QUARTERLY OF APPLIED MATHEMATICS
VOLUME LX, NUMBER 1
MARCH 2002, PAGES 37–73

A GEOMETRIC EVOLUTION PROBLEM

By

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Abstract. A traditional approach to compression moulding of polymers involves the study of a generalized Hele-Shaw flow of a power-law fluid, and leads to the p-Poisson equation for the instantaneous pressure in the fluid. By studying the convex dual of an equivalent extremal problem, one may let the power-law index of the fluid tend to zero. The solution of the resulting extremal problem, referred to as the asymptotically dual problem, is known to have the property that the flow is always directed towards the closest point on the boundary.

In this paper we use this property to introduce the concept of boundary velocity in the case of piecewise C^2 domains with only convex corners, and we also give an explicit solution to the asymptotically dual problem in this case. This involves the study of certain topological properties of the ridge of planar domains.

With use of the boundary velocity, we define a geometric evolution problem and the concept of classical solutions of it. We prove a uniqueness theorem and use a comparison principle to study the persistence of corners. We actually estimate "waiting times" for corners, in terms of geometric quantities of the initial domain.

1. Introduction. In this paper we will present a geometric model for the mould-filling pattern in the case of compression moulding of polymers. Since certain properties of the final part depend on this pattern, such a model is an important tool in order to optimize shape and placement of the initial charge.

1.1. *Physical background.* Our model is based on a classical paper by Lee, Folgar, and Tucker [13]. By studying the so-called generalized Hele-Shaw flow for a fluid of power-law type, physical conservation laws will lead to a nonlinear boundary value problem for the pressure in the fluid. The main steps in this procedure are described below.

Assume that, at a certain time t during the compression, the polymer occupies a cylindrical domain $M_t = \Omega_t \times (0, h(t))$, where Ω_t is a planar domain and $h(t)$ is assumed "small" compared to the dimensions of Ω_t. As the gap closes (with closing speed $-\dot{h}(t)$) there will be a flow in M_t with velocity $V = (u, v, w)$.

Received October 11, 1999.
2000 *Mathematics Subject Classification.* Primary 53C44.
This paper is part of a licentiate thesis, supervised by Professor Gunnar Aronsson, Department of Mathematics, Linköping University, Sweden.

Let Ω be any smooth subdomain of Ω_t. The flow can pass the boundary of $\Omega \times (0, h(t))$ only through $\partial\Omega \times (0, h(t))$. Let $N = (m, n)$ be the outward unit normal to $\partial\Omega$ and introduce the averages

$$\overline{u} = \frac{1}{h(t)} \int_0^{h(t)} u\, dz \quad \text{and} \quad \overline{v} = \frac{1}{h(t)} \int_0^{h(t)} v\, dz.$$

Then the total flow out of $\Omega \times (0, h(t))$ is

$$\begin{aligned} \int_{\partial\Omega \times (0,h(t))} (u, v, w) \cdot (m, n, 0)\, dA &= \int_{\partial\Omega} \left(\int_0^{h(t)} (um + vn)\, dz \right) ds \\ &= h(t) \int_{\partial\Omega} (\overline{u}, \overline{v}) \cdot N\, ds = h(t) \int_\Omega \operatorname{div} \overline{V}\, dA, \end{aligned}$$

where $\overline{V} = (\overline{u}, \overline{v})$. Assuming incompressibility, the total "production" in $\Omega \times (0, h(t))$ is $-\dot{h}(t) \int_\Omega dA$. Since Ω was arbitrary in Ω_t, we get that mass conservation is equivalent to

$$h \operatorname{div} \overline{V} = -\dot{h}. \tag{1}$$

Let $\mathbf{T}$ be the *Cauchy stress tensor*, $\mathbf{b}$ the *body force* and ρ the *density*. Cauchy's theorem states that momentum balance is equivalent to the condition

$$\operatorname{div} \mathbf{T} + \mathbf{b} = \rho \dot{V}$$

with $\mathbf{T}$ symmetric (see, for instance, ch. 14, p. 101, in [8]). Here $\dot{V}$ denotes the material time derivative of the spatial field V. In our case, $\mathbf{b}$ is just the gravity, which is assumed negligible. One also assumes that inertia forces are small compared to viscous forces, and hence neglects the term $\rho\dot{V}$ as well (see p. 116 and Appendix A in [13]). Thus we have $\operatorname{div} \mathbf{T} = 0$.

Next, we use the constitutive model for a *generalized Newtonian fluid*, which by definition means that

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D},$$

where p is the *pressure*, $\mathbf{I}$ the identity tensor, μ the *viscosity* and $\mathbf{D}$ the *stretching tensor*, that is,

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \quad \text{where } \mathbf{L} = \operatorname{grad} V.$$

By *Hele-Shaw flow* one means a flow satisfying a collection of simplifying assumptions, or approximations, motivated by the extreme geometry one has at flows in narrow gaps. First of all we impose a no-slip condition at the plates, that is, we assume $V = 0$ at $z = 0$ and at $z = h(t)$. Now the following simplifications are made. (For more detailed motivations and further references, we once again refer to [13].)

1. The no-slip condition, together with the fact that $h(t)$ is much smaller than the typical dimensions of Ω_t, suggests that u and v are much larger than w, and therefore $w = 0$ is assumed.

2. By the same argument as in 1, u and v should change more rapidly in the z-direction than in the x- and y-directions. Hence the derivatives of u and v with respect to x and

y are neglected. Thus the stretching tensor is represented by the matrix

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & 0 & u_z \\ 0 & 0 & v_z \\ u_z & v_z & 0 \end{pmatrix}.$$

3. We assume the flow to be symmetric with respect to the plane $z = h/2$. Thus we should have $u_z(x,y,h/2) = v_z(x,y,h/2) = 0$. We also assume that the pressure p is independent of the z-coordinate.

Using these assumptions we now obtain

$$\begin{aligned} \operatorname{div}\mathbf{T} &= -\operatorname{grad} p + 2\operatorname{div}(\mu\mathbf{D}) \\ &= \left(-p_x + \frac{\partial}{\partial z}(\mu u_z), -p_y + \frac{\partial}{\partial z}(\mu v_z), \frac{\partial}{\partial x}(\mu u_z) + \frac{\partial}{\partial y}(\mu v_z)\right), \end{aligned}$$

and hence $\operatorname{div}\mathbf{T} = 0$ gives

$$\begin{aligned} p_x &= \frac{\partial}{\partial z}(\mu u_z), \\ p_y &= \frac{\partial}{\partial z}(\mu v_z), \\ 0 &= \frac{\partial}{\partial x}(\mu u_z) + \frac{\partial}{\partial y}(\mu v_z). \end{aligned}$$

Since p was assumed independent of z we get, for any $z_0 \in [0,h]$,

$$\begin{aligned} &\mu(x,y,z)u_z(x,y,z) \\ &\quad= \int_{z_0}^{z} p_x(x,y)\,d\zeta + \mu(x,y,z_0)u_z(x,y,z_0) \\ &\quad= (z - z_0)p_x(x,y) + \mu(x,y,z_0)u_z(x,y,z_0), \end{aligned}$$

and since $u_z(x,y,h/2) = 0$ that

$$\mu(x,y,z)u_z(x,y,z) = (z - h/2)p_x(x,y). \tag{2}$$

Thus,

$$u(x,y,z) = \int_{z_0}^{z} \frac{p_x(x,y)}{\mu(x,y,\zeta)}(\zeta - h/2)\,d\zeta + u(x,y,z_0).$$

Putting $z_0 = h$, the term $u(x,y,z_0)$ will vanish, due to the no-slip condition, and we obtain

$$u(x,y,z) = -p_x(x,y)\int_{z}^{h} \frac{\zeta - h/2}{\mu(x,y,\zeta)}\,d\zeta.$$

Using the symmetry of the flow with respect to $z = h/2$, we obtain

$$\begin{aligned}
\overline{u} &= \frac{1}{h}\int_0^h u\,dz = \frac{2}{h}\int_{h/2}^h u\,dz \\
&= -\frac{2}{h}p_x\int_{h/2}^h \left(\int_z^h \frac{\zeta - h/2}{\mu(x,y,\zeta)}\,d\zeta\right) dz \\
&= -\frac{2}{h}p_x\int_{h/2}^h \left(\int_{h/2}^{\zeta} \frac{\zeta - h/2}{\mu(x,y,\zeta)}\,dz\right) d\zeta \\
&= -\frac{2}{h}p_x\int_{h/2}^h \frac{(\zeta - h/2)^2}{\mu(x,y,\zeta)}\,d\zeta \\
&= -\frac{p_x}{h}\int_0^h \frac{(z - h/2)^2}{\mu(x,y,z)}\,dz
\end{aligned}$$

and similarly for $\overline{v}$.

Introducing the *flow conductance*,

$$S = \int_0^h \frac{(z - h/2)^2}{\mu(x,y,z)}\,dz,$$

we can write

$$\overline{V} = -\frac{1}{h}S\nabla p, \tag{3}$$

where $\nabla p = (p_x, p_y)$. Combining (1) and (3) we have that conservation of mass and momentum gives

$$\operatorname{div}(S\nabla p) = \dot{h}. \tag{4}$$

By definition, a *power-law fluid* is a generalized Newtonian fluid for which

$$\mu = K\dot{\gamma}^{n-1},$$

where n and K are material constants. n is assumed positive and is called the *power-law index*. $\dot{\gamma}$ is an invariant that measures the rate of deformation, namely,

$$\dot{\gamma} = \sqrt{2\mathbf{D} : \mathbf{D}} = \sqrt{2\sum d_{ij}^2},$$

where the d_{ij} are the components of $\mathbf{D}$. In our case this means that $\dot{\gamma} = \sqrt{u_z^2 + v_z^2}$. Note that μ is constant if $n = 1$, which corresponds to the Newtonian case. We will focus on the *pseudoplastic* case, when $0 < n < 1$.

Equation (2) implies

$$\begin{aligned}
\dot{\gamma}^2 &= u_z^2 + v_z^2 \\
&= \left(\frac{p_x}{\mu}(z - h/2)\right)^2 + \left(\frac{p_y}{\mu}(z - h/2)\right)^2 \\
&= \frac{(z - h/2)^2}{\mu^2}(p_x^2 + p_y^2),
\end{aligned}$$

and hence

$$\mu = K\frac{|z-h/2|^{n-1}}{\mu^{n-1}}|\nabla p|^{n-1}.$$

Solving for μ we obtain

$$\mu = K^{1/n}|z-h/2|^{1-1/n}|\nabla p|^{1-1/n}.$$

Using the above expression for the viscosity, the flow conductance S becomes

$$\begin{aligned} S &= \int_0^h \frac{(z-h/2)^2}{\mu}\,dz \\ &= \int_0^h \frac{(z-h/2)^2}{K^{1/n}|z-h/2|^{1-1/n}|\nabla p|^{1-1/n}}\,dz \\ &= \frac{1}{2^{1+1/n}}\frac{h^{2+1/n}}{(2+1/n)K^{1/n}}|\nabla p|^{1/n-1}. \end{aligned}$$

Note that $1/n - 1 > 0$ since $n < 1$. Put $s = 1/n$ (so $s > 1$) and let

$$C = \frac{1}{2^{1+s}(2+s)K^s}.$$

Then

$$S = C|\nabla p|^{s-1}h^{s+2},$$

and Eq. (4) transforms to

$$\operatorname{div}(|\nabla p|^{s-1}\nabla p) = \frac{\dot h}{Ch^{s+2}}. \tag{5}$$

Further, according to (3), the flow $\overline{V}$ is then given by

$$\overline{V} = -Ch^{s+1}|\nabla p|^{s-1}\nabla p. \tag{6}$$

The right-hand side in (5) is constant in space for each fixed time t. Since its value does not affect p except for a multiplicative factor, the actual value is not important for the flow pattern. Indeed, different functions h merely correspond to different time variables. Since a change of time variables may simplify the governing equations (5) and (6), we introduce a new time $\tau = \theta(t)$ such that $\theta(0) = 0$.

Let us first write $p(x, y, t) = R(t)P(x, y, t)$, where

$$R = \left(\frac{-\dot h}{Ch^{s+2}}\right)^{1/s}.$$

Then (5) takes the form

$$\operatorname{div}(|\nabla P|^{s-1}\nabla P) = -1,$$

and (6) becomes

$$\overline{V} = -Ch^{s+1}R^s|\nabla P|^{s-1}\nabla P = \frac{\dot h}{h}|\nabla P|^{s-1}\nabla P.$$

Let $\overline{W}$ denote the velocity with respect to the variables x, y, and τ. Then

$$\overline{W} = \overline{V}\frac{dt}{d\tau} = \frac{\dot{h}}{h}|\nabla P|^{s-1}\nabla P\frac{1}{\dot{\theta}}.$$

Choosing θ such that $\dot{\theta} = -\dot{h}/h$, this means that

$$\overline{W} = -|\nabla P|^{s-1}\nabla P.$$

Thus, returning to our original notation, Eqs. (5) and (6) can be replaced by

$$\operatorname{div}(|\nabla p|^{s-1}\nabla p) = -1 \tag{7}$$

and

$$\overline{V} = -|\nabla p|^{s-1}\nabla p.$$

Equation (7) is called the $(s+1)$-*Poisson equation.*

So far we have not discussed boundary conditions at $\partial\Omega_t \times (0, h)$. At parts in contact with walls or obstacles in the mould we must have $\frac{\partial p}{\partial N} = 0$ since the polymer flow cannot penetrate the walls. (Here N is the outward unit normal of $\partial\Omega_t$.) However, in this paper we will only consider the case with *free boundary.* Then the pressure along $\partial\Omega_t$ must equal the atmospheric pressure p_0. By studying $p - p_0$ instead, we may simply assume $p = 0$ along $\partial\Omega_t$.

Let us summarize. With a suitable redefinition of time, we have found that the flow $\overline{V}$, at a time t when the polymer occupies a region $M_t = \Omega_t \times (0, h(t))$, is given by

$$\overline{V} = -|\nabla p|^{s-1}\nabla p, \tag{8}$$

where p is the solution of the boundary value problem

$$\begin{aligned} \operatorname{div}(|\nabla p|^{s-1}\nabla p) &= -1 \quad \text{in } \Omega_t, \\ p &= 0 \qquad \text{on } \partial\Omega_t, \end{aligned} \tag{9}$$

and $n = 1/s$ is the power-law index of the polymer.

(A kinematic boundary condition along $\partial\Omega_t$, expressing conservation of volume, will be included in Sec. 3.)

1.2. *Passage to the limit.* The Newtonian case corresponds to $n = 1$. Then $s = 1$ and we obtain the classical Poisson equation. However, in practice, typical values of n lie in the range from 0.1 to 0.3 (so-called *strongly sheer-thinning fluids*). By studying the dual of an extremal problem equivalent to (9) it is possible to let $n \to 0$. Below we state the main steps in this procedure. For details we refer to a paper by Janfalk [10].

First of all, to write (9) in a more classical form, we write u instead of p and let $p = s + 1$. We also omit the index t; so now the gapwise average velocity is

$$\overline{V} = -|\nabla u|^{p-2}\nabla u, \tag{10}$$

where u is the solution of the boundary value problem

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= -1 \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{11}$$

and $p = 1/n + 1$.

From now on, derivatives and differential equations are interpreted in the weak sense. Further, $x \in \mathbb{R}^2$ will denote the space variable, and dx is the two-dimensional Lebesgue measure. $\Omega \subset \mathbb{R}^2$ is assumed to be a bounded domain with Lipschitz boundary $\partial\Omega$. Define the functional

$$J_p(w) = \frac{1}{p}\int_\Omega |\nabla w|^p\,dx - \int_\Omega w\,dx$$

on the Sobolev space $\mathcal{A}_p = W_0^{1,p}(\Omega)$, $2 < p < \infty$. Note that, since $\Omega \subset \mathbb{R}^2$ and $p > 2$, every function in $\mathcal{A}_p$ is continuous.

Let $u \in \mathcal{A}_p$ be a solution (in the weak sense) to (11). Then u solves the problem $(\mathcal{P}_p)$: Find $u_p \in \mathcal{A}_p$ such that

$$J_p(u_p) = \inf_{u\in\mathcal{A}_p} J_p(u). \qquad (\mathcal{P}_p)$$

This is a convex extremal problem and hence there is a dual problem $(\mathcal{P}_q^*)$ in the sense of convex analysis, namely: Find $r_q \in \mathcal{A}_q^*$ such that

$$J_q^*(r_q) = \sup_{r\in\mathcal{A}_q^*} J_q^*(r), \qquad (\mathcal{P}_q^*)$$

where J_q^* is the functional defined by

$$J_q^*(r) = -\frac{1}{q}\int_\Omega |r|^q\,dx$$

and

$$\mathcal{A}_q^* = \{r \in L^q(\nabla, \mathbb{R}^2) : \operatorname{div} r \equiv 1\}.$$

Of course $1/p + 1/q = 1$. The condition $\operatorname{div} r \equiv 1$ is interpreted in a weak sense, namely,

$$\int_\Omega r\cdot\nabla\varphi\,dx = -\int_\Omega \varphi\,dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Theorem 3.1 in [10] reads as follows.

THEOREM 1.1. The problems $(\mathcal{P}_p)$ and $(\mathcal{P}_q^*)$ are uniquely solvable. Further, if u_p and r_q are their respective solutions, then

$$J_p(u_p) = J_q^*(r_q),$$
$$r_q = -|\nabla u_p|^{p-2}\nabla u_p \quad \text{a.e.},$$

and

$$\int_\Omega |\nabla u_p|^p\,dx = \int_\Omega |r_q|^q\,dx = \int_\Omega u_p\,dx.$$

This theorem thus states that the solution r_q to $(\mathcal{P}_q^*)$ can be identified with the flow $\overline{V}$, given by Eq. (10).

Now let $n \to 0$. Then $p \to \infty$ and $q \to 1$. Since it is sensible to replace q by 1 in the dual problem $(\mathcal{P}_q^*)$, we now get the *asymptotically dual problem* $(\mathcal{P}_1^*)$: Find $v_1 \in \mathcal{A}_1^*$ such that

$$\int_\Omega |v_1|\,dx = \inf_{v\in\mathcal{A}_1^*}\int_\Omega |v|\,dx. \qquad (\mathcal{P}_1^*)$$

Theorem 1.1 means that a solution v_1 to $(\mathcal{P}_1^*)$ can be seen as a counterpart of the flow $\overline{V}$.

The following result is included in Theorem 4.1 in [11], p. 172.

THEOREM 1.2. Let $\psi(x) = \operatorname{dist}(x, \partial\Omega)$. Then

$$\inf_{v\in\mathcal{A}_1^*} \int_\Omega |v|\,dx = \int_\Omega \psi\,dx.$$

Further, if v is an extremal to $(\mathcal{P}_1^*)$, then there is a nonnegative scalar function F such that $v(x) = -F(x)\nabla\psi(x)$ for almost every $x \in \Omega$.

In [11], Janfalk has proved the existence and uniqueness of a solution to $(\mathcal{P}_1^*)$ in the case when Ω is a polygonal domain. There is also given an explicit expression for F. He has also sketched a proof in the case when Ω is a bounded domain with piecewise C^3 boundary [12].

2. The limit case. The property $v(x) = -F(x)\nabla\psi(x)$ means that *the flow is directed towards the closest point on the boundary.* In this section this very important observation will be combined with the condition $\operatorname{div} v = 1$, in order to construct an extremal to $(\mathcal{P}_1^*)$. The values of this extremal at the boundary $\partial\Omega$ lead to the concept of *boundary velocity* (Def. 2.19), which will be used (in Sec. 3) to describe the movement of the free boundary in the time-dependent evolution problem.

2.1. *The ridge of a planar domain.* The fact that the flow is directed towards the closest boundary point indicates that the location of points with no unique closest boundary point plays an important role. The set of all such points appears in the literature under names like ridge, skeleton, medial axis, and central set. We will use the ridge concept of Evans and Harris [6]. The distance function ψ will from now on be denoted d, that is, $d(x) = \operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x-y|$.

DEFINITION 2.1 (Ridge). For a domain $\Omega \subset \mathbb{R}^2$, which does not contain any half-plane, we define

1. For $x \in \Omega$, the *near set* $N(x) = \{y \in \partial\Omega : d(x) = |x-y|\}$.

2. For $x \in \Omega$ and $y \in N(x)$, let $\lambda = \sup\{t : y \in N(y + t(x-y))\}$ and define the *ridge point* of x as $p(x) = y + \lambda(x-y)$ and the *ridge* of Ω as $P(\Omega) = \{p(x) : x \in \Omega\}$.

3. The function $r : \Omega \to (0,\infty)$ by $r = d \circ p$.

The definition is illustrated in Fig. 1. Note that, since Ω does not contain any half-plane, λ in part 3 of the definition is always finite. Further, if $\operatorname{card} N(x) > 1$, then $\lambda = 1$ for every choice of $y \in N(x)$ and hence $p(x) = x$. Thus p is well defined.

We collect some basic properties of near sets and distance functions in the following lemma. They correspond to Lemma 3.1, Theorem 3.3, and Lemma 3.4 in [6], pp. 149–150, where proofs also can be found.

LEMMA 2.2. 1. If $y \in N(x)$, then $y \in N(z)$ for every z on the line segment joining x and y.

2. The near set $N(x)$ is compact for each $x \in \Omega$.

3. Let $\{x_n\}$ be a sequence in Ω and let $y_n \in N(x_n)$ for all n. If $x_n \to x \in \Omega$ and $y_n \to y$, then $y \in N(x)$.

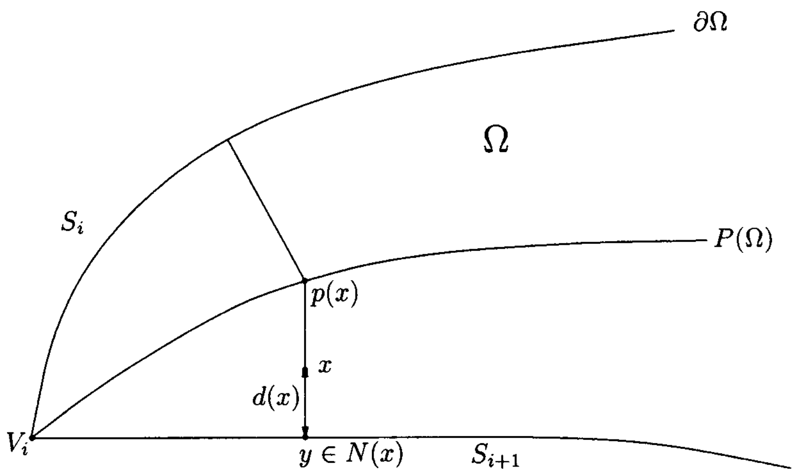

FIG. 1. Ridge definition

4. d is differentiable at $x \in \Omega$ if and only if $\operatorname{card} N(x) = 1$. If d is differentiable at $x \in \Omega$, then the gradient is $\nabla d(x) = (x - y)/|x - y|$, where $y \in N(x)$. Further, ∇d is continuous on its domain of definition.

Proposition 3.7 in [6], p. 151, states the following.

PROPOSITION 2.3 (Upper semi-continuity of r). The function r is upper semi-continuous on Ω.

The question of continuity of p and r is much more complicated. Indeed, these functions need not be continuous. However, according to [6] (Theorem 3.9, p. 152), the following is true.

THEOREM 2.4. The functions p and r are continuous on Ω if and only if the ridge $P(\Omega)$ is closed relative to Ω.

By skeleton we will mean the following.

DEFINITION 2.5 (Skeleton). For a domain Ω we define the *skeleton* R_0 as

$$R_0(\Omega) = \{x \in \Omega : \operatorname{card} N(x) \geq 2\}.$$

In [11] it is proven that if Ω is a polygonal domain, then $P(\Omega)$ is closed relative to Ω (Theorem 3.5, p. 170). Thus, in this case, the functions p and r are continuous. Further, it is proven that $R_0(\Omega) = P(\Omega)$ (Theorem 3.4, p. 170).

It is always true that $R_0(\Omega) \subset P(\Omega)$ but the opposite relation need not hold. For instance, the foci of an elliptic disc belong to the ridge but not to the skeleton. However, the following general results can be found in a recent paper by Fremlin [7] (Theorem 3B, Corollary 3C, and Proposition 3N).

THEOREM 2.6. If Ω is a domain that does not contain any half-plane, then

1. $R_0(\Omega) \subset P(\Omega) \subset \overline{R_0(\Omega)}$,

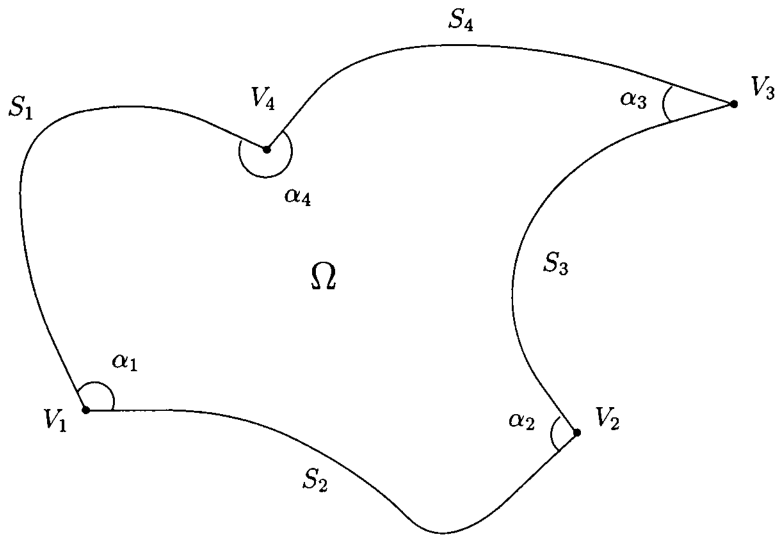

FIG. 2. A piecewise C^2 domain: V_4 re-entrant vertex, the others convex

2. $P(\Omega)$ is connected,
3. $P(\Omega)$ has zero two-dimensional measure.

2.2. *Piecewise C^2 domains.* We will restrict attention to domains with piecewise C^2 boundary. We make the following definition.

DEFINITION 2.7 (Piecewise C^2 domain). An open connected subset $\Omega \subset \mathbb{R}^2$, which does not contain any half-plane, will be called a *piecewise C^2 domain* if its boundary $\partial\Omega$ consists of a finite number of connected parts, each of which consists of a finite number of C^2 closed arcs. We assume $\partial\Omega$ to be positively oriented. If $S_1, S_2, \dots, S_m$ are the boundary arcs of a connected component of $\partial\Omega$, we assume their numbering to be in accordance with the orientation. With the exception that the terminal point of S_i coincides with the initial point of S_{i+1}, $i = 1, 2, \dots, m-1$ (and for $i = m$ as well if the boundary component is bounded) all arcs are assumed mutually disjoint.

Note that we require neither boundedness nor simple connectedness of a piecewise C^2 domain. By a closed arc we mean an arc containing its end points (if it has any).

From now on Ω will always denote a piecewise C^2 domain, unless otherwise stated.

DEFINITION 2.8. Let $S_1, S_2, \dots, S_m$ be the boundary arcs of one of the connected components of $\partial\Omega$. Then the terminal point of S_i (which coincides with the initial point of S_{i+1}) will be denoted V_i and called a *vertex.* The interior angle between S_i and S_{i+1} at V_i will be denoted α_i. A vertex V_i will be called *convex* if $0 \leq \alpha_i < \pi$, *flat* if $\alpha_i = \pi$ and *re-entrant* if $\pi < \alpha_i \leq 2\pi$. A non-flat vertex V_i with $0 < \alpha_i < 2\pi$ will also be referred to as a *corner.*

Note that convex vertices are the only boundary points that do not belong to any near set $N(x)$.

On every boundary arc S_i of a piecewise C^2 domain, the signed curvature κ_i and the center of curvature σ_i are well defined and continuous (for definitions and some

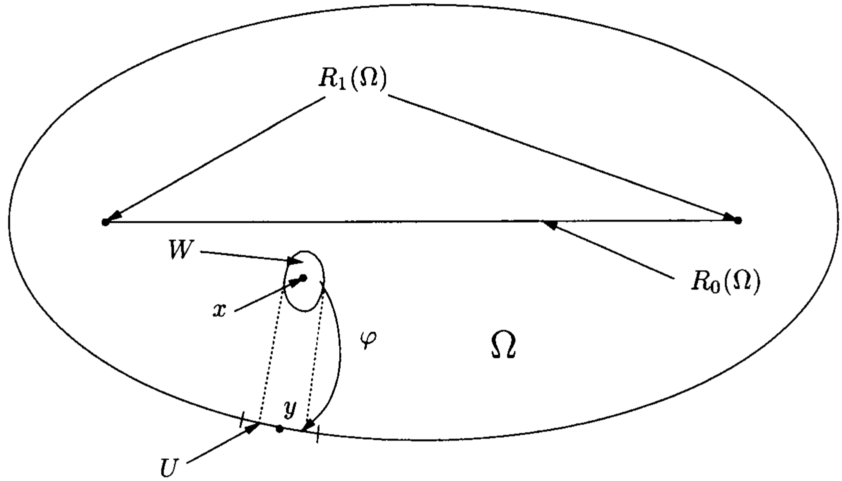

FIG. 3. Illustration of Lemma 2.10

elementary results, see the appendix). This defines the signed curvature κ and the center of curvature σ at every boundary point but the vertices (since κ_i and κ_{i+1} may differ at V_i).

DEFINITION 2.9. For a piecewise C^2 domain Ω we define

$$R_1(\Omega) = \{x \in \Omega : \operatorname{card} N(x) = 1 \text{ and } x = \sigma_i(y) \text{ where } y \in N(x) \cap S_i\}.$$

Thus $x \in R_1(\Omega)$ if x is the center of curvature of $\partial\Omega$ at a point y in the near set $N(x)$.

LEMMA 2.10. Let Ω be a piecewise C^2 domain and let

$$x \in \Omega \setminus \{R_0(\Omega) \cup R_1(\Omega)\}.$$

Then, for every neighbourhood $U \subset \partial\Omega$ of $y \in N(x)$, there is a neighbourhood $W \subset \Omega$ of x and a map $\varphi : W \to \partial\Omega$ with the following properties.

1. $\{\varphi(z)\} = N(z)$ for all $z \in W$.
2. $\varphi(W) \subset U$.
3. φ is continuous on W.
4. $W \cap P(\Omega) = \varnothing$.

Proof. Since $x \notin R_0(\Omega)$ we have $\operatorname{card} N(x) = 1$ and hence $\{y\} = N(x)$, where y is not a convex vertex. Assume that $y \in S_i$. If y is not a re-entrant vertex either, then by Lemma A.4 and the fact that $x \notin R_1(\Omega)$, we have $\kappa_i(y)d(x) < 1$.

1. It is sufficient to prove that $\operatorname{card} N(z) = 1$ in a neighbourhood W of x. If no such neighbourhood exists, we can choose a sequence $\{x_n\}$ in Ω such that $x_n \to x$ and $\operatorname{card} N(x_n) \geq 2$ for all n. Thus there are $u_n, v_n \in N(x_n)$ such that $u_n \neq v_n$ for all n. Choose convergent subsequences $\{u'_n\}$ and $\{v'_n\}$ and put $u = \lim u'_n, v = \lim v'_n$. Then it follows from part 3 of Lemma 2.2 that $u, v \in N(x)$ and thus $u = v = y$.

If y is not a vertex, we can assume that all u'_n and v'_n belong to S_i. Further, since $u'_n, v'_n \in \partial B(x'_n, d(x'_n))$ and $B(x'_n, d(x'_n)) \subset \Omega$, there must be a point $w'_n \in \partial\Omega$ between

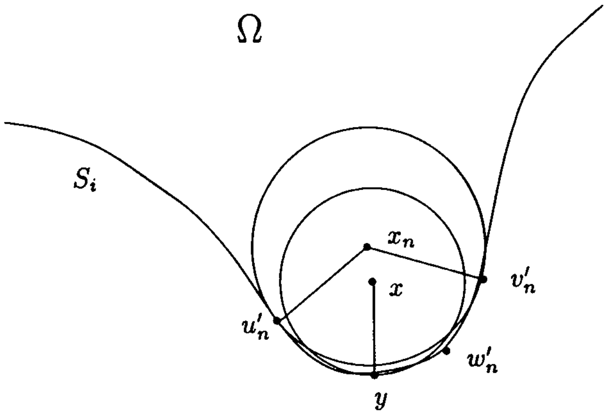

FIG. 4. Proof of part 1 of Lemma 2.10

u_n' and v_n' such that $\kappa_i(w_n') \geq 1/d(x_n')$ (see Fig. 4). Then $w_n' \to y$ and since κ_i and d are continuous it follows that $\kappa_i(y)d(x) \geq 1$, which is a contradiction.

If $y = V_i$ is a re-entrant vertex and $x = y + d(x)n_i(y)$, the same argument applies. If V_i is flat we have to allow v_n' to belong to S_{i+1}. But since, in that case, $x = y + d(x)n_i(y) = y + d(x)n_{i+1}(y)$ and both κ_i and κ_{i+1} are continuous, the same argument as above implies that $\kappa_i(y)d(x) \geq 1$ or $\kappa_{i+1}(y)d(x) \geq 1$, which is also a contradiction. What remains then is the case when V_i is a re-entrant vertex and $x \neq y + d(x)n_i(y)$, $x \neq y + d(x)n_{i+1}(y)$. In this case x is an interior point of the sector with apex at V_i bounded by the lines $y + sn_i(y)$ and $y + sn_{i+1}(y)$. Hence x_n' is in this sector for large n and we must have $u_n' = v_n' = y$, which is the contradiction that finally settles part 1 of the lemma.

2. Let W and φ be chosen in accordance with 1. If there is no neighbourhood $W_0 \subset W$ of x such that $\varphi(W_0) \subset U$, then there is a sequence $\{x_n\}$ in W such that $x_n \to x$ and $\varphi(x_n) \notin U$. Put $u_n = \varphi(x_n)$ and choose a convergent subsequence $\{u_n'\}$. Clearly, its limit u belongs to $N(x) \setminus \{y\}$. Thus $\operatorname{card} N(x) > 1$, which is a contradiction.

3. Assume W and φ to satisfy 1 and 2 and let $x_n \to x_0 \in W$. Put $y_n = \varphi(x_n)$ and choose a convergent subsequence y_n' with limit y_0. Then $y_0 \in N(x_0)$ by part 3 of Lemma 2.2 and hence $y_0 = \varphi(x_0)$ by 1 above. Thus, the continuity of φ follows.

4. Choose W and φ in accordance with 1. Since W is open and $\operatorname{card} N(z) = 1$ for all $z \in W$, it follows that $W \cap \overline{R_0(\Omega)} = \varnothing$. But according to part 1 of Theorem 2.6, $P(\Omega) \subset \overline{R_0(\Omega)}$, and hence we have $W \cap P(\Omega) = \varnothing$.

The lemma is proved. □

LEMMA 2.11. Let Ω be a piecewise C^2 domain and let $x \in \partial\Omega$. Then x is a limit point to $P(\Omega)$ if and only if x is a convex vertex.

Proof. If x is a convex vertex it is obvious that x is a limit point to $P(\Omega)$. On the other hand, let $x \in \overline{P(\Omega)}$ and assume that x is not a convex vertex. Choose a sequence $\{y_n\}$ in $R_0(\Omega)$ such that $y_n \to x$. (This is possible since $P(\Omega) \subset \overline{R_0(\Omega)}$ by Theorem 2.6.) Let $u_n, v_n \in N(y_n)$ be such that $u_n \neq v_n$. Since $d(y_n) \to 0$, we have $\lim u_n = \lim v_n = x$.

Thus we can choose $w_n \in \partial\Omega$ between u_n and v_n such that $\kappa(w_n) \geq 1/d(y_n) \to \infty$, which contradicts the fact that κ is bounded near x. □

Lemma 2.10 has some important implications. First we have a theorem that geometrically characterises the ridge.

THEOREM 2.12 (Ridge decomposition). Let Ω be a piecewise C^2 domain. Then $P(\Omega) = R_0(\Omega) \cup R_1(\Omega)$.

Proof. Clearly $R_0(\Omega) \subset P(\Omega)$. Let $x \in R_1(\Omega)$ and $y \in N(x)$. Assume that $x \notin P(\Omega)$. Then there is a $z \in \Omega$ such that $y \in N(z), d(z) > d(x)$ and $z = y + d(z)n_i(y)$. Then $\kappa_i(y)d(z) > \kappa_i(y)d(x) = 1$, which contradicts Lemma A.4. Thus $R_1(\Omega) \subset P(\Omega)$ and we have proved $R_0(\Omega) \cup R_1(\Omega) \subset P(\Omega)$.

On the other hand, let $x \in \Omega \setminus \{R_0(\Omega) \cup R_1(\Omega)\}$. Then, by part 4 of Lemma 2.10, there is a neighbourhood W of x such that $W \cap P(\Omega) = \varnothing$. Thus $x \notin P(\Omega)$ and hence $P(\Omega) \subset R_0(\Omega) \cup R_1(\Omega)$. □

REMARK. In [4], Caffarelli and Friedman define the ridge of a piecewise C^3 domain as the points of Ω that have no neighbourhood where the distance function has Lipschitz continuous gradient. They prove that the ridge, in this sense, equals the union of $R_0(\Omega)$ and $R_1(\Omega)$. Thus Theorem 2.12 above implies that their ridge concept is equivalent to that of Evans and Harris [6], at least for piecewise C^3 domains.

Finally we can prove that the ridge is relatively closed.

THEOREM 2.13 (Ridge closedness). Let Ω be a piecewise C^2 domain. Then $P(\Omega)$ is closed relative to Ω and the functions p and r are continuous on Ω.

Proof. Let $x \notin P(\Omega)$. By Theorem 2.12 we have $x \notin R_0(\Omega) \cup R_1(\Omega)$ and hence, according to part 4 of Lemma 2.10, there is a neighbourhood W of x such that $W \cap P(\Omega) = \varnothing$. Thus $\Omega \setminus P(\Omega)$ is open and hence $P(\Omega)$ is closed relative to Ω. Theorem 2.4 now implies the continuity of p and r. □

REMARK. For a bounded C^2 domain (hence without vertices), the closedness of $P(\Omega)$ is covered by Theorem 1, p. 2, in a paper by Milman and Waksman [14].

2.3. *Ridge distance.* The distance function d can be extended to $\overline{\Omega}$. ∇d will then exist and be continuous at all boundary points but the vertices. In fact, ∇d will be the inward unit normal to $\partial\Omega$.

Assume that $x \in \partial\Omega$ is not a vertex. Then there is a $y \in \Omega$ such that $x \in N(y)$ and the ridge point $p(y)$ is independent of the actual choice of such a y. Therefore, we can continue p to x by defining $p(x) = p(y)$. This also defines the function $r = d \circ p$ at $x : r(x) = |p(x) - x|$. If x is a convex vertex then, by Lemma 2.11, x is a limit point of the ridge and we define $r(x) = 0$. Thus, if $\partial\Omega$ has no re-entrant vertices, we have a natural continuation of r to $\partial\Omega$, and the following definition makes sense.

DEFINITION 2.14 (Ridge distance). Let Ω be a piecewise C^2 domain without re-entrant vertices. The restriction of r to the boundary $\partial\Omega$ is called the *ridge distance* and is denoted D.

Note that $D(x)$ is *not* the distance from x to the ridge $P(\Omega)$ but the distance from x to the ridge point $p(x)$. Note also that if $x \in N(y)$, then $d(y) \leq D(x)$ with equality if and only if $y = p(x)$.

Let us formulate an immediate corollary to Lemma A.4. Put $\kappa(x) = \infty$ if $x = V_i$ is a convex vertex and $\kappa(x) = \max\{\kappa_i(x), \kappa_{i+1}(x)\}$ if V_i is flat. Let us also agree that $0 \cdot \infty = 0$.

COROLLARY 2.15 (Curvature and ridge distance). Let Ω be a piecewise C^2 domain without re-entrant vertices. Then $\kappa(x)D(x) \leq 1$ for every $x \in \partial\Omega$.

The next theorem is the key to Sec. 2.4.

THEOREM 2.16 (Continuity of ridge distance). Let Ω be a piecewise C^2 domain without re-entrant vertices. Then the ridge distance D is continuous on $\partial\Omega$.

Proof. Let $x \in \partial\Omega$ and let $\{x_n\} \subset \partial\Omega$ be a sequence such that $x_n \to x$. Since $\{D(x_n)\}$ is bounded, the continuity follows if we can prove that $D(x)$ is its only limit point. Choose a subsequence $\{x'_n\}$ such that $\{D(x'_n)\}$ converges and let D^* denote the limit. We may assume that x'_n is not a vertex and hence put $y'_n = p(x'_n)$. Then $x'_n \in N(y'_n)$ and $d(y'_n) = D(x'_n)$ for all n. Let $\{y''_n\}$ be a convergent subsequence of $\{y'_n\}$ and put $y = \lim y''_n$.

Assume that x is not a vertex. Then, by Lemma 2.11, $y \in \Omega$ and Theorem 2.13 implies that $y \in P(\Omega)$. By Lemma 2.2 we have $x \in N(y)$ and hence we must have $y = p(x)$ and $d(y) = D(x)$. Thus the continuity of d yields

$$D^* = \lim D(x''_n) = \lim d(y''_n) = d(y) = D(x).$$

If x is a convex vertex, then $D(x) = 0$ and the continuity follows by elementary geometry. □

REMARK. A proof in the case when Ω is bounded and has no vertices is also given in [14] (Theorem 2, p. 3).

2.4. *Boundary velocity.* Let ω be a subset of the boundary $\partial\Omega$ and let $E(\omega)$ be the points of Ω that have a closest boundary point in ω. If v is a solution to $(\mathcal{P}_1^*)$, we know that the flow in $E(\omega)$ is directed towards ω. The condition $\operatorname{div} v \equiv 1$ thus implies that the flow through ω should equal the area of $E(\omega)$. If we let ω shrink to $x \in \partial\Omega$, we should therefore be able to determine the velocity $v(x)$, that is, the *velocity of the free boundary* $\partial\Omega$ *at* x, by a limit procedure.

To be more precise, let $\omega \subset \partial\Omega$ be an open arc and introduce

$$E(\omega) = \{x \in \Omega : \operatorname{dist}(x, \omega) < \operatorname{dist}(x, \partial\Omega \setminus \omega)\}.$$

Since the distance functions are continuous, it follows that $E(\omega)$ is open, hence measureable. We denote its measure by $\mu(\omega)$ and write $l(\omega)$ for the length of the arc ω.

DEFINITION 2.17 (Area density). Let $x \in \partial\Omega$ and let $\{\omega_k\}_{k=1}^\infty$ be a shrinking sequence of open arcs $\omega_k \subset \partial\Omega$ such that $\{x\} = \bigcap_{k=1}^\infty \omega_k$. We define the *area density* $m(x)$ at x by

$$m(x) = \lim_{k\to\infty} \frac{\mu(\omega_k)}{l(\omega_k)},$$

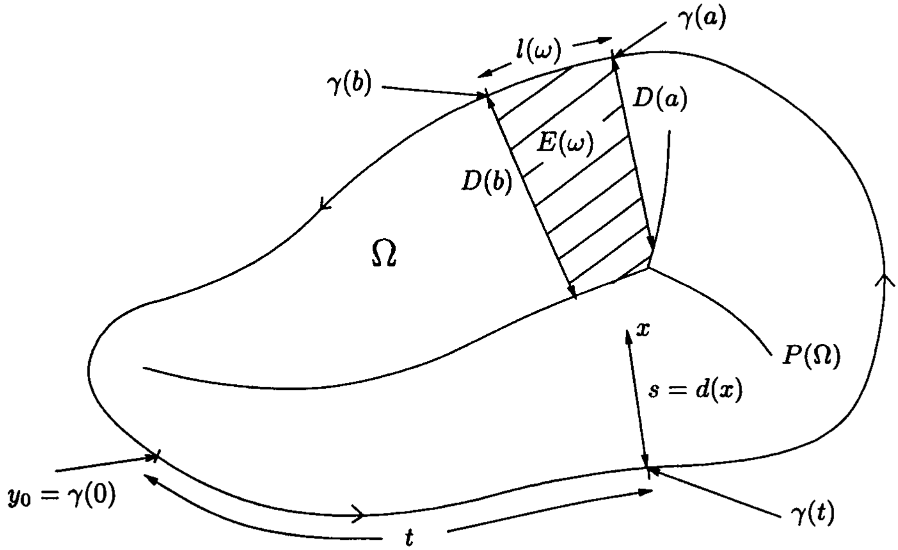

FIG. 5. The new orthogonal coordinates s and t

provided the limit exists and is independent of the actual choice of such a sequence $\{\omega_k\}_{k=1}^{\infty}$.

In order to prove the existence of $m(x)$ and to derive an explicit expression for m, we use the ridge distance to introduce new coordinates in Ω as follows. Assume that Ω only has convex vertices; so re-entrant and flat vertices are no longer allowed. Let Γ be a connected component of $\partial\Omega$ and choose a reference point $y_0 \in \Gamma$. Let $\gamma : I \to \Gamma$, where I is an interval containing 0, be a positively oriented parametrization of Γ with respect to arc length such that $\gamma(0) = y_0$ (if Γ is unbounded, then $I = \mathbb{R}$). Let us identify D and $D \circ \gamma$. Then, for every $x \in E(\Gamma) \setminus P(\Omega)$, there are unique $t \in I$ and $s \in (0, D(t))$ such that $x = \gamma(t) + sn(t)$, where n is the left unit normal to γ. (Since $\gamma(t)$ cannot be a convex vertex, and since we have no re-entrant vertices, $n(t)$ is well defined.) This defines curvilinear coordinates s and t on $E(\Gamma) \setminus P(\Omega)$. If $\omega = \gamma((a, b))$, then $E(\omega) \setminus P(\Omega)$ corresponds to the set

$$E'(\omega) = \{(s, t) : a < t < b, 0 < s < D(t)\}$$

in the (s, t)-plane.

According to Lemma A.3,

$$\frac{\partial x}{\partial t}(s, t) = (1 - s\kappa(t))\gamma'(t) \quad \text{and} \quad \frac{\partial x}{\partial s}(s, t) = n(t),$$

and it follows that the new coordinates are orthogonal. Thus, using the fact that $P(\Omega)$ is a null set (Theorem 2.6), and assuming that ω contains no vertices (so that κ is defined and continuous on (a,b)), we obtain

$$\begin{aligned}\mu(\omega) &= \int_{E(\omega)} dx = \int_{E(\omega)\setminus P(\Omega)} dx \\ &= \iint_{E'(\omega)} \left|\frac{\partial x}{\partial t}(s,t)\right| \left|\frac{\partial x}{\partial s}(s,t)\right| ds\,dt \\ &= \int_a^b \int_0^{D(t)} (1 - s\kappa(t))\,ds\,dt \\ &= \int_a^b (D(t) - D(t)^2\kappa(t)/2)\,dt,\end{aligned}$$

where the integrand is continuous. Now we can state the following.

THEOREM 2.18 (Area density). Let Ω be a piecewise C^2 domain with only convex vertices. Then the area density m is well defined and continuous on $\partial\Omega$ and given by

$$m(x) = \begin{cases} D(x) - D(x)^2\kappa(x)/2 & \text{if } x \text{ is not a vertex,} \\ 0 & \text{if } x \text{ is a vertex.} \end{cases} \tag{12}$$

Proof. Let $x_0 = \gamma(t_0) \in \partial\Omega$ and let $\{\omega_k\}$ be as in Def. 2.17, that is, $\omega_k = \gamma((a_k, b_k))$ with $a_k \nearrow t_0$ and $b_k \searrow t_0$ as $k \to \infty$. We can assume that $x_0 = \gamma(t_0)$ is the only possible vertex in each of the sets ω_k. If x_0 is a convex vertex, then $D - D^2\kappa/2$ tends to zero at x_0 since D tends to zero and κ is bounded. Otherwise both D and κ are continuous at x_0. Thus the argument above implies

$$\begin{aligned}m(x_0) &= \lim_{k\to\infty} \frac{\mu(\omega_k)}{l(\omega_k)} \\ &= \lim_{k\to\infty} \frac{1}{b_k - a_k} \int_{a_k}^{b_k} (D(t) - D(t)^2\kappa(t)/2)\,dt \\ &= \lim_{t\to t_0} (D(t) - D(t)^2\kappa(t)/2) \\ &= \begin{cases} D(x_0) - D(x_0)^2\kappa(x_0)/2 & \text{if } x_0 \text{ is not a vertex,} \\ 0 & \text{if } x_0 \text{ is a vertex,} \end{cases}\end{aligned}$$

and the theorem is proved. □

Thus we have motivated the following definition, which will be the main tool in our study of the time-dependent evolution problem.

DEFINITION 2.19 (Boundary velocity). Let Ω be a piecewise C^2 domain with only convex vertices. Then the *boundary velocity* v is defined on $\partial\Omega$ by

$$v(x) = \begin{cases} -m(x)\nabla d(x) & \text{if } x \text{ is not a vertex,} \\ 0 & \text{if } x \text{ is a vertex.} \end{cases} \tag{13}$$

Note that Theorem 2.18 guarantees that the boundary velocity v is well defined and continuous on $\partial\Omega$, since $|\nabla d| \equiv 1$ on its domain of definition, and since m tends to zero at the vertices (which is exactly the points where ∇d does not exist).

Some simple properties of signed curvature, ridge distance, and area density are collected in Lemmas A.5 and A.6 in the appendix.

2.5. *Explicit solution to* $(\mathcal{P}_1^*)$. What remains is to show that the boundary velocity, as defined above, really is connected to an extremal to $(\mathcal{P}_1^*)$.

Once again Ω is a piecewise C^2 domain with only convex vertices. Take $x \in \Omega \setminus P(\Omega)$ and let Ω_1 be a connected component of

$$\{x_1 \in \Omega : d(x_1) > d(x)\}$$

such that $x \in \partial\Omega_1$. If Ω_1 is a piecewise C^2 domain, then we have a well-defined boundary velocity v_1 on $\partial\Omega_1$ and

$$v_1(x) = -(D_1(x) - D_1(x)^2\kappa_1(x)/2)\nabla d_1(x)$$

since x cannot be a vertex on $\partial\Omega_1$. Further, there is a unique $y \in \partial\Omega$ such that $|x-y| = d(x)$, which can be written $y = x - d(x)\nabla d(x)$. Since x lies on an arc of $\partial\Omega_1$, which is a parallel curve to the arc of $\partial\Omega$ on which y lies, it follows from Lemma A.3 and Lemma A.5 that

$$D_1(x) = D(y) - d(x) \quad \text{and} \quad \kappa_1(x) = \frac{\kappa(y)}{1 - d(x)\kappa(y)}.$$

Thus we have

$$v_1(x) = -\left(D(y) - d(x) - \frac{1}{2}(D(y) - d(x))^2\frac{\kappa(y)}{1 - d(x)\kappa(y)}\right)\nabla d(x). \tag{14}$$

The right-hand side of (14) is well defined for every $x \in \Omega \setminus P(\Omega)$; so we define v_1 on $\Omega \setminus P(\Omega)$ by (14). Since y depends continuously on x (part 3 of Lemma 2.10) and since both D and κ are continuous at y (since y cannot be a vertex), it follows that v_1 is a continuous function on $\Omega \setminus P(\Omega)$. We want to extend it to Ω. The boundedness of κ and the fact that $\kappa \le 1/D$ implies that

$$\left|(D(y) - d(x))^2\frac{\kappa(y)}{1 - d(x)\kappa(y)}\right| \le (D(y) - d(x))^2\frac{C}{1 - d(x)/D(y)} = CD(y)(D(y) - d(x))$$

for some constant C. Clearly $D(y) - d(x) \to 0$ when $\Omega \setminus P(\Omega) \ni x \to x_0 \in P(\Omega)$; so v_1 becomes a continuous function on Ω if we define $v_1(x_0) = 0$ at every $x_0 \in P(\Omega)$.

THEOREM 2.20 (Explicit solution to $(\mathcal{P}_1^*)$). Let Ω be a bounded piecewise C^2 domain with only convex vertices, and define v_1 on Ω as above. Then v_1 is an extremal to $(\mathcal{P}_1^*)$. Further, if we let $F = |v_1|$, then $v_1 = -F\nabla d$ a.e. in Ω.

Proof. First we need to prove that $v_1 \in \mathcal{A}_1^*$, that is,

$$\int_\Omega v_1 \cdot \nabla\varphi\, dx = -\int_\Omega \varphi\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. If this is true, then it is true with φ replaced by d as well, since d can be approximated sufficiently well in $C_0^\infty(\Omega)$ if Ω is bounded. Thus it follows that

$$\int_\Omega d\, dx = \int_{\Omega\setminus P(\Omega)} F\nabla d \cdot \nabla d\, dx = \int_\Omega |v_1|\, dx$$

and hence, by Theorem 1.2, v_1 is an extremal.

Let S_i be a boundary arc of $\partial\Omega$ and let γ be a parametrization of S_i with respect to arc length such that $\gamma(0) = V_{i-1}$ and $\gamma(l) = V_i$. (S_i is bounded since Ω is.) Now we follow the argument preceding Theorem 2.18. For every $x \in E(S_i) \setminus P(\Omega)$ there are unique t and s such that $0 < t < l$, $0 < s < D(t)$, and $x = \gamma(t) + s\nabla d(t)$. In these new coordinates s and t the area measure was found to be $(1 - s\kappa(t))\,ds\,dt$.

Put $y = x - d(x)\nabla d(x)$, that is, $y = \gamma(t)$. It is easily verified that $F = |v_1|$ can be written as

$$F(x) = \frac{1}{1 - \kappa(y)d(x)} \int_{d(x)}^{D(y)} (1 - \kappa(y)\zeta)\,d\zeta,$$

which in the new coordinates s and t becomes

$$F(s,t) = \frac{1}{1 - s\kappa(t)} \int_{s}^{D(t)} (1 - \zeta\kappa(t))\,d\zeta.$$

Let $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
&\int_{E(S_i)\setminus P(\Omega)} v_1(x)\cdot\nabla\varphi(x)\,dx \\
&\quad= -\int_{E(S_i)\setminus P(\Omega)} F(x)\nabla d(x)\cdot\nabla\varphi(x)\,dx \\
&\quad= -\int_0^l \int_0^{D(t)} F(s,t)\frac{\partial\varphi}{\partial s}(s,t)(1 - s\kappa(t))\,ds\,dt \\
&\quad= -\int_0^l \int_0^{D(t)} \left\{\int_s^{D(t)} (1 - \zeta\kappa(t))\,d\zeta\right\} \frac{\partial\varphi}{\partial s}(s,t)\,ds\,dt \\
&\quad= -\int_0^l \left\{\left[\int_s^{D(t)} (1 - \zeta\kappa(t))\,d\zeta\varphi(s,t)\right]_{s=0}^{D(t)} \right.\\
&\qquad\qquad \left. - \int_0^{D(t)} -(1 - s\kappa(t))\varphi(s,t)\,ds\right\} dt \\
&\quad= -\int_0^l \int_0^{D(t)} (1 - s\kappa(t))\varphi(s,t)\,ds\,dt \\
&\quad= -\int_{E(S_i)\setminus P(\Omega)} \varphi(x)\,dx.
\end{aligned}$$

Now it follows that $v_1 \in \mathcal{A}_1^*$, because $\Omega = (\bigcup_i E(S_i)) \cup P(\Omega)$, and $P(\Omega)$ is a null set (Theorem 2.6), which also implies that $v_1 = -F\nabla d$ a.e. in Ω. □

The following corollary is immediate.

COROLLARY 2.21. Let Ω be a piecewise C^2 domain with only convex vertices and let v_1 be the extremal in Theorem 2.20. Then

$$\lim_{x\to y\in\partial\Omega} |v_1(x)| = m(y),$$

that is, the extremal v_1 can be extended continuously to $\partial\Omega$ such that its restriction to $\partial\Omega$ equals the boundary velocity as defined in Def. 2.19.

We end this section with a theorem which states that the velocity of the center of mass of Ω is zero.

THEOREM 2.22. Let Ω be a bounded piecewise C^2 domain with only convex vertices and let v_1 be the extremal in Theorem 2.20. Then

$$\int_{\Omega} v_1(x)\,dx = 0.$$

Proof. Applying the same change of coordinates as in the proof of Theorem 2.20, we obtain in a similar way that

$$\int_{\Omega} v_1(x)\,dx = -\int_{\Omega\setminus P(\Omega)} d(x)\nabla d(x)\,dx.$$

Let $D^* = \max_{x\in\partial\Omega} D(x)$ and for $s \in (0, D^*)$ put

$$\Omega_s = \{x \in \Omega : d(x) > s\}.$$

$\nabla d(x)$ exists at almost every point $x \in \partial\Omega_s$. Moreover, $\nabla d(x)$ is the inward unit normal to $\partial\Omega_s$ at all points where it exists. Since $\partial\Omega_s$ is a closed curve (or a union of closed curves), it follows that

$$\int_{\partial\Omega_s\setminus P(\Omega)} \nabla d(x)\,dS = 0$$

for all $s \in (0, D^*)$, and we obtain

$$\int_{\Omega\setminus P(\Omega)} d(x)\nabla d(x)\,dx = \int_0^{D^*} s\left(\int_{\partial\Omega_s\setminus P(\Omega)} \nabla d(x)\,dS\right) ds = 0,$$

which proves the theorem. □

3. The evolution problem. In this section we will study the *evolution problem*, that is, given the domain initially occupied by the polymer, determine the domain it occupies at every later time during the compression. For this purpose we first of all need to define a suitable solution concept. We will restrict ourselves to so-called classical solutions and derive certain properties, such as uniqueness, for this class.

3.1. *Classical solutions.* By a solution of the evolution problem we roughly mean an expanding one-parameter family $\{\Omega(t)\}_{t\in[0,T)}$ of piecewise C^2 domains $\Omega(t)$ in the plane, consistent with formula (13). Since this is a point-wise relation between continuous quantities, we talk about *classical solutions of the evolution problem.*

Concerning vertices, we accept a finite number of convex vertices, but the set of vertices must not increase. It is also consistent with Eq. (13) to require that a convex vertex does not move. The interior angle of a vertex may however increase to π, and thus a boundary point may cease to be a vertex from a certain moment and then start to move.

Before we specify the conditions above more exactly, we note that a boundary arc S of a bounded piecewise C^2 domain is always of one of the following three types.

Type 1: S is an arc joining two vertices V_i and V_j.

Type 2: S is an arc for which the end points meet at the same vertex V_i.

Type 3: S is a closed curve without vertices.

By a *classical solution of the evolution problem with initial data* Ω_0, we mean a set function $\Omega(t)$, defined on some interval $[0, T)$, which satisfies the six conditions stated below. We will use the notation $\{\Omega(t)\}_{t\in[0,T)}$ for such a function.

Condition 1: $\Omega(0) = \Omega_0$.

Condition 2: $\{\Omega(t)\}_{t\in[0,T)}$ is an expanding family, that is, if $0 \leq t_1 \leq t_2 < T$, then $\Omega(t_1) \subset \Omega(t_2)$.

Condition 3: With only two kinds of exceptions (described below), $\Omega(t)$ is for every $t \in [0, T)$ a bounded piecewise C^2 domain with only convex vertices.

Let $\mathcal{V}_t$ be the set of vertices of $\partial\Omega(t)$. Then $\mathcal{V}_0$ is finite because of condition 3. That vertices are not allowed to move, and new vertices not to occur, is accomplished by the following.

Condition 4: The family $\{\mathcal{V}_t\}_{t\in[0,T)}$ is shrinking, that is, if $0 \leq t_1 \leq t_2 < T$, then $\mathcal{V}_{t_1} \supset \mathcal{V}_{t_2}$.

To proceed we need the important concept of waiting time. We make the following definition.

DEFINITION 3.1 (Waiting time). For every $x \in \mathcal{V}_0$ we define the *waiting time* t_x^* by $t_x^* = \sup\{t \in [0, T) : x \in \mathcal{V}_t\}$.

Let $t_1, t_2, \ldots, t_M$ be the distinct waiting times of the vertices of $\partial\Omega(0)$ ordered such that

$$0 \leq t_1 < t_2 < \cdots < t_M < T.$$

Thus, at every t_i at least one vertex disappears. At the precise moment when a vertex disappears, it is allowed to be a flat vertex, but this is the only circumstance under which non-convex vertices are allowed to exist. This is the first kind of exception to condition 3. If the disappearing vertex is the point where two arcs of the first type meet, then the number of arcs is reduced and it is understood that the two arcs are replaced by a new arc of the first or second type. If the disappearing vertex is the end point of an arc of the second type, then the number of arcs is not reduced but this arc is replaced by an arc of the third type.

We also allow "holes" in $\Omega(t)$ to be "filled" and hence disappear, that is, we allow arcs of the third type to collapse to points and then disappear. Let $\tau_1, \tau_2, \ldots, \tau_N$ be the times when this happens, ordered so that

$$0 < \tau_1 < \tau_2 < \cdots < \tau_N < T.$$

Thus, at every τ_j at least one arc disappears. At the precise moment when this happens, $\Omega(t_j)$ is punctured and hence not a piecewise C^2 domain, but this is the only circumstance under which $\Omega(t)$ is allowed not to be a piecewise C^2 domain. This is the second kind of exception to condition 3.

Despite these exceptions, we emphasize that condition 3 implies that coalescence is excluded.

Let $[a, b] \subset [0, T)$ be an interval that contains no t_i and no τ_j. Then the number of vertices, arcs and connected components of $\partial\Omega(t)$ is constant for $t \in [a, b]$ and all vertices are convex. We will need parametrizations of the boundary arcs and some kind of continuity property for the expansion. Therefore, for any such interval we require the following.

Condition 5: There exists a constant $C > 0$ and for every boundary arc $S(t)$ a C^2 function $\gamma : [0,1] \times [a,b] \to \mathbb{R}^2$ such that

(i) $\sigma \mapsto \gamma(\sigma, t)$ is a one-to-one parametrization of $S(t)$ for every $t \in [a,b]$,

(ii) $|\gamma'_\sigma| \geq C$ on $[0,1] \times [a,b]$,

(iii) $\gamma(0,t) \equiv V_i$ and $\gamma(1,t) \equiv V_j$ if S is of the first type,
$\gamma(0,t) \equiv V_i \equiv \gamma(1,t)$ if S is of the second type,
$\gamma(0,t) \equiv \gamma(1,t)$ if S is of the third type.

Moreover, γ can be extended continuously to $[0,1] \times [a_0, b_0]$, where $[a_0, b_0]$ is any interval on which the arc $S(t)$ is defined.

Finally, the movement of the boundary must be consistent with (13), that is, the boundary curve's intersection with its own instantaneous normal must move with a velocity that equals the boundary velocity as given by Def. 2.19. This can be expressed in terms of the above-introduced parametrizations. For this purpose, let m_t be the area density on $\partial\Omega(t)$ according to Def. 2.17. The consistency relation is formulated as follows.

Condition 6: Let $t \in [0,T)$ and $x \in \partial\Omega(t) \setminus \{y \in \mathcal{V}_0 : t^*_y = t\}$ be arbitrary and let γ be a parametrization of the arc $S(t)$ with properties as in condition 5. Then

$$\gamma'_t(\sigma, t) \cdot n_t(x) = m_t(x), \tag{15}$$

where σ is such that $\gamma(\sigma, t) = x$ and $n_t(x)$ is the outward unit normal to $S(t)$ at x.

A few remarks are in order.

REMARKS.

1. The first condition implies that the initial data Ω_0 must be a piecewise C^2 domain without re-entrant vertices. This may seem too restrictive, and with small changes of the definition it is possible to allow a finite number of re-entrant vertices on $\partial\Omega_0$. Since the area density is unbounded at a re-entrant vertex, such a vertex should cease to exist immediately.

2. As we have already mentioned, the third condition means that a convex vertex does not move as long as it exists, which is consistent with the sixth condition, since $m_t(x) = 0$ if x is a convex vertex. This motivates why t^*_x is called the waiting time of x. At this stage it is not clear whether t^*_x really can be strictly positive or not. We will return to this question in the next section.

3. Since we require a vertex to disappear immediately when it becomes flat, it follows that $x \in \partial\Omega(t)$, $t > 0$, is a vertex only if x is a *convex* vertex of $\partial\Omega_0$.

4. The fifth condition imposes a kind of regularity on the expansion of $\Omega(t)$ and also provides us with "nice" parametrizations of the boundary arcs. These parametrizations become very important in the proof of the comparison principle (Theorem 3.6).

5. The sixth condition states that the "normal velocity" of the boundary equals the boundary velocity introduced in Sec. 2.4. Note, however, that we do not say anything about this velocity at flat vertices, since the area density is undefined at such points. If x is a convex vertex, there are two possible choices of the arc S and hence the normal $n_t(x)$. But since in this case both $\gamma'_t(\sigma, t)$ and $m_t(x)$ vanish, the relation (15) still makes sense. It is therefore sensible to write

$$v_t(x) = (\gamma'_t(\sigma, t) \cdot n_t(x)) n_t(x) = m_t(x) n_t(x)$$

for all $x \in \partial\Omega(t)$ that are not flat vertices, since this implies $|v_t(x)| = m_t(x)$.

6. It is a consequence of condition 5 that a classical solution $\Omega(t)$ is a set with Lipschitzian growth, at least locally in space and time for $t \neq \tau_j$.

So far we have restricted attention to bounded classical solutions. After small modifications we can omit the boundedness requirement in the third condition. What happens then is that we get a fourth and a fifth type of boundary arcs, both unbounded, namely with or without an end point. These can be treated very much like the arcs of types 2 and 3 respectively. In the fifth condition the interval $[0,1]$ needs to be replaced by $[0,\infty)$ and $(-\infty,\infty)$ respectively when it comes to these new types of arcs.

3.2. *Particular solutions.* Below we list some simple but important particular solutions of the evolution problem. It is possible to derive less trivial solutions from the theory of the porous medium equation. For a few words about this, see Sec. 3.5.

The expanding circle. Let $\Omega_0 = B(r_0)$ be an open disk of radius r_0. Due to the symmetry of Ω_0, a classical solution should be of the form $\Omega(t) = B(r(t))$. Equation (15) becomes $r'(t) = m_t(x)$ and by (12) we have

$$m_t(x) = r(t) - \frac{1}{2}(r(t))^2\frac{1}{r(t)} = \frac{r(t)}{2}.$$

Thus we must have $r' = r/2$, that is, $r(t) = r_0e^{t/2}$. This classical solution we call the *expanding circle* and we denote it by $\mathcal{E}_{r_0}(t) = \Omega(t)$.

The infinite sector. Let

$$\Omega_0 = \{(\xi,\eta) \in \mathbb{R}^2 : 0 < |\eta|/\xi < \tan\alpha_0\},$$

where $0 < \alpha_0 < \pi/2$, that is, Ω_0 is a sector with interior angle $2\alpha_0$. We seek a solution of the form

$$\Omega(t) = \{(\xi,\eta) \in \mathbb{R}^2 : 0 < |\eta|/\xi < \tan\alpha(t)\},$$

with $0 < \alpha(t) < \pi/2$. At $x = (\xi, \xi\tan\alpha(t)) \in \partial\Omega(t)$, the "normal velocity" of the boundary is

$$\xi\alpha'(t)/\cos\alpha(t)$$

and the area density becomes

$$m_t(x) = \xi\tan\alpha(t)/\cos\alpha(t).$$

Hence Eq. (15) becomes $\alpha' = \tan\alpha$, which implies

$$\alpha(t) = \arcsin((\sin\alpha_0)e^t).$$

This gives a solution for $0 \leq t < -\ln(\sin\alpha_0)$, which we call the *infinite sector.* We denote it by $\mathcal{I}_{\alpha_0}(t) = \Omega(t)$. Note that $\mathcal{I}_{\alpha_0}(t)$ approaches the right half plane as $t \to -\ln(\sin\alpha_0)$, in the sense that $\alpha(t) \to \pi/2$ and

$$\bigcup_{t\in(0,-\ln(\sin\alpha_0))} \mathcal{I}_{\alpha_0}(t) = \{(\xi,\eta) \in \mathbb{R}^2 : \xi > 0\}.$$

The thickening band. Let Ω_0 be a band of width $2b_0$ and seek a solution

$$\Omega(t) = \{(\xi,\eta) \in \mathbb{R}^2 : |\eta| < b(t)\}.$$

Then the consistence relation (15) becomes $b'(t) = m_t(x)$ where $m_t(x) = b(t)$ for all $x \in \partial\Omega(t)$. Hence $b(t) = b_0e^t$ makes $\Omega(t)$ a classical solution for all $t \geq 0$.

The annulus. Consider the *annulus* $\Omega(t) = B(R(t)) \setminus \overline{B(r(t))}$. We obtain a solution if we choose the functions r and R such that

$$R' = \frac{R-r}{2} - \frac{1}{2}\left(\frac{R-r}{2}\right)^2 \frac{1}{R},$$
$$-r' = \frac{R-r}{2} + \frac{1}{2}\left(\frac{R-r}{2}\right)^2 \frac{1}{r}.$$

Multiplying these relations by R and r respectively, their sum and difference become

$$(R^2 - r^2)' = R^2 - r^2,$$
$$(R^2 + r^2)' = (R-r)^2/2.$$

We cannot give explicit expressions for r and R but we see that $R(t)^2 - r(t)^2 = (R_0^2 - r_0^2)e^t$. This is a classical solution as long as $r(t) > 0$. If t^* is such that $r(t^*) = 0$ and $R_1 = R(t^*)$, then we can choose $r(t) = 0$ and $R(t) = R_1 e^{(t-t^*)/2}$ for $t \geq t^*$. With this choice, $\Omega(t)$ becomes a classical solution for $t \geq t^*$ as well.

Note that $r' \to -\infty$ when $r \to 0$, that is, when $t \to t^*_-$. At the same time we have that $R' \to 3R_1/8$, whereas $R' \to R_1/2$ when $t \to t^*_+$. Thus R' has a jump discontinuity at the time when the annulus becomes a disk. This is caused by the discontinuous development of the ridge at $t = t^*$.

3.3. *A waiting time estimate.* Now we turn our attention to a comparison principle, involving the expanding circle and the infinite sector. Consider any classical solution $\Omega(t)$ with $\Omega(0) \supset\supset \mathcal{E}_{r_0}(0)$ (that is, $\Omega(0) \supset \overline{\mathcal{E}_{r_0}(0)}$). We want to prove that $\Omega(t) \supset\supset \mathcal{E}_{r_0}(t)$ for all $t > 0$. Let t be any fixed time for which this is true and put

$$\delta = \operatorname{dist}(\partial\mathcal{E}_{r_0}, \partial\Omega),$$

where we omit the dependence on t. Take $x_1 \in \partial\mathcal{E}_{r_0}$ and $x_2 \in \partial\Omega$ such that $|x_1 - x_2| = \delta$. Clearly, x_1 is not a vertex. Let us put subscripts 1 and 2 on functions defined on $\partial\mathcal{E}_{r_0}$ and $\partial\Omega$, respectively. At the points x_1 and x_2 we have $v_1 \| v_2$ and since

$$D_1 = r \quad \text{and} \quad \kappa_1 = 1/r,$$
$$D_2 \geq D_1 + \delta \quad \text{and} \quad \kappa_2 \leq 1/D_2,$$

we get

$$|v_1| = r/2 \quad \text{and} \quad |v_2| \geq D_2/2 \geq (r+\delta)/2,$$

and hence that

$$|v_2| - |v_1| \geq \delta/2 > 0.$$

This actually implies that δ is an increasing function; so $\delta(t) > 0$ for all t and the inclusion follows. (See comment below on differentiability.)

In the same manner, $\Omega(0) \subset\subset \mathcal{I}_{\alpha_0}(0)$ implies $\Omega(t) \subset\subset \mathcal{I}_{\alpha_0}(t)$, if we only require that $\Omega(t)$ is bounded for every fixed t (this is always true if $\Omega(0)$ is bounded due to Theorem 3.4 below). Indeed, for fixed t, put

$$\delta = \operatorname{dist}(\partial\Omega, \partial\mathcal{I}_{\alpha_0}),$$

and let $x_2 \in \partial\Omega(t)$ and $x_3 \in \partial\mathcal{I}_{\alpha_0}(t)$ be such that $\delta = |x_2 - x_3|$ (this is where the boundedness of Ω is needed). All entities related to the infinite sector are now denoted by a subscript 3. Then

$$D_3 \geq D_2 + \delta \quad \text{and} \quad \kappa_2 \geq \kappa_3 = 0;$$

so

$$|v_3| = D_3 \quad \text{and} \quad |v_2| = D_2 - D_2^2\kappa_2/2 \leq D_2,$$

and hence

$$|v_3| - |v_2| \geq D_3 - D_2 \geq \delta > 0.$$

This is true, also, if x_2 is a vertex. We formulate this result as a lemma.

LEMMA 3.2. Let $\{\Omega(t)\}_{t\in[0,T)}$ be a bounded classical solution such that

$$\mathcal{E}_{r_0}(0) \subset\subset \Omega(0) \subset\subset \mathcal{I}_{\alpha_0}(0).$$

Then

$$\mathcal{E}_{r_0}(t) \subset\subset \Omega(t) \subset\subset \mathcal{I}_{\alpha_0}(t)$$

for all $t \in [0, \min(T, -\ln\sin\alpha_0))$.

In the arguments above, one has to convince oneself that δ is differentiable, at least for almost every t, and that the derivative $\delta'(t)$ can be estimated by comparing boundary velocities at points where the distance $\delta(t)$ is obtained. This is a little technical, and since this kind of argument has to be repeated in the proof of the general comparison principle (Theorem 3.6) we postpone that discussion until then.

Lemma 3.2 can be sharpened a little.

THEOREM 3.3. Let $\{\Omega(t)\}_{t\in[0,T)}$ be a bounded classical solution with

$$\mathcal{E}_{r_0}(0) \subset \Omega(0) \subset \mathcal{I}_{\alpha_0}(0).$$

Then

$$\mathcal{E}_{r_0}(t) \subset \Omega(t) \subset \mathcal{I}_{\alpha_0}(t)$$

for all $t \in [0, \min(T, -\ln\sin\alpha_0))$.

Proof. For any $\delta > 0$, let $\mathcal{E}_{r_0-\delta}$ be an expanding circle centered at the same point as $\mathcal{E}_{r_0}$ (see Fig. 6). Then $\mathcal{E}_{r_0-\delta}(0) \subset\subset \Omega(0)$ and by the lemma above, $\mathcal{E}_{r_0-\delta}(t) \subset\subset \Omega(t)$, and hence

$$\mathcal{E}_{r_0}(t) = \bigcup_{\delta>0} \overline{\mathcal{E}_{r_0-\delta}(t)} \subset \Omega(t).$$

Let $\mathcal{I}_{\alpha_0}^{\delta}$ be a translation of $\mathcal{I}_{\alpha_0}$ a distance δ along its axis of symmetry such that $\mathcal{I}_{\alpha_0}^{\delta} \supset \mathcal{I}_{\alpha_0}$ (see Fig. 6). Then

$$\Omega(0) \subset\subset \mathcal{I}_{\alpha_0}^{\delta}(0),$$

and by Lemma 3.2,

$$\Omega(t) \subset \bigcap_{\delta>0} \mathcal{I}_{\alpha_0}^{\delta}(t) = \overline{\mathcal{I}_{\alpha_0}(t)},$$

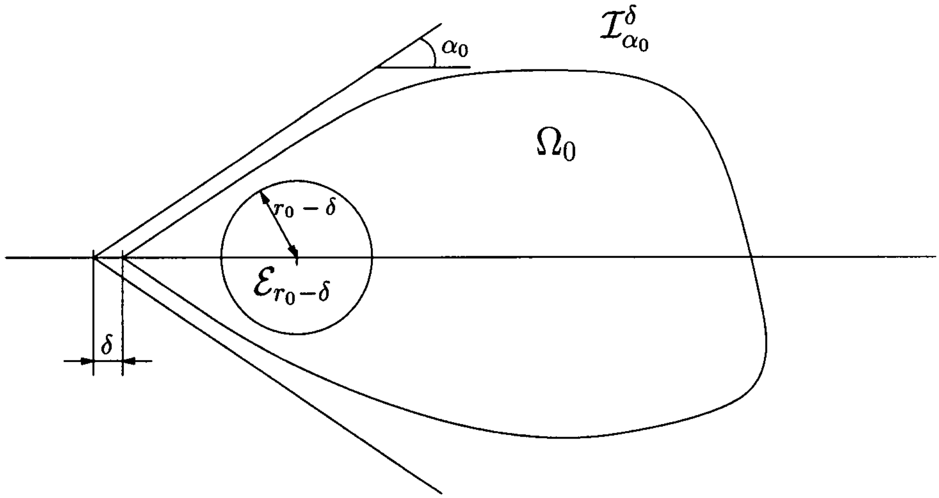

FIG. 6. Illustration of the proof of Theorem 3.3

which proves the theorem, since $\Omega(t)$ is an open set. □

The arguments above can be repeated with an expanding circle circumscribing $\Omega(0)$ if $\Omega(0)$ is bounded. If t is such that $\Omega(t) \subset\subset \mathcal{E}_{r_0}(t)$ and $\delta(t)$ is defined as above, we can choose x_1 on $\partial\Omega(t)$ and x_2 on the circle such that $\delta(t) = |x_1 - x_2|$. Comparing boundary velocities we get

$$\begin{aligned} |v_2| - |v_1| &= D_2 - D_2^2\kappa_2/2 - D_1 + D_1^2\kappa_1/2 \\ &\geq (D_2 - D_1) - (D_2 + D_1)(D_2 - D_1)\kappa_2/2 \\ &= (D_2 - D_1)(1 - (D_1 + D_2)\kappa_2/2), \end{aligned}$$

where the last expression clearly is strictly positive since

$$(D_1 + D_2)\kappa_2/2 < D_2\kappa_2 \leq 1.$$

Further, $|v_2| - |v_1| > 0$ is trivially true if x_1 is a vertex. Thus we have the following theorem.

THEOREM 3.4 (Uniform boundedness). Let $\{\Omega(t)\}_{t\in[0,T)}$ be a classical solution with $\Omega(0)$ bounded. Then $\Omega(t)$ is uniformly bounded for $t \in [0, T)$.

An important application of Theorem 3.3 is to estimate the time a corner persists, that is, the waiting time.

COROLLARY 3.5 (Waiting time estimate). Let $\{\Omega(t)\}_{t\in[0,T)}$ be a bounded classical solution with $\Omega(0)$ convex and $x \in \partial\Omega(0)$ a vertex with interior angle $2\alpha_0$. Then

$$\min(T, \ln(1/\sin\alpha_0)) \leq t_x^* \leq 2\ln(1/\sin\alpha_0).$$

Proof. Comparison with the infinite sector directly gives the estimate $t_x^* \geq \ln(1/\sin\alpha_0)$. Comparison with an included circle of radius r_0, with its center a distance R from the vertex, gives $t_x^* \leq 2\ln(R/r_0)$. Proper choice of the circle finally gives $t_x^* \leq 2\ln(1/\sin\alpha_0)$. □

REMARK. If the convexity requirement is replaced by the inclusion $\Omega(0) \subset \mathcal{I}_{\beta_0}(0)$, where the infinite sector $\mathcal{I}_{\beta_0}(0)$ has its apex at x, then the estimate becomes

$$\ln(1/\sin\beta_0) \leq t_x^* \leq 2\ln(1/\sin\alpha_0).$$

In this way we can give a lower bound for the waiting time even if x is a cusp.

The importance of this corollary is not only the actual form on the bounds but also the fact that the waiting time is always strictly positive and finite. Thus a convex vertex remains a vertex, without moving, for some positive, finite time.

3.4. *Comparison principle and uniqueness.* After the question of existence, that of uniqueness is one of the most important. This will follow from a comparison principle to be proved in this section.

The main idea is (as in the previous section) to estimate the derivative of the function $\delta(t)$ by comparing boundary velocities at suitable points. For this part of the proof it might be possible to argue as in the proof of Lemma 3.2 to prove that, for bounded classical solutions, $\Omega_1(0) \subset\subset \Omega_2(0)$ implies $\Omega_1(t) \subset\subset \Omega_2(t)$. If we do not allow any vertices, the continuity of the expansions then gives that $\Omega_1(0) \subset \Omega_2(0)$ implies $\Omega_1(t) \subset \Omega_2(t)$.

However, we want to consider two general solutions; so we choose a slightly different approach for this part of the proof. (The main difference is that we will make a proof by contradiction instead of a direct one. The estimates we need to do are roughly the same.) Before we can compare boundary velocities, though, we have to study the differentiability of $\delta(t)$.

THEOREM 3.6 (Comparison principle). Let $\{\Omega_1(t)\}_{t\in[0,T)}$ and $\{\Omega_2(t)\}_{t\in[0,T)}$ be bounded classical solutions. If $\Omega_1(0) \subset \Omega_2(0)$, then $\Omega_1(t) \subset \Omega_2(t)$ for all $t \in [0,T)$.

Proof. To make the proof easier to follow, we first study the case without vertices and we also assume $\partial\Omega_2(t)$ to be connected, since otherwise we can apply all arguments below to each connected component separately. We give the main outline with some of the most important steps stated as assertions. Then we prove those assertions before we end with an argument that covers the case with vertices as well. Further, to clarify notation we put a "bar" on all vectors and vector-valued functions.

Introduce (see Fig. 7)

$$\delta(t) = \max_{\overline{x}_1 \in \partial\Omega_1(t)} \operatorname{dist}(\overline{x}_1, \Omega_2(t)).$$

Thus $\delta(0) = 0$ and we want to prove $\delta(t) = 0$ for all $t \in [0,T)$. Assume that this is false. Then (since δ is continuous, according to assertion 1 below) there exist t_0 and t_1 such that $\delta(t_0) = 0$ and $\delta(t) > 0$ for $t_0 < t < t_1$. This will lead to a contradiction.

ASSERTION 1. δ is Lipschitz continuous, hence absolutely continuous and differentiable almost everywhere.

Let $\overline{\gamma}(\cdot,t) : \sigma \mapsto \overline{\gamma}(\sigma,t)$ be a parametrization of $\partial\Omega_2(t)$, and let $\overline{n}(\sigma,t)$ be the outward unit normal to $\partial\Omega_2(t)$ at $\overline{\gamma}(\sigma,t)$. If $\overline{\gamma}(\sigma,t) \in \Omega_1(t)$ we define $l(\sigma,t)$ to be the distance

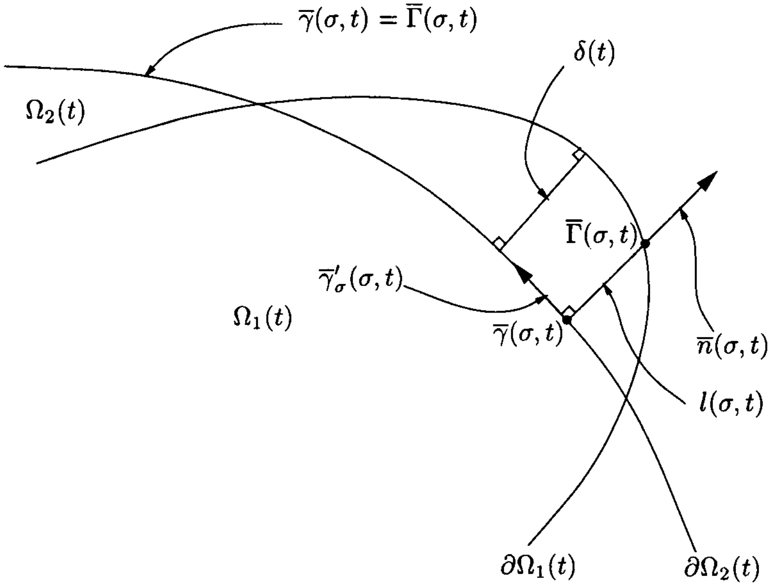

FIG. 7. Definition of $\delta, \overline{\gamma}, \overline{n}, l$, and $\overline{\Gamma}$

from $\overline{\gamma}(\sigma,t)$ to $\partial\Omega_1(t)$, measured in the positive direction of $\overline{n}(\sigma,t)$. Otherwise we put $l(\sigma,t)=0$. Thus, if $|t_1-t_0|$ is small enough, it is clear that

$$\delta(t) = \max_\sigma l(\sigma,t) \tag{16}$$

and that

$$\overline{\Gamma}(\sigma,t) = \overline{\gamma}(\sigma,t) + l(\sigma,t)\overline{n}(\sigma,t)$$

is a parametrization of $\partial(\Omega_1(t)\cup\Omega_2(t))$. Further, with a proper choice of the functions $\overline{\gamma}(\cdot,t)$, the following two assertions hold. (For an example of the set $S(t)$ introduced below, see Fig. 8.)

ASSERTION 2. Let $S(t) = \{\sigma : \delta(t) = l(\sigma,t)\}$. Then

$$\delta'(t) = \max_{\sigma\in S(t)} \frac{\partial l}{\partial t}(\sigma,t) \tag{17}$$

for almost every $t \in (t_0,t_1)$.

ASSERTION 3. Let $m_1(\cdot,t)$ and $m_2(\cdot,t)$ be the area densities on $\partial\Omega_1(t)$ and $\partial\Omega_2(t)$ respectively. Then, for $\sigma \in S(t)$, we have

$$\frac{\partial l}{\partial t}(\sigma,t) = m_1(\overline{x}_1,t) - m_2(\overline{x}_2,t),$$

where $\overline{x}_2 = \overline{\gamma}(\sigma,t)$ and $\overline{x}_1 = \overline{\Gamma}(\sigma,t)$.

Assertions 2 and 3 enable us to estimate $\delta'(t)$ by the use of area densities, evaluated at certain points.

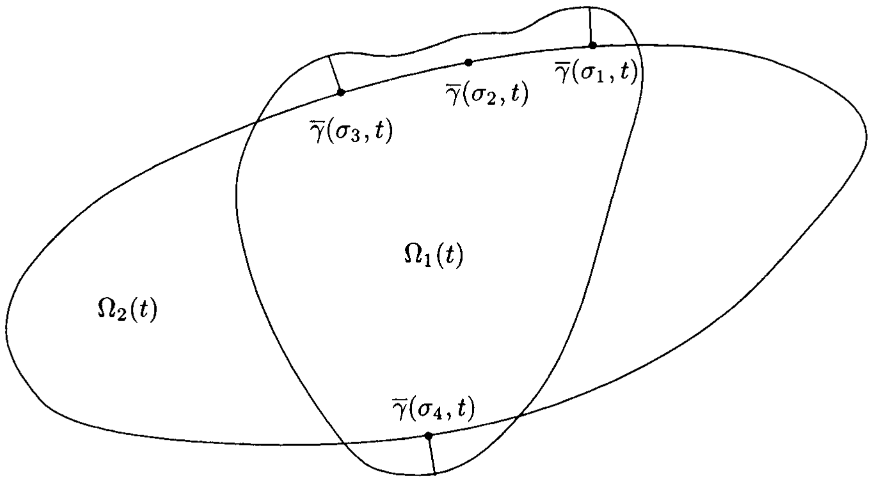

FIG. 8. An example where $S(t) = \{\sigma_1, \sigma_3, \sigma_4\}$

ASSERTION 4. There is a constant K such that

$$m_1(\overline{x}_1, t) - m_2(\overline{x}_2, t) \leq K\delta(t)$$

for every $t \in (t_0, t_1)$, if for every t we choose $\overline{x}_1$ and $\overline{x}_2$ as in Assertion 3.

The assertions 1 4 imply $\delta'(t) \leq K\delta(t)$ for almost every $t \in (t_0, t_1)$. Thus

$$\frac{d}{dt}(\delta(t)e^{-Kt}) \leq 0$$

for almost every $t \in (t_0, t_1)$ and the absolute continuity of δ, together with $\delta(t_0) = 0$, implies

$$\delta(t)e^{-Kt} \leq \delta(t_0)e^{-Kt_0} = 0,$$

that is, $\delta(t) \leq 0$ for all $t \in (t_0, t_1)$, and we have reached a contradiction.

Now we prove the four assertions.

Proof of Assertion 1. By the fifth condition in the definition of classical solution, both $\Omega_1(t)$ and $\Omega_2(t)$ are sets with Lipschitzian growth. Since $\delta(t)$ measures how far outside $\Omega_2(t)$ that $\Omega_1(t)$ has reached, it is clear that δ itself must be a Lipschitz function on a suitably restricted interval. □

Proof of Assertion 2. Since $\Omega_2(t)$ is a classical solution without vertices and with connected boundary, the fifth condition in the definition of a classical solution provides us with a parametrization of the boundary. To be more precise, there exists a C^2 function $\overline{\gamma} : [0,1] \times [t_0, t_1] \to \mathbb{R}^2$ which for every fixed $t \in [t_0, t_1]$ is a parametrization of $\partial\Omega_2(t)$.

This choice of $\overline{\gamma}$ makes the function $l(\sigma, t)$ regular enough on $[0,1] \times (t_0, t_1)$ to state that

$$\delta'(t) = \frac{d}{dt} \max_{\sigma \in [0,1]} l(\sigma, t) = \max_{\sigma \in S(t)} \frac{\partial l}{\partial t}(\sigma, t)$$

for almost every $t \in (t_0, t_1)$. (See, for instance, Theorem 3.3, p. 76, in [15].) □

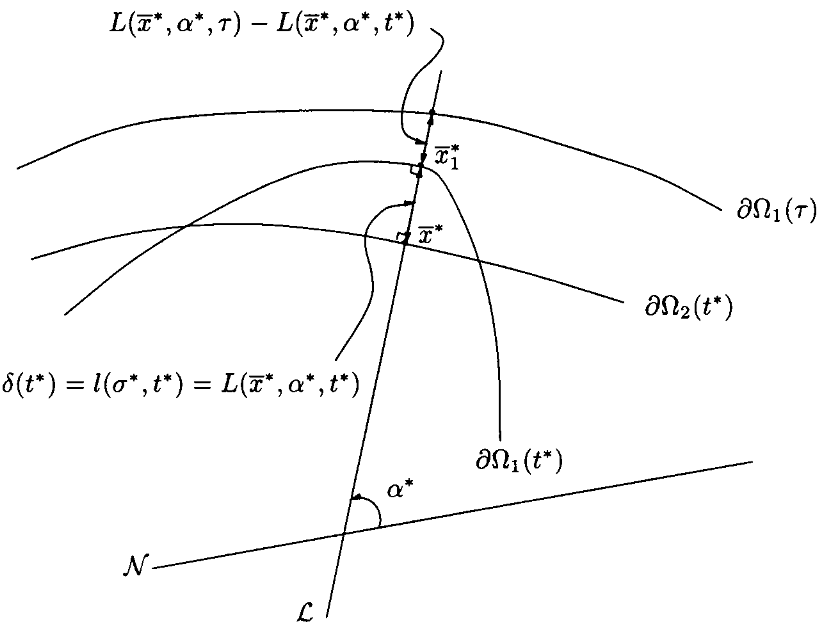

FIG. 9. Illustration of the argument concerning the third term in (18)

Proof of Assertion 3. Let $L(\overline{x}, \alpha, \tau)$ be the distance from $\overline{x} \in \mathbb{R}^2$ to the set $\partial\Omega_1(\tau)$ measured in a direction that makes an angle α with some fixed reference line $\mathcal{N}$. If $\alpha(\sigma, t)$ is the angle between $\overline{n}(\sigma, t)$ and $\mathcal{N}$, then $\alpha(\cdot, t)$ is a C^1 function on $[0, 1]$. Now we can write

$$l(\sigma, t) = L(\overline{\gamma}(\sigma, t), \alpha(\sigma, t), t).$$

Next, let $t^* \in (t_0, t_1)$ and $\sigma^* \in S(t^*)$. Then L is continuously differentiable in a neighbourhood of $(\overline{x}^*, \alpha^*, \tau^*) = (\overline{\gamma}(\sigma^*, t^*), \alpha(\sigma^*, t^*), t^*)$. Let ∇L denote the gradient of L with respect to its first argument. Then by the chain rule,

$$\frac{\partial l}{\partial t} = \langle \nabla L, \overline{\gamma}_t' \rangle + \frac{\partial L}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial L}{\partial \tau} \frac{\partial \tau}{\partial t}. \tag{18}$$

We want to evaluate this expression at (σ^*, t^*). For this purpose, let $\mathcal{L}$ denote the straight line through $\overline{x}^*$ that is perpendicular to $\partial\Omega_2(t^*)$ (hence parallel to $\overline{n}(\sigma^*, t^*)$). We study the three terms in (18) in reversed order.

In the third term in (18) we have $\partial\tau/\partial t = 1$ since $\tau = t$. Put $\overline{x}_1^* = \overline{\Gamma}(\sigma^*, t^*)$. Then it is immediate that

$$\frac{\partial L}{\partial \tau}(\overline{x}^*, \alpha^*, t^*) = m_1(\overline{x}_1^*, t^*),$$

since $\{\Omega_1(t)\}_{t\in[0,T)}$ is a classical solution. (See Fig. 9 for an illustration.) $\mathcal{L}$ and $\partial\Omega_1(t^*)$ are perpendicular at their intersection point $\overline{x}_1^*$ since $\sigma^* \in S(t^*)$. Hence

$$\frac{\partial}{\partial \alpha} L(\overline{x}^*, \alpha, t^*)_{|_{\alpha=\alpha^*}} = 0$$

by geometrical reasons and the second term in (18) vanishes.

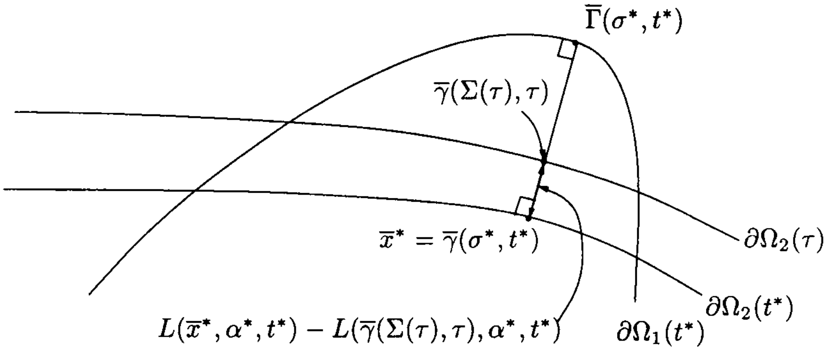

FIG. 10. Illustration of the argument concerning the first term in (18)

The first term in (18) can be written

$$\langle \nabla L(\overline{x}^*, \alpha^*, t^*), \overline{\gamma}'_t(\sigma^*, t^*)\rangle = \frac{\partial}{\partial t} L(\overline{\gamma}(\sigma^*, t), \alpha^*, t^*)_{|_{t=t^*}}.$$

For arbitrary t, choose $\Sigma(t)$ such that $\overline{\gamma}(\Sigma(t), t)$ is the intersection point between $\mathcal{L}$ and $\partial\Omega_2(t)$. Then clearly (see Fig. 10)

$$\frac{\partial}{\partial t} L(\overline{\gamma}(\Sigma(t), t), \alpha^*, t^*)_{|_{t=t^*}} = -m_2(\overline{x}^*, t^*).$$

But

$$\begin{aligned}
&\frac{\partial}{\partial t} L(\overline{\gamma}(\Sigma(t), t), \alpha^*, t^*)_{|_{t=t^*}} \\
&\quad = \frac{\partial}{\partial t} L(\overline{\gamma}(\Sigma(t), t^*), \alpha^*, t^*)_{|_{t=t^*}} + \frac{\partial}{\partial t} L(\overline{\gamma}(\sigma^*, t), \alpha^*, t^*)_{|_{t=t^*}} \\
&\quad = \frac{\partial}{\partial t} L(\overline{\gamma}(\sigma^*, t), \alpha^*, t^*)_{|_{t=t^*}},
\end{aligned}$$

where the last equality follows from orthogonality.

Thus we have found that

$$\frac{\partial l}{\partial t}(\sigma^*, t^*) = m_1(\overline{x}_1^*, t^*) - m_2(\overline{x}_2^*, t^*),$$

where $\overline{x}_2^* = \overline{x}^*$, and the third assertion is proved. □

Proof of Assertion 4. For any $t \in (t_0, t_1)$, we consider any $\sigma \in S(t)$ and put $\overline{x}_1 = \overline{\Gamma}(\sigma, t)$ and $\overline{x}_2 = \overline{\gamma}(\sigma, t)$. First of all, we need to relate ridge distances and signed curvatures to each other. Let (see Fig. 11)

$$\Omega_{\delta(t)} = \{x : \mathrm{dist}(x, \Omega_2(t)) < \delta(t)\}.$$

If $|\delta(t)\kappa_2| \leq C < 1$, then $\Omega_{\delta(t)}$ itself is a piecewise C^2 domain without re-entrant vertices, and $\partial\Omega_{\delta(t)}$ is a parallel curve to $\partial\Omega_2(t)$. Further, $\overline{x}_1 \in \partial\Omega_1(t) \cap \partial\Omega_{\delta(t)}$ and is not a vertex, neither on $\partial\Omega_1(t)$ nor on $\partial\Omega_{\delta(t)}$. According to Lemma A.3, Lemma A.5, and Lemma A.6 we have

$$D_1(\overline{x}_1) \leq D_{\delta(t)}(\overline{x}_1) = D_2(\overline{x}_2) + \delta(t)$$

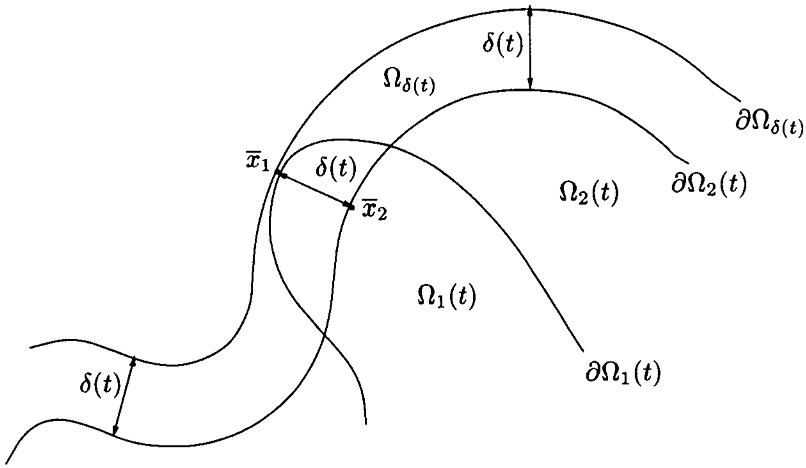

FIG. 11. Definition of $\Omega_{\delta(t)}$

and

$$\kappa_1(\overline{x}_1) \geq \kappa_{\delta(t)}(\overline{x}_1) = \kappa_2(\overline{x}_2)/(1 + \delta(t)\kappa_2(\overline{x}_2)),$$

where D_1, D_2, κ_1, and κ_2 are the ridge distances and signed curvatures on $\partial\Omega_1(t)$ and $\partial\Omega_2(t)$ respectively. To simplify the notation we introduce the polynomial

$$\Psi(D, \kappa) = D - D^2\kappa/2.$$

First we note that

$$\frac{\partial\Psi}{\partial D} = 1 - D\kappa \quad \text{and} \quad \frac{\partial\Psi}{\partial\kappa} = -D^2/2;$$

so Ψ is decreasing as a function of κ. As a function of D, Ψ is increasing if $D\kappa \leq 1$, otherwise decreasing. However, at a point on the boundary of a piecewise C^2 domain, which is not a vertex, we always have $D > 0$ and $D\kappa \leq 1$. Second, we note that

$$\Psi(D, \kappa_1 - \kappa_2) = \Psi(D, \kappa_1) + D^2\kappa_2/2.$$

Now we have that

$$m_1(\overline{x}_1) - m_2(\overline{x}_2) = \Psi(D_1(\overline{x}_1), \kappa_1(\overline{x}_1)) - \Psi(D_2(\overline{x}_2), \kappa_2(\overline{x}_2)).$$

We get two cases, and for notational convenience we omit the dependence of $t, \overline{x}_1$, and $\overline{x}_2$.

1. $\kappa_2 \geq 0$. Then, at $\kappa = \kappa_2$, we have $\partial\Psi/\partial D \leq 1$ for any $D > 0$ and $\partial\Psi/\partial D \geq 0$ for any $D \leq D_2$. Hence

$$\Psi(D_1, \kappa_2) \leq \begin{cases} \Psi(D_2, \kappa_2) + D_1 - D_2 & \text{if } D_1 > D_2, \\ \Psi(D_2, \kappa_2) & \text{if } D_1 \leq D_2. \end{cases}$$

Thus $\Psi(D_1,\kappa_2) \leq \Psi(D_2,\kappa_2) + \delta$ and we obtain

$$\begin{aligned}\Psi(D_1,\kappa_1) &\leq \Psi\left(D_1, \frac{\kappa_2}{1+\delta\kappa_2}\right)\\ &\leq \Psi(D_1, \kappa_2(1-\delta\kappa_2))\\ &= \Psi(D_1,\kappa_2) + \delta D_1^2\kappa_2^2/2\\ &\leq \Psi(D_2,\kappa_2) + (1 + D_1^2\kappa_2^2/2)\delta,\end{aligned}$$

where the second inequality follows from the fact that

$$\frac{1}{1+\delta\kappa_2} \geq 1 - \delta\kappa_2.$$

2. $\kappa_2 < 0$. Now we have

$$\frac{\partial\Psi}{\partial D}(D,\kappa_2) = 1 - D\kappa_2 \leq 1 - (D_2+\delta)\kappa_2 \leq 1 + C - D_2\kappa_2$$

for D between D_1 and D_2 and

$$\frac{\partial\Psi}{\partial D}(D,\kappa_2) \geq 1$$

for all $D > 0$; so

$$\Psi(D_1,\kappa_2) \leq \begin{cases}\Psi(D_2,\kappa_2) + (D_1 - D_2)(1 + C - D_2\kappa_2) & \text{if } D_1 > D_2,\\ \Psi(D_2,\kappa_2) & \text{if } D_1 \leq D_2.\end{cases}$$

Hence

$$\begin{aligned}\Psi(D_1,\kappa_1) &\leq \Psi\left(D_1, \frac{\kappa_2}{1+\delta\kappa_2}\right)\\ &= \Psi\left(D_1, \kappa_2 - \frac{\delta\kappa_2^2}{1+\delta\kappa_2}\right)\\ &= \Psi(D_1,\kappa_2) + \delta\frac{D_1^2\kappa_2^2}{2(1+\delta\kappa_2)}\\ &\leq \Psi(D_2,\kappa_2) + \left(1 + C - D_2\kappa_2 + \frac{D_1^2\kappa_2^2}{2(1+\delta\kappa_2)}\right)\delta.\end{aligned}$$

Thus, by choosing

$$K \geq \max\left\{1 + D_1^2\kappa_2^2/2, 1 + C - D_2\kappa_2 + \frac{D_1^2\kappa_2^2}{2(1+\delta\kappa_2)}\right\},$$

we have the estimate

$$m_1(\overline{x}_1, t) - m_2(\overline{x}_2, t) \leq K\delta(t).$$

Such a K can be chosen independently of t, since ridge distances and signed curvatures are uniformly bounded, if only $|\delta\kappa_2| \leq C < 1$ uniformly for $t < t_1$, which is achieved by choosing t_1 small enough. The fourth assertion is proved. □

What remains is the case of vertices. Assertion 1 is independent of whether there are vertices or not. Since vertices do not move, there can be no vertices on the part of $\partial\Omega_1(t)$ that is "outside" $\Omega_2(t)$. Hence we need not care about vertices on $\partial\Omega_1(t)$.

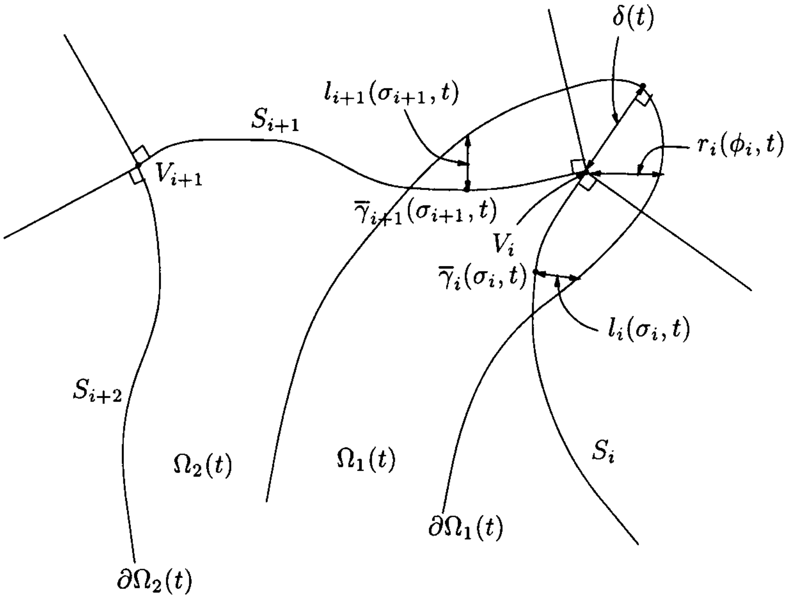

FIG. 12. Illustration of the case with vertices

Let $S_i(t)$, $i = 1, 2, \ldots, n$, be the boundary arcs of $\partial\Omega_2(t)$ ordered as in the definition of piecewise C^2 domains in Sec. 2.2. Now we can introduce functions $\overline{\gamma}_i, l_i, \overline{n}_i, \alpha_i$, and $\overline{\Gamma}_i$ for each i.

Let $\overline{V}_i$ be the vertex where $S_i(t)$ and $S_{i+1}(t)$ meet. If t_1 is small enough, $\overline{V}_i$ will be a vertex for every $t < t_1$ since vertices do not move. For each vertex $\overline{V}_i$ on $\partial\Omega_2(t)$, there is a sector (depending on t) bounded by the normals $\overline{n}_i$ and $\overline{n}_{i+1}$ at $\overline{V}_i$. In every such sector, we can introduce polar coordinates (r_i, ϕ_i), where we always measure the angle ϕ_i counterclockwise from some fixed reference line. Hence there is a function $r_i(\phi_i, t)$ such that if $\partial\Omega_1(t)$ intersects this sector, then that part of $\partial\Omega_1(t)$ is parametrized by $\overline{V}_i + r_i(\phi_i, t)(\cos\phi_i, \sin\phi_i)$. (See Fig. 12 for an illustration.)

With the aid of all the functions l_i and r_i, we can construct a function l such that $l = l_i$ along each S_i. Then (16) and (17) still hold and the fourth assertion provides us with a constant K such that

$$m_1(\overline{x}_1, t) - m_2(\overline{x}_2, t) \leq K\delta(t)$$

for all $t \in (t_0, t_1)$ and all pairs $\overline{x}_1 \in \partial\Omega_1(t)$ and $\overline{x}_2 \in \partial\Omega_2(t) \setminus (\bigcup_i \{\overline{V}_i\})$ such that $\delta(t) = |\overline{x}_1 - \overline{x}_2|$.

Assume that for some $t = t^*$ we have $\delta(t^*) = r_i(\phi^*, t^*)$. Let $\overline{x}_2^* = \overline{V}_i$ and put $\overline{x}_1^* = \overline{V}_i + r_i(\phi_i^*, t^*)(\cos\phi_i^*, \sin\phi_i^*)$ (see Fig. 13). Then, since $\overline{V}_i$ does not move, it follows, by an argument analogous to the investigation of the third term in (18), that

$$\frac{\partial r_i}{\partial t}(\phi_i^*, t^*) = m_1(\overline{x}_1^*, t^*).$$

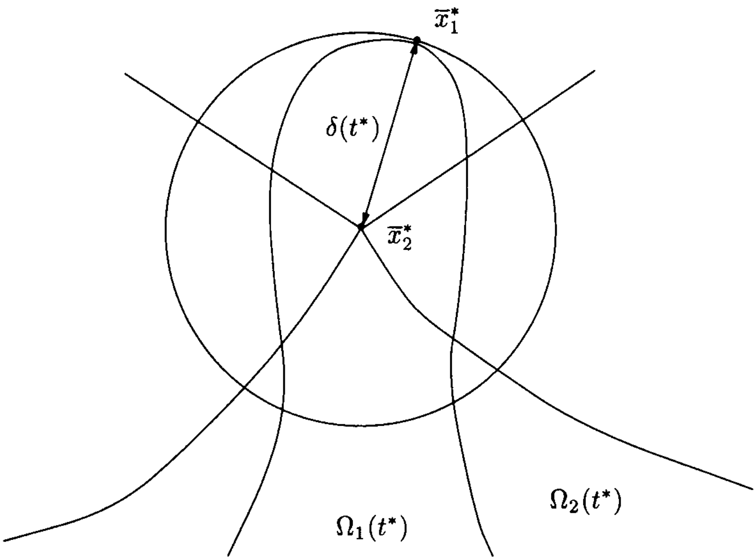

FIG. 13. $\kappa_1(\overline{x}_1^*, t^*)$ cannot be less than $1/\delta(t^*)$

But at $\overline{x}_1^*$ the signed curvature cannot be less than $1/\delta(t^*)$ (see Fig. 13) and hence the ridge distance cannot be greater than $\delta(t^*)$. Thus

$$m_1(\overline{x}_1^*, t^*) = D_1(\overline{x}_1^*, t^*) - D_1(\overline{x}_1^*, t^*)^2 \kappa_1(\overline{x}_1^*, t^*)/2 \leq \delta(t^*)$$

and it suffices to choose $K' = \max\{1, K\}$ to obtain $\delta'(t) \leq K'\delta(t)$ for almost every $t \in (t_0, t_1)$. The theorem is proved. □

It is an immediate consequence of the comparison principle that classical solutions are uniquely determined by their initial data.

THEOREM 3.7 (Uniqueness). Let $\{\Omega_1(t)\}_{t\in[0,T)}$ and $\{\Omega_2(t)\}_{t\in[0,T)}$ be bounded classical solutions. If $\Omega_1(0) = \Omega_2(0)$, then $\Omega_1(t) = \Omega_2(t)$ for all $t \in [0, T)$.

3.5. *Concluding remarks.* Theorem 3.6 (the comparison principle), Theorem 3.3, and Theorem 3.4 all deal with properties that a classical solution inherits from initial data. There are other properties as well, which we also believe to be inherited from initial data. We conjecture that convexity is such a property and that being starshaped is another, although we have no complete proofs yet.

Let us conclude this paper with a few comments about the question of existence of classical solutions. Unfortunately, we have not been able to prove any general existence result. However, there is a possibility of constructing classical solutions from (explicit) solutions of the porous medium equation. For details concerning the connection between our evolution problem and the porous medium equation we refer to [3]. It is worth mentioning though, that Evans has sketched another approach to the evolution problem ([5]) involving a weak solution concept, and it seems as if weak solutions exist under general assumptions on the initial data. This is, however, beyond the aim of this paper.

Appendix.

DEFINITION A.1 (Signed curvature). Let γ be a C^2 curve parametrized with respect to arc length t. The *signed curvature* κ of γ is defined by

$$\kappa(x) = \gamma''(t) \cdot n(x),$$

where $n(x)$ is the left unit normal at $x = \gamma(t)$. Further, the *center of curvature* is denoted by σ and is given by

$$\sigma(x) = \gamma(t) + \frac{1}{\kappa(x)} n(x),$$

provided $\kappa(x) \neq 0$.

Note that $|\kappa|$ is the (unsigned) curvature of γ and that κ is positive if the center of curvature lies in the direction of n, that is, to the left of the curve; otherwise κ is negative, or zero.

DEFINITION A.2 (Parallel curve). Let γ be a C^1 curve in the plane with left unit normal n and let δ be a constant. The curve γ_δ, defined by

$$\gamma_\delta(t) = \gamma(t) + \delta n(x),$$

where $x = \gamma(t)$, is called a *parallel curve* to γ. (See Fig. 14.)

The following result can be found in elementary texts on the analytic geometry of curves (see, for instance, [9], p. 411).

LEMMA A.3. Let γ_1 be a C^2 curve with signed curvature κ_1 and let $\gamma_2 = \gamma_\delta$ be a parallel curve to γ_1 with δ such that $1 - \delta\kappa_1(x) > 0$ for all $x = \gamma_1(t)$. Then γ_2 is a C^2 curve and for all $x = \gamma_1(t)$ and $y = \gamma_2(t)$ we have

1. $\gamma_2'(t) = (1 - \delta\kappa_1(x))\gamma_1'(t)$.
2. The signed curvature κ_2 of γ_2 is $\kappa_2(y) = \kappa_1(x)/(1 - \delta\kappa_1(x))$.

LEMMA A.4. Let Ω be a piecewise C^2 domain and let $x \in \Omega, y \in N(x)$. If $y \in S_i$ and $x = y + d(x)n_i(y)$, where n_i is the inward unit normal to $\partial\Omega$ along S_i, then $\kappa_i(y)d(x) \leq 1$. (See Fig. 15.)

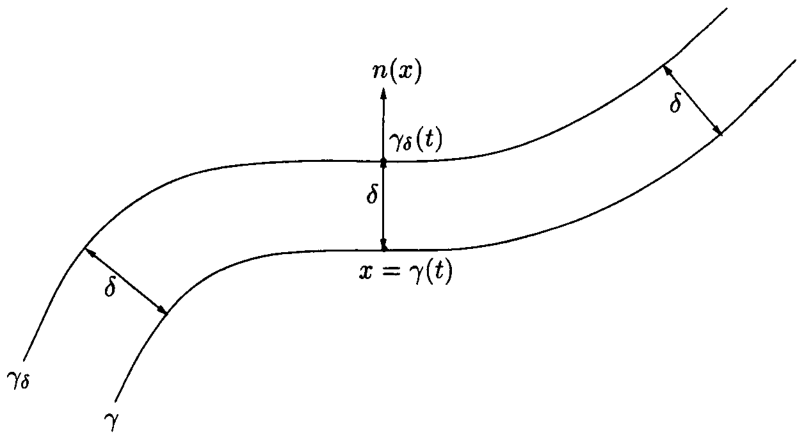

FIG. 14. Parallel curve

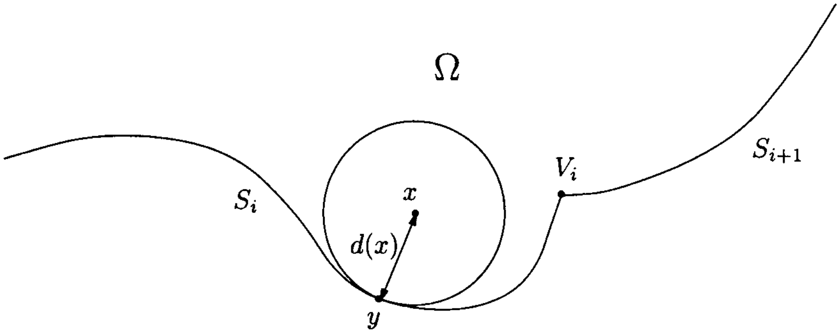

FIG. 15. Illustration of Lemma A.4

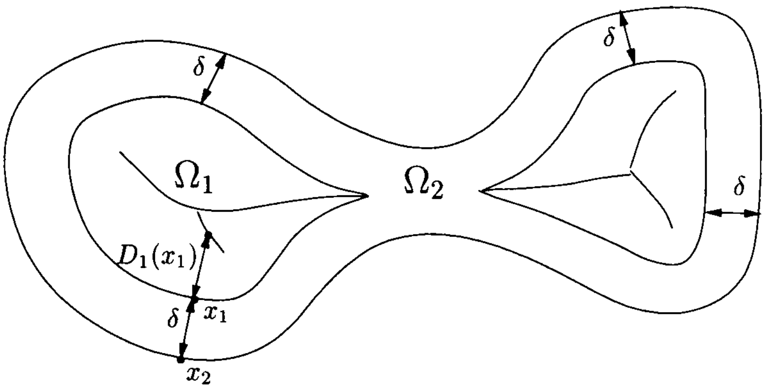

FIG. 16. Illustration of Lemma A.5

Proof. If $\kappa_i(y) \le 0$, the inequality is trivial. Assume $\kappa_i(y) > 0$. Since the open disk $B = B(x, d(x))$ is contained in Ω, and tangent to $\partial\Omega$ at y, the curvature of ∂B cannot be less than $\kappa_i(y)$, that is, $\kappa_i(y) \le 1/d(x)$. Thus $\kappa_i(y)d(x) \le 1$. □

LEMMA A.5. Let Ω_1 and Ω_2 be piecewise C^2 domains with only convex vertices such that Ω_1 is a connected component of the set $\{y \in \Omega_2 : d_2(y) > \delta\}$, where δ is a constant such that

$$0 < \delta < \max_{x \in \partial\Omega_2} D_2(x).$$

If $x_1 \in \partial\Omega_1$ and $x_2 \in N_2(x_1)$, then $D_1(x_1) = D_2(x_2) - \delta$. (See Fig. 16.)

Proof. For the sets R_0 and R_1 introduced in Definitions 2.5 and 2.9, it is easily seen that $R_0(\Omega_1) = R_0(\Omega_2) \cap \Omega_1$ and $R_1(\Omega_1) = R_1(\Omega_2) \cap \Omega_1$, and hence, by Theorem 2.12, $P(\Omega_1) = P(\Omega_2) \cap \Omega_1$. Since the boundary arcs to which x_1 and x_2 belong are parallel curves, it is immediate that $D_1(x_1) = D_2(x_2) - \delta$. □

LEMMA A.6. Let Ω_1 and Ω_2 be piecewise C^2 domains with only convex vertices. If $\Omega_1 \subset \Omega_2$ and $x \in \partial\Omega_1 \cap \partial\Omega_2$, then

1. $D_1(x) \le D_2(x)$.

2. $\kappa_1(x) \geq \kappa_2(x)$ if x is not a vertex on $\partial\Omega_1$ or $\partial\Omega_2$.
3. $m_1(x) \leq m_2(x)$.

Proof. 1 and 2 are trivial and imply 3 when combined with formula (12) in Theorem 2.18 and the inequality in Corollary 2.15. □

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