$, thus in the following we shall denote, for all $a \in \mathcal{C}$, by $\varLambda_a := \{\lambda \in \mathbf{\Lambda} \mid \widehat{\lambda}(a) = a\}$ hence $\mathbf{\Lambda} = \bigcup_{a \in \mathcal{C}} \varLambda_a$.

For the K-loop $(\mathbf{H}, \oplus, o)$ we shall provide a geometric interpretation in section 5.

We conclude the section with the following geometric proposition that will be useful in the proof of theorem 5.1.2.

Proposition 3.2 *Let* $\mathbf{K}^* X \in \mathrm{PSL}_2(\mathbf{K}) \setminus \{\mathbf{1}\}$, $a \in \mathcal{C}$ *and* $\alpha = \omega(a)$. *Then* $\widehat{\mathbf{K}^* X}$ *fixes* a *if and only if* $\mathbf{K}^* X \in \alpha$.

Proof First note that each $\widehat{\mathbf{K}^* X}$ is a projectivity of $\mathrm{PG}(3, \mathbf{K})$ that fixes the quadric $\mathcal{Q}$ with its reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ (see [15, Props 3.5 and 3.6]), preserves the polarity induced by $\mathcal{Q}$ and fixes the point $\mathbf{1}$ and its polar plane π_∞. Moreover each point of the line $l := \overline{\mathbf{1}, \mathbf{K}^* X}$ through the points $\mathbf{1}$ and $\mathbf{K}^* X$ is fixed and, denoting by $l' = \omega(l)$ the conjugate line, each point $u \in l' \cap \mathcal{C}$ is fixed as well. In fact let r_1 and r_2 be the two lines of $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively through the point u. Since $u \in l'$ the line l is contained in the polar plane of u (which is the tangent plane to $\mathcal{Q}$ in u), so it has non-empty intersection with both r_1 and r_2 hence $\widehat{\mathbf{K}^* X}(r_1) = r_1$ and $\widehat{\mathbf{K}^* X}(r_2) = r_2$, thus $\{u\} = r_1 \cap r_2$ is fixed as well.

Let us now assume $\mathbf{K}^* X \in \alpha$. Hence $a \in l' \cap \mathcal{C}$ and the result follows.

Conversely assume $\widehat{\mathbf{K}^* X}(a) = a$. If a is the only fixed point of $\widehat{\mathbf{K}^* X}$ on $\mathcal{C}$, then $\mathbf{K}^* X$ is a limit rotation of $\varLambda_a \subseteq \alpha$, thus the result is proven, hence we can assume that $\widehat{\mathbf{K}^*X}$ has a further fixed point $b \in \mathcal{C}$. If, by contradiction, we assume $\mathbf{K}^*X \notin \alpha$ then $a \notin l'$ and, by the previous part of this proof, the projectivity $\widehat{\mathbf{K}^*X}$ fixes each point of $l' \cap \mathcal{C}$. Thus $\widehat{\mathbf{K}^*X}_{|\mathcal{C}}$ has at least three distinct fixed points so $\widehat{\mathbf{K}^*X}_{|\mathcal{C}} = \mathrm{id}$. This entails that $\widehat{\mathbf{K}^*X}_{|\pi_\infty} = \mathrm{id}$, therefore $\widehat{\mathbf{K}^*X} = \mathrm{id}$ since each point of the line $l \not\subseteq \pi_\infty$ is fixed as well, and we are done. □

4 Algebraic properties of the section loops of $\mathcal{G}/\mathcal{D}$

Now we relate some algebraic properties of the loops arising from sections of the coset space $\mathcal{G}/\mathcal{D}$ to geometric properties of $\mathrm{PG}(3, \mathbf{K}) \setminus \mathcal{Q}$.

Proposition 4.1 *Let $\mathcal{D}'$ be a conjugate of $\mathcal{D}$ and let L be a transversal both of $\mathcal{G}/\mathcal{D}$ and $\mathcal{G}/\mathcal{D}'$. Then the left loops $(L, *)$ and $(L, *')$ induced by $\mathcal{D}$ and $\mathcal{D}'$ respectively are isotopic.*

Proof If we denote by $o = \mathcal{D} \cap \pi_\infty$ and $o' = \mathcal{D}' \cap \pi_\infty$, then by remark 3.1 the loops $(L, *)$ and $(L, *')$ are isomorphic to $\mathcal{L}(\mathbf{H}, L, o)$ and $\mathcal{L}(\mathbf{H}, L, o')$ respectively, and these loops are well known to be isotopic (see e.g. [14, prop 3.4]). □

Remark 4.2 Note that if (P, Γ) is a regular permutation set which is also invariant (namely $\gamma \Gamma \gamma^{-1} = \Gamma$ for each $\gamma \in \Gamma$), then for any $o, o' \in P$ the loop derivations $\mathcal{L}(P, \Gamma, o)$ and $\mathcal{L}(P, \Gamma, o')$ are in fact isomorphic. For, consider the map $\alpha = \widetilde{o\, o'} \in \Gamma$, then $\Gamma = \alpha \Gamma \alpha^{-1}$ and $\alpha(o) = o'$, thus by [13, Thm 3.2] α is the required isomorphism.

This happens, for instance, in the case of a K-loop, which can always be derived from an invariant regular set of involutions (see [2], [6]), or in the case of the limit rotation loop of the hyperbolic plane, which is left conjugacy closed (see [4, (4.1.1)]).

In the following for a matrix $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{K})$ let us consider the adjoint matrix $\mathrm{adj}\, X := \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$ and let $\nu : \mathrm{PG}(3, \mathbf{K}) \to \mathrm{PG}(3, \mathbf{K});\ \mathbf{K}^*X \mapsto \mathbf{K}^* \,\mathrm{adj}\, X$. Note that this map is a projectivity that fixes the quadric $\mathcal{Q}$ and whenever $\mathbf{K}^*X \in \mathrm{PGL}_2(\mathbf{K})$ we have $\nu(\mathbf{K}^*X) = \mathbf{K}^*X^{-1}$.

Proposition 4.3 *Let $(\mathbf{H}, \cdot)$ be a loop such that $L := \lambda(\mathbf{H}) \subseteq \mathcal{G}$. Then the following are equivalent:*

1. *$(\mathbf{H}, \cdot)$ fulfils the left inverse property;*
2. *$(\mathbf{H}, \cdot)$ is a loop with inverses;*
3. *L is fixed by the projectivity ν.*

Proof "1 ⇒ 2" is trivial and "3 ⇒ 1" is [7, (3.1.1)], thus it remains to prove that "2 ⇒ 3". Consider the loop $(L, *)$ isomorphic to $(\mathbf{H}, \cdot)$ and let $\mathbf{K}^*A \in L$ and $\mathbf{K}^*X$ be its (unique) inverse in L. Then

$$AX\mathcal{D} \cap L = (\mathbf{K}^*A) * (\mathbf{K}^*X) = \mathbf{1} = (\mathbf{K}^*X) * (\mathbf{K}^*A) = XA\mathcal{D} \cap L$$

hence, since each left coset of $\mathcal{D}$ intersects L precisely in one point, $AX\mathcal{D} = XA\mathcal{D} = \mathcal{D}$ or equivalently $\mathbf{K}^*AX \in \mathcal{D}$ and $\mathbf{K}^*XA \in \mathcal{D}$, thus

$$\mathbf{K}^*X = A^{-1}\mathcal{D} \cap L = \mathcal{D}A^{-1} \cap L.$$

The cosets $A^{-1}\mathcal{D}$ and $\mathcal{D}A^{-1}$ coincide if and only if $\mathcal{D} = A\mathcal{D}A^{-1}$, that is if and only if $\mathbf{K}^*A \in \mathcal{D}$, thus if and only if $\mathbf{K}^*A = \mathbf{1}$; hence we can assume $A^{-1}\mathcal{D} \neq \mathcal{D}A^{-1}$. Under this assumption, since both $A^{-1}\mathcal{D}$ and $\mathcal{D}A^{-1}$ are lines in $\mathrm{PG}(3, \mathbf{K})$, we have $|A^{-1}\mathcal{D} \cap \mathcal{D}A^{-1}| \leqslant 1$, whence

$$\mathbf{K}^*X = A^{-1}\mathcal{D} \cap \mathcal{D}A^{-1} = \mathbf{K}^*A^{-1},$$

proving that $\mathbf{K}^*A^{-1} \in L$. □

Remark 4.4 Note that if L is a transversal of $\mathcal{G}/\mathcal{D}$ which contains a point $\mathbf{K}^*A$ internal to $\mathbf{\Lambda}$, then the line $\overline{\mathbf{1}, \mathbf{K}^*A}$ is conjugate to $\mathcal{D}$ and has at least two distinct intersections with L, hence by [7, (2.7)] $(L, *)$ is not a loop (it can possibly be a left loop).

5 Planar transversals of $\mathcal{G}/\mathcal{D}$

In this Section we aim at describing the behaviour of the planes π through the point $\mathbf{1}$ with respect to the possibility of obtaining loops. By Section 2, $L_\pi := \pi \cap \mathcal{G}$ is a left loop if and only if it is a transversal of $\mathcal{G}/\mathcal{D}$ and this happens exactly when L_π does not contain any left coset of $\mathcal{D}$. We distinguish the following three cases:

1. π does not contain any generatrix of the cone $\mathbf{\Lambda}$. This is equivalent to requiring that the pole p of π in the polarity ω is an internal point of the conic $\mathcal{C}$. In this case π is called an *external plane* (w.r.t. $\mathbf{\Lambda}$).
2. π contains two distinct generatrices of $\mathbf{\Lambda}$, equivalently its pole p is an external point of $\mathcal{C}$. In this case π is called a *secant plane*.
3. π contains exactly one generatrix of the cone $\mathbf{\Lambda}$, i.e. π is tangent to $\mathbf{\Lambda}$ and to $\mathcal{Q}$ and π is the polar plane of a point $p \in \mathcal{C}$, hence π is a *tangent plane*.

The following holds true.

Theorem 5.1 *Let $L_\pi = \pi \cap \mathcal{G}$, then $(L_\pi, *)$ is a left loop if and only if π is an external or a tangent plane. In particular:*

1. *if π is an external plane then L_π is a loop;*
2. *if π is the tangent plane in the point $a \in \mathcal{C}$ then L_π is a group, namely $L_\pi = N_{\mathcal{G}}(\Lambda_a)$ (i.e. the normalizer of Λ_a in $\mathcal{G}$).*

Proof First note that $\mathcal{D}$ and all its left cosets are lines of $\mathrm{PG}(3,\mathbf{K})$ with no intersection with $\mathcal{Q}$. Moreover it is known that the set of all the left cosets of the line $\mathcal{D}$ in $\mathrm{PGL}_2(\mathbf{K})$ together with one regulus of $\mathcal{Q}$ is an elliptic linear congruence of $\mathrm{PG}(3,\mathbf{K})$, hence each plane π contains precisely a line of that congruence, which either lies completely in $\mathcal{G}$ or has no points at all in $\mathcal{G}$.

If π is a secant plane, let us denote by l_1 and l_2 the two generatrices of the cone $\mathbf{\Lambda}$ in π and by $g\mathcal{D}$ the unique line of the elliptic congruence contained in π. Since $g\mathcal{D}$ intersects the two lines l_1 and l_2 in $\mathcal{G}$ then $g\mathcal{D} \subseteq \pi \cap \mathcal{G}$, thus L_π cannot be a transversal of $\mathcal{G}/\mathcal{D}$. Since the same considerations hold also for any other line $\mathcal{D}' = G\mathcal{D}G^{-1}$ for any $\mathbf{K}^*G \in \mathcal{G}$, we conclude that L_π cannot be endowed with the left-loop operation "$*$".

If π is an external plane, any of its lines must contain points not in $\mathcal{G}$, hence the unique line of the congruence contained in π has no points in $\mathcal{G}$, so L_π is a transversal of the coset space $\mathcal{G}/\mathcal{D}$, i.e. $(L_\pi, *)$ is a left loop. Moreover, since the same holds for all the conjugates $G\mathcal{D}G^{-1}$ of $\mathcal{D}$, where $\mathbf{K}^*G \in \mathcal{G}$, the transversal $(L_\pi, *)$ is a loop.

Finally if π is a tangent plane in the point $a \in \mathcal{C}$, the unique line of the elliptic congruence is in $\mathcal{Q}$, so again it has no point at all in $\mathcal{G}$ and L_π is a transversal of $\mathcal{G}/\mathcal{D}$. Moreover by proposition 3.2 we have that $\mathbf{K}^*X \in L_\pi$ if and only if $\widehat{\mathbf{K}^*X}$ fixes the point a, and since the point $\mathbf{1}$ and its polar plane π_∞ are fixed as well, this is equivalent to the fact that the line $\Lambda_a = \overline{\mathbf{1}, a}$ is fixed, i.e. $\mathbf{K}^*X \in N_{\mathcal{G}}(\Lambda_a)$. □

Among all possible external planes, a particular situation is the one described by the following

Proposition 5.2 *Let $(\mathbf{H}, \oplus, o)$ be the K-loop of the hyperbolic plane. Then*

$$\lambda(\mathbf{H}) = \{\mathbf{K}^*A \in \mathcal{G} \mid A = A^t\} = \pi_s \cap \mathcal{G} =: L_s$$

where π_s is the polar plane of the point $o \in \mathbf{H}$ (whence π_s is an external plane), thus the section loop $(L_s, *)$ is isomorphic to the K-loop of the hyperbolic plane.

Proof First note that the polar plane of the point $o = \mathbf{K}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to $\mathcal{Q}$ is the plane π_s of equation $x_2 - x_3 = 0$, corresponding precisely to the symmetric matrices. Recall now that for the K-loop $(\mathbf{H}, \oplus, o)$ for all $a \in \mathbf{H}$ it holds $\lambda_a = \widetilde{o\,a}\,\widetilde{o}$ where $\widetilde{o\,a}$ and $\widetilde{o}$ are central symmetries, thus they belong to $\mathbf{H}$ and in particular $\widetilde{o} = o$ hence $\lambda(\mathbf{H}) = \mathbf{H}\,o$. An easy computation shows that $\mathbf{H}\,o = L_s$, thus, by 3.1, the result follows. □

Theorem 5.3 *Let π be any external plane and L_π equipped with the operation "$*$" induced by the coset space $\mathcal{G}/\mathcal{D}$. Then $(L_\pi, *)$ is isomorphic to the K-loop $(\mathbf{H}, \oplus, o)$ of the hyperbolic plane.*

Proof Consider the point $o_\pi \in \mathbf{H}$ pole of the plane π and the map $\lambda_{o_\pi} \in \lambda(\mathbf{H})$ mapping o to o_π. Since $\widehat{\lambda}_{o_\pi}$ fixes $\mathcal{Q}$ and $\mathbf{1}$, it preserves the polarity induced by $\mathcal{Q}$ and hence maps the plane π_s to π. Moreover since $\mathcal{D}$ is the line through

1 and o, we have $\widehat{\lambda}_{o_\pi}(\mathcal{D}) = \mathcal{D}'$ where $\mathcal{D}'$ is the line through **1** and o_π. Thus the map $\widehat{\lambda}_{o_\pi}$ fulfils the hypotheses of [7, (2.7.6)], the loops $(L_s, *)$ and $(L_\pi, *')$ where $*'$ is obtained as section of the coset space $\mathcal{G}/\mathcal{D}'$, are isomorphic and, by 5.2, they are isomorphic to the K-loop of the hyperbolic plane. Moreover the loop $(L_\pi, *')$ and the loop $(L_\pi, *)$ are isotopic by proposition 4.1, but since the first one is a K-loop, this isotopism is in fact an isomorphism (see Remark 4.2). □

6 Automorphisms of the section loops

The following proposition, which generalizes [12, Prop 4.4], characterizes a convenient subgroup of the group of automorphisms of any loop $(\mathbf{H}, \cdot)$ defined on the hyperbolic plane such that its left multiplications are direct motions.

Proposition 6.1 *Let* $(\mathbf{H}, \cdot)$ *be a loop with* $\lambda(\mathbf{H}) \subseteq \mathcal{G}$ *and consider in* $\operatorname{Aut}\mathcal{G}$ *the subgroup*

$$T := \{\beta \in \operatorname{Aut}\mathcal{G} \mid \beta(\lambda(\mathbf{H})) = \lambda(\mathbf{H}) \text{ and } \beta(\mathcal{D}) = \mathcal{D}\}.$$

Then T *is isomorphic to a subgroup of* $\operatorname{Aut}(\mathbf{H}, \cdot)$*. If moreover* $\mathcal{G}$ *is generated by* $\lambda(\mathbf{H})$*, then* T *is isomorphic to the whole group* $\operatorname{Aut}(\mathbf{H}, \cdot)$*.*

Proof By [7, (2.7.6)] the map $\chi : T \to \operatorname{Aut}(\mathbf{H}, \cdot)$; $\beta \mapsto \varphi\beta\varphi^{-1}$ is the required monomorphism.

Assume now that $\lambda(\mathbf{H})$ generates $\mathcal{G}$. Then the map $\widehat{\cdot} :\ \operatorname{Aut}(\mathbf{H}, \cdot) \to \operatorname{Aut}\mathcal{G}$; $\alpha \mapsto \widehat{\alpha}$, where

$$\widehat{\alpha} : \begin{cases} \mathcal{G} \to & \mathcal{G} \\ f \mapsto \alpha \circ f \circ \alpha^{-1}, \end{cases}$$

is a monomorphism by [7, (2.5)]. For all $\alpha \in \operatorname{Aut}(\mathbf{H}, \cdot)$ and $a, x \in \mathbf{H}$ we have

$$\left(\widehat{\alpha}(\lambda_a)\right)(x) = \alpha \circ \lambda_a \circ \alpha^{-1}(x) = \alpha(a \cdot \alpha^{-1}(x)) = \alpha(a) \cdot x = \lambda_{\alpha(a)}(x),$$

thus $\widehat{\alpha}(\lambda(\mathbf{H})) = \lambda(\mathbf{H})$. Moreover, for all $\mathbf{K}^*A \in \mathcal{G}$, a point $x \in \mathbf{H}$ is fixed by $\widehat{\mathbf{K}^*A}$ if and only if the point $\alpha(x)$ is fixed by $\widehat{\alpha}(\mathbf{K}^*A)$, thus

$$\begin{aligned} \widehat{\alpha}(\mathcal{D}) &= \widehat{\alpha}\big(\{\mathbf{K}^*A \in \mathcal{G} \mid \widehat{\mathbf{K}^*A}(x) = x \Longleftrightarrow x = o\}\big) = \\ &= \{\mathbf{K}^*A \in \mathcal{G} \mid \widehat{\mathbf{K}^*A}(x) = x \Longleftrightarrow x = \alpha(o)\}. \end{aligned}$$

Since $\alpha(o) = o$ we have $\widehat{\alpha}(\mathcal{D}) = \mathcal{D}$, thus $\widehat{\operatorname{Aut}(\mathbf{H}, \cdot)} \subseteq T$.

Finally for all $\alpha \in \operatorname{Aut}(\mathbf{H}, \cdot)$ and $x \in \mathbf{H}$ we have

$$\left(\chi(\widehat{\alpha})\right)(x) = \varphi \circ \widehat{\alpha} \circ \varphi^{-1}(x) = \varphi \circ \widehat{\alpha}(\lambda_x) = \varphi\left(\lambda_{\alpha(x)}\right) = \alpha(x),$$

thus $\chi \circ \widehat{\cdot} = \mathrm{id}$, proving that $T = \widehat{\operatorname{Aut}(\mathbf{H}, \cdot)}$. □

In the particular cases of the limit rotation loop and of the K-loop of the hyperbolic plane we can provide also a geometric description of the group $\mathrm{Aut}(\mathbf{H},\cdot)$. In the following, according to our notation, we shall consider the groups $\widehat{\mathrm{PGL}_2(\mathbf{K})}$, $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{D}}$ consisting of inner automorphisms with respect to elements of the corresponding groups and, if we denote by $\mathbf{K}^*J := \mathbf{K}^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathrm{PGL}_2(\mathbf{K})$, then we get $\mathrm{PGL}_2(\mathbf{K}) = \mathcal{G} \rtimes \langle \mathbf{K}^*J\rangle$. Moreover let $\overline{\mathbf{K}}$ be the group of pure semilinear collineations of $\mathrm{PG}(3,\mathbf{K})$, namely made up of elements $\overline{\alpha} \in \overline{\mathbf{K}}$ such that

$$\overline{\alpha} : \begin{cases} PG(3,\mathbf{K}) & \to PG(3,\mathbf{K}) \\ \mathbf{K}^*(x_1,x_2,x_3,x_4) & \mapsto \mathbf{K}^*(x_1^\alpha,x_2^\alpha,x_3^\alpha,x_4^\alpha)\end{cases}$$

where $\alpha \in \mathrm{Aut}(\mathbf{K},+,\cdot)$.

The automorphism group of the limit rotation loop is determined in [12, Thm. 4.5] (for the proof see also [11]) as follows:

Theorem 6.2 *Let $(\mathbf{H},\cdot)$ be the limit rotation loop of the hyperbolic plane $\mathbf{H}$ over a Euclidean field $\mathbf{K}$. Then*

$$\mathrm{Aut}(\mathbf{H},\cdot) \cong \widehat{\mathcal{D}} \rtimes \overline{\mathbf{K}}.$$

To deal with the case of the K-loop of the hyperbolic plane we need to employ the setting introduced in [1] and [3], in particular to prove the following result.

Lemma 6.3 *Let $(\mathbf{H},\oplus)$ be the K-loop of the hyperbolic plane and $L_s = \lambda(\mathbf{H}) = \pi_s \cap \mathcal{G}$. Then L_s generates the whole group $\mathcal{G}$.*

Proof We start by proving that the left inner mapping group $\Delta = \mathcal{D}$. Clearly $\Delta \subseteq \mathcal{D}$, thus consider $\mathbf{K}^*D \in \mathcal{D} \setminus \mathbf{1}$, which is a non-trivial rotation of $\mathbf{H}$ of angle γ (see e.g. [1, p. 155]). It is known (see e.g. [1, § 42]) that there exists a triangle (x_1,x_2,x_3) in $\mathbf{H}$ such that its angular defect is precisely γ. By [3, (2.3), (3.5) and (4.4.1)] there exist $a,b \in \mathbf{H}$ such that either $\mathbf{K}^*D = \delta_{a,b} \in \Delta$ or $\mathbf{K}^*D^{-1} = \delta_{a,b} \in \Delta$, thus $\mathcal{D} \subseteq \Delta$. Since by definition $\Delta \subseteq \langle L_s\rangle$ and $\mathcal{G} = L_s\mathcal{D}$ we are done. □

Theorem 6.4 *Let $(\mathbf{H},\oplus)$ be the K-loop of the hyperbolic plane and $L_s = \lambda(\mathbf{H}) = \pi_s \cap \mathcal{G}$. Then*

$$\mathrm{Aut}(\mathbf{H},\oplus) \cong (\widehat{\mathcal{D}} \rtimes \langle \widehat{\mathbf{K}^*J}\rangle) \rtimes \overline{\mathbf{K}}.$$

Proof According to proposition 6.1 to describe $\mathrm{Aut}(\mathbf{H},\oplus)$ it is enough to characterize geometrically the subgroup T of $\mathrm{Aut}\,\mathcal{G}$. The result follows by noticing that for all $\mathbf{K}^*G \in \mathcal{G}$ it holds $\widehat{\mathbf{K}^*G}(\mathcal{D}) = \mathcal{D}$ if and only if $\mathbf{K}^*G \in \mathcal{D}$. Moreover $\widehat{\mathbf{K}^*J}(o) = o$, hence $\widehat{\mathbf{K}^*J}(\mathcal{D}) = \mathcal{D}$. Since each element of $\widehat{\mathcal{D}} \rtimes \langle\widehat{\mathbf{K}^*J}\rangle$ fixes the quadric $\mathcal{Q}$ and the point o, it fixes also $\pi_s = \omega(o)$ and hence $\widehat{\mathcal{D}} \rtimes \langle\widehat{\mathbf{K}^*J}\rangle = \{\widehat{\mathbf{K}^*A} \in \widehat{\mathrm{PGL}_2(\mathbf{K})} \mid \widehat{\mathbf{K}^*A}(L_s) = L_s \text{ and } \widehat{\mathbf{K}^*A}(\mathcal{D}) = \mathcal{D}\}$.

Finally note that each $\overline{\alpha} \in \overline{\mathbf{K}}$ preserves both the plane π_s and the subgroup $\mathcal{D}$ as a straightforward computation shows. □

References

1. Hartshorne, R.: Geometry: Euclid and beyond. Undergraduate Texts in Mathematics. Springer-Verlag, New York (2000)
2. Karzel, H.: Loops related to geometric structures. Quasigroups Related Systems **15**(1), 47–76 (2007)
3. Karzel, H., Marchi, M.: Relations between the K-loop and the defect of an absolute plane. Results Math. **47**(3-4), 305–326 (2005). DOI 10.1007/BF03323031. URL http://dx.doi.org/10.1007/BF03323031
4. Karzel, H., Pasotti, S., Pianta, S.: A class of fibered loops related to general hyperbolic planes. Aequationes Math. **87**(1), 31–42 (2014). DOI 10.1007/s00010-012-0164-8
5. Karzel, H., Sörensen, K., Windelberg, D.: Einführung in die Geometrie. Vandenhoeck & Ruprecht, Göttingen (1973). Studia mathematica/Mathematische Lehrbücher, Taschenbuch 1, Uni-Taschenbücher, No. 184
6. Karzel, H., Wefelscheid, H.: A geometric construction of the K-loop of a hyperbolic space. Geom. Dedicata **58**(3), 227–236 (1995). DOI 10.1007/BF01263454. URL http://dx.doi.org/10.1007/BF01263454
7. Kiechle, H.: Theory of K-loops, *Lecture Notes in Mathematics*, vol. 1778. Springer-Verlag, Berlin (2002)
8. Kreuzer, A.: Inner mappings of Bruck loops. Math. Proc. Cambridge Philos. Soc. **123**(1), 53–57 (1998). DOI 10.1017/S0305004197001771. URL http://dx.doi.org/10.1017/S0305004197001771
9. Nagy, P.T., Strambach, K.: Loops as invariant sections in groups, and their geometry. Canad. J. Math. **46**(5), 1027–1056 (1994). DOI 10.4153/CJM-1994-059-8. URL http://dx.doi.org/10.4153/CJM-1994-059-8
10. Nagy, P.T., Strambach, K.: Loops in group theory and Lie theory, *de Gruyter Expositions in Mathematics*, vol. 35. Walter de Gruyter & Co., Berlin (2002). DOI 10.1515/9783110900583. URL http://dx.doi.org/10.1515/9783110900583
11. Pasotti, S., Pianta, S.: The limit rotation loop of a hyperbolic plane. Applied Mathematical Sciences **7**(117-120), 5863–5878 (2013). DOI 10.12988/ams.2013.38430. URL http://www.scopus.com/inward/record.url?eid=2-s2.0-84886734118partnerID=40md5=4e0279e5946aebc55367e1fbfeac9317
12. Pasotti, S., Pianta, S.: Corrigendum to "S. Pasotti, S. Pianta, The limit rotation loop of a hyperbolic plane, Applied Mathematical Sciences, 7 (2013), no. 117-120, 5863-5878" . Applied Mathematical Sciences **8**(93-96), 4793–4795 (2014). DOI 10.12988/ams.2014.41498
13. Pasotti, S., Zizioli, E.: Loops, regular permutation sets and colourings of directed graphs. J. Geom. DOI 10.1007/s00022-014-0230-6. To appear
14. Pasotti, S., Zizioli, E.: Slid product of loops: a generalization. Results Math. **65**(1-2), 193–212 (2014)
15. Pianta, S., Zizioli, E.: Collineations of geometric structures derived from quaternion algebras. J. Geom. **37**(1-2), 142–152 (1990). DOI 10.1007/BF01230367. URL http://dx.doi.org/10.1007/BF01230367

References

1. Hartshorne, R.: Geometry: Euclid and beyond. Undergraduate Texts in Mathematics. Springer-Verlag, New York (2000)
2. Karzel, H.: Loops related to geometric structures. Quasigroups Related Systems **15**(1), 47–76 (2007)
3. Karzel, H., Marchi, M.: Relations between the K-loop and the defect of an absolute plane. Results Math. **47**(3-4), 305–326 (2005). DOI 10.1007/BF03323031.
4. Karzel, H., Pasotti, S., Pianta, S.: A class of fibered loops related to general hyperbolic planes. Aequationes Math. **87**(1), 31–42 (2014). DOI 10.1007/s00010-012-0164-8
5. Karzel, H., Sörensen, K., Windelberg, D.: Einführung in die Geometrie. Göttingen (1973)

6. Karzel, H., Wefelscheid, H.: A geometric construction of the K-loop of a hyperbolic space. Geom. Dedicata **58**(3), 227–236 (1995). DOI 10.1007/BF01263454.
7. Kiechle, H.: Theory of K-loops, *Lecture Notes in Mathematics*, vol. 1778. Springer-Verlag, Berlin (2002)
8. Kreuzer, A.: Inner mappings of Bruck loops. Math. Proc. Cambridge Philos. Soc. **123**(1), 53–57 (1998). DOI 10.1017/S0305004197001771.
9. Nagy, P.T., Strambach, K.: Loops as invariant sections in groups, and their geometry. Canad. J. Math. **46**(5), 1027–1056 (1994). DOI 10.4153/CJM-1994-059-8.
10. Nagy, P.T., Strambach, K.: Loops in group theory and Lie theory, *de Gruyter Expositions in Mathematics*, vol. 35. Walter de Gruyter & Co., Berlin (2002). DOI 10.1515/9783110900583.
11. Pasotti, S., Pianta, S.: Corrigendum to "S. Pasotti, S. Pianta, The limit rotation loop of a hyperbolic plane, Applied Mathematical Sciences, 7 (2013), no. 117-120, 5863-5878" . Applied Mathematical Sciences To appear
12. Pasotti, S., Pianta, S.: The limit rotation loop of a hyperbolic plane. Applied Mathematical Sciences **7**(117-120), 5863–5878 (2013). DOI 10.12988/ams.2013.38430.
13. Pasotti, S., Zizioli, E.: Loops, regular permutation sets and colourings of directed graphs. J. Geom. DOI 10.1007/s00022-014-0230-6. To appear
14. Pasotti, S., Zizioli, E.: Slid product of loops: a generalization. Results Math. **65**(1-2), 193–212 (2014)
15. Pianta, S., Zizioli, E.: Collineations of geometric structures derived from quaternion algebras. J. Geom. **37**(1-2), 142–152 (1990). DOI 10.1007/BF01230367.