

A geometric heat flow for vector fields

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Abstract We introduce and study a geometric heat flow to find Killing vector fields on closed Riemannian manifolds with positive sectional curvature. We study its various properties, prove the global existence of the solution to this flow, discuss its convergence and possible applications, and its relation to the Navier-Stokes equations on manifolds and Kazdan-Warner-Bourguignon-Ezin identity for conformal Killing vector fields. We also provide two new criteria on the existence of Killing vector fields. A similar flow to finding holomorphic vector fields on Kähler manifolds will be studied by Li and Liu (2014).

Keywords geometric heat flow, Killing vector fields, Yano's theorem, Navier-Stokes equations, Kazdan-Warner-Bourguignon-Ezin identity

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1 A geometric heat flow for vector fields

Recently, we have witnessed the power of geometric flows in studying lots of problems in geometry and topology. In this paper, we introduce a geometric heat flow for vector fields on a Riemannian manifold and study its various properties.

Throughout this paper, we adopt the Einstein summation and notions as those in [3]. All manifolds and vector fields are smooth; a manifold is said to be *closed* if it is compact and without boundary. We shall often raise and lower indices for tensor fields.

1.1 Deformation tensor field of a vector field

Let (M, g) be a closed and orientable Riemannian manifold. To a vector field X we associate its deformation $(0, 2)$ -tensor field $\text{Def}(X)$, which is an obstruction of X to be Killing and is locally defined by

$$(\text{Def}(X))_{ij} := \frac{\nabla_i X_j + \nabla_j X_i}{2}, \quad (1.1)$$

where ∇ denotes the Levi-Civita connection of g . Equivalently, it is exactly (up to a constant factor) the Lie derivative of g along the vector field X , i.e., $\mathcal{L}_X g$. We say that X is a *Killing vector field* if $\text{Def}(X) = 0$. Consider the L^2 -norm of $\text{Def}(X)$:

$$\mathfrak{L}(X) := \int_M |\text{Def}(X)|^2 dV, \quad (1.2)$$

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where dV stands for the volume form of g and $|\cdot|$ means the norm of $\text{Def}(X)$ with respect to g . It is clear that the critical point X of \mathfrak{L} satisfies

$$\Delta X^i + \nabla^i \text{div}(X) + R^i_j X^j = 0. \quad (1.3)$$

Here and henceforth, $\Delta := g^{ij} \nabla_i \nabla_j$ is the Laplace-Beltrami operator of g and R_{ij} denotes the Ricci curvature of g . In fact,

$$\begin{aligned} \frac{d}{dt} \mathfrak{L}(X_t) &= 2 \int_M \langle \text{Def}(X_t), \partial_t \text{Def}(X_t) \rangle dV \\ &= 2 \int_M (\nabla^i (X_t)^j + \nabla^j (X_t)^i) (\nabla_i \partial_t (X_t)_j + \nabla_j \partial_t (X_t)_i) dV \\ &= - \int_M [\Delta (X_t)^i \cdot \partial_t (X_t)_i + \nabla^j \nabla^i (X_t)^j \cdot \partial_t (X_t)_i] dV \\ &= - \int_M [\Delta (X_t)^i + \nabla^i \text{div}(X_t) + R^i_j (X_t)^j] \partial_t (X_t)^i dV. \end{aligned}$$

1.2 A geometric heat flow for vector fields

Motivated by (1.3), we introduce a geometric heat flow for vector fields

$$\partial_t (X_t)^i = \Delta (X_t)^i + \nabla^i \text{div}(X_t) + R^i_j (X_t)^j, \quad X_0 = X, \quad (1.4)$$

where X is a fixed vector field on M and $\partial_t := \frac{\partial}{\partial t}$ is the time derivative. If we define Ric^\sharp , the $(1, 1)$ -tensor field associated to Ric , by

$$g(\text{Ric}^\sharp(X), Y) := \text{Ric}(X, Y),$$

where X, Y are two vector fields, then Ric^\sharp is an operator on the space of vector fields, denoted by $C^\infty(TM)$, and the flow (1.4) can be rewritten as

$$\partial_t X_t = \Delta X_t + \nabla \text{div}(X_t) + \text{Ric}^\sharp(X_t). \quad (1.5)$$

In 1952, Yano [15–17] showed that a vector field $X = X^i \frac{\partial}{\partial x^i}$ is a Killing vector field if and only if it satisfies

$$\Delta X^i + R^i_j X^j = 0, \quad \text{div}(X) = 0. \quad (1.6)$$

His result depends on an integral formula, now called *Yano's integral formula*,

$$0 = \int_M [\text{Ric}(X, X) - |\nabla X|^2 + 2|\text{Def}(X)|^2 - |\text{div}(X)|^2] dV, \quad (1.7)$$

which holds for any vector field X . This integral formula lets us define so-called the *Bochner-Yano integral* for every vector field X :

$$\mathcal{E}(X) := \int_M [|\nabla X|^2 + |\text{div}(X)|^2 - \text{Ric}(X, X)] dV. \quad (1.8)$$

Consequently, Yano's integral formula implies that $\mathcal{E}(X)$ is always nonnegative and $\mathcal{E}(X) = 2\mathfrak{L}(X)$ for every vector field X . On the other hand, Watanabe [13] proved that X is a Killing vector field if and only if $\mathcal{E}(X) = 0$, and hence if and only if $\mathfrak{L}(X) = 0$.

Yano's equations (1.6) induces a system of equations, called the *Bochner-Yano flow*

$$\partial_t (X_t)^i = \Delta (X_t)^i + R^i_j (X_t)^j, \quad \text{div}(X_t) = 0. \quad (1.9)$$

Notice that Yano's equation (1.6) (resp., Bochner-Yano flow (1.9)) is a special case of (1.3) (resp., flow (1.4)).

Proposition 1.1. *If X_t is the solution to the flow (1.4), then*

$$\mathcal{E}(X_t) \geq 0, \tag{1.10}$$

$$\frac{d}{dt}\mathcal{E}(X_t) = -2 \int_M |\partial_t X_t|^2 dV \leq 0, \tag{1.11}$$

$$\mathcal{E}(X_t) = -\frac{d}{dt} \left(\frac{1}{2} \int_M |X_t|^2 dV \right). \tag{1.12}$$

Consequently, $\mathcal{E}(X_t)$ is monotone nonincreasing and $\int_M |X_t|^2 dV$ is also monotone nonincreasing.

Proof. The first one directly follows from (1.7). Since the flow (1.4) is the gradient flow of the functional \mathcal{E} , we prove the second one. To prove (1.12), we use the formula

$$\frac{1}{2} \Delta |X|^2 = \langle X, \Delta X \rangle + |\nabla X|^2$$

to deduce that

$$\begin{aligned} \mathcal{E}(X_t) &= \int_M \left[\frac{1}{2} \Delta |X_t|^2 - (X_t)_i \Delta (X_t)^i + |\operatorname{div}(X_t)|^2 - \operatorname{Ric}(X_t, X_t) \right] dV \\ &= - \int_M [(X_t)_i \Delta (X_t)^i + (X_t)_i \nabla^i \operatorname{div}(X_t) + (X_t)_i \cdot R^i_j (X_t)^j] dV \\ &= - \int_M (X_t)_i [\Delta (X_t)^i + \nabla^i \operatorname{div}(X_t) + R^i_j (X_t)^j] dV \\ &= -\frac{1}{2} \int_M \partial_t |X_t|^2 dV. \end{aligned}$$

Hence, the conclusion is obvious. □

Corollary 1.2. *If X_t is the solution to the flow (1.4) for $t \in [0, T]$, then we have*

$$\int_0^T \int_M |\partial_t X_t|^2 dV dt \leq \frac{1}{2} \mathcal{E}(X). \tag{1.13}$$

Proof. For any T , we have

$$-2 \int_0^T \int_M |\partial_t X_t|^2 dV dt = \mathcal{E}(X_T) - \mathcal{E}(X) \geq -\mathcal{E}(X),$$

since \mathcal{E} is nonnegative. This proves (1.13). □

1.3 Evolution equations

To study the long time existence and the convergence of the geometric heat flow (1.4), we prove its several associated evolution equations.

Lemma 1.3. *If X_t is the solution to (1.4), then*

$$\partial_t |X_t|^2 = \Delta |X_t|^2 - 2|\nabla X_t|^2 + 2 \langle X_t, \nabla \operatorname{div}(X_t) \rangle + 2\operatorname{Ric}(X_t, X_t). \tag{1.14}$$

Proof. Calculate

$$\begin{aligned} \partial_t |X_t|^2 &= 2(X_t)_i \partial_t (X_t)^i \\ &= 2(X_t)_i (\Delta (X_t)^i + \nabla^i \operatorname{div}(X_t) + R^i_j (X_t)^j) \\ &= \Delta |X_t|^2 - 2|\nabla X_t|^2 + 2 \langle X_t, \nabla \operatorname{div}(X_t) \rangle + 2\operatorname{Ric}(X_t, X_t), \end{aligned}$$

which proves (1.4). □

Lemma 1.4. If X_t is the solution to (1.4), then

$$\begin{aligned} \partial_t |\nabla X_t|^2 &= \Delta |\nabla X_t|^2 - 2 |\nabla^2 X_t|^2 - 4 R_{ijk\ell} \nabla^i (X_t)^k \cdot \nabla^j (X_t)^\ell \\ &\quad - 2 R_{ij} \nabla^i (X_t)^k \cdot \nabla^j (X_t)_k + 2 R_{ij} \nabla^k (X_t)^i \cdot \nabla_k (X_t)^j \\ &\quad + 2 \langle \text{Def}(X_t), \nabla \nabla \text{div}(X_t) \rangle \\ &\quad + 2 (\nabla_i R_{jk} - \nabla^\ell R_{likj}) (X_t)^k \nabla^i (X_t)^j. \end{aligned} \quad (1.15)$$

Proof. From the definition of the flow, we have

$$\begin{aligned} \partial_t |\nabla X_t|^2 &= 2 \nabla^i (X_t)_j \cdot \nabla_i \partial_t (X_t)^j \\ &= 2 \nabla^i (X_t)_j \cdot \nabla_i (\Delta (X_t)^j + \nabla^j \text{div}(X_t) + R^j_k (X_t)^k). \end{aligned}$$

We use the Ricci identity to deduce that

$$\begin{aligned} \nabla_i \Delta (X_t)^j &= g^{pq} \nabla_i \nabla_p \nabla_q (X_t)^j \\ &= g^{pq} [\nabla_p \nabla_i \nabla_q (X_t)^j - R_{ipq}{}^r \nabla_r (X_t)^j + R_{ipr}{}^j \nabla_q (X_t)^r] \\ &= \nabla^q [\nabla_q \nabla_i (X_t)^j + R_{iqr}{}^j (X_t)^r] - R_{ir} \nabla^r (X_t)^j + R_{ipr}{}^j \nabla^p (X_t)^r \\ &= \Delta \nabla_i (X_t)^j + \nabla^q (R_{iqr}{}^j (X_t)^r) - R_{ir} \nabla^r (X_t)^j + R_{ipr}{}^j \nabla^p (X_t)^r \\ &= \Delta \nabla_i (X_t)^j + \nabla^q R_{iqr}{}^j \cdot (X_t)^r + 2 R_{iqr}{}^j \nabla^q (X_t)^r - R_{ir} \nabla^r (X_t)^j. \end{aligned}$$

Plugging it into the equation for $\partial_t |\nabla X_t|^2$, we arrive at

$$\begin{aligned} \partial_t |\nabla X_t|^2 &= 2 \nabla^i (X_t)_j [\Delta \nabla_i (X_t)^j + \nabla^q R_{iqr}{}^j (X_t)^r + 2 R_{iqr}{}^j \nabla^q (X_t)^r \\ &\quad - R_{ir} \nabla^r (X_t)^j + \nabla_i \nabla^j \text{div}(X_t) + (X_t)^k \nabla_i R^j_k + R^j_k \nabla_i (X_t)^k] \\ &= \Delta |\nabla X_t|^2 - 2 |\nabla^2 X_t|^2 + 2 \nabla^q R_{iqr}{}^j \nabla^i (X_t)_j \cdot (X_t)^r \\ &\quad + 4 R_{iqr}{}^j \nabla^q (X_t)^r \nabla^i (X_t)^j - 2 R_{ir} \nabla^r (X_t)^j \nabla^i (X_t)_j \\ &\quad + 2 \nabla^i (X_t)_j \cdot \nabla_i \nabla^j \text{div}(X_t) + 2 \nabla_i R^j_k \cdot (X_t)^k \nabla^i (X_t)_j + 2 R^j_k \nabla^i (X_t)^k \nabla^i (X_t)_j \\ &= \Delta |\nabla X_t|^2 - 2 |\nabla^2 X_t|^2 - 4 R_{qirj} \nabla^q (X_t)^r \nabla^i (X_t)^j \\ &\quad - 2 R_{ir} \nabla^r (X_t)^j \nabla^i (X_t)_j + 2 R_{jk} \nabla_i (X_t)^k \nabla^i (X_t)^j \\ &\quad + 2 \nabla^i (X_t)_j \cdot \nabla_i \nabla^j \text{div}(X_t) + 2 \nabla_i R_{jk} \cdot (X_t)^k \nabla^i (X_t)^j \\ &\quad - 2 \nabla^q R_{qirj} (X_t)^r \nabla^i (X_t)^j. \end{aligned}$$

Changing the indices yields the desired result. \square

By the Bianchi identity, the above lemma can be written as the following corollary.

Corollary 1.5. If X_t is the solution to the flow (1.4), then

$$\begin{aligned} \partial_t |\nabla X_t|^2 &= \Delta |\nabla X_t|^2 - 2 |\nabla^2 X_t|^2 - 4 R_{ijk\ell} \nabla^i (X_t)^k \nabla^j (X_t)^\ell \\ &\quad - 2 R_{ij} \nabla^i (X_t)^k \nabla^j (X_t)_k + 2 R_{ij} \nabla^k (X_t)^i \nabla_k (X_t)^j \\ &\quad + 2 (\nabla_i R_{jk} - \nabla_j R_{ki} + \nabla_k R_{ij}) (X_t)^k \nabla^i (X_t)^j \\ &\quad + 2 \langle \text{Def}(X_t), \nabla \nabla \text{div}(X_t) \rangle. \end{aligned}$$

Lemma 1.6. (1) If X_t is the solution to the flow (1.4), then

$$\partial_t \text{div}(X_t) = 2 \Delta \text{div}(X_t) + \langle X_t, \nabla R \rangle + 2 R_{ij} \nabla^i (X_t)^j. \quad (1.16)$$

(2) If X_t is the solution to the flow (1.4), then

$$\begin{aligned} \partial_t |\text{div}(X_t)|^2 &= 2 \Delta |\text{div}(X_t)|^2 - 4 |\nabla \text{div}(X_t)|^2 \\ &\quad + 2 \text{div}(X_t) \langle X_t, \nabla R \rangle + 4 \text{div}(X_t) \cdot R_{ij} \nabla^i (X_t)^j \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \int_M |\operatorname{div}(X_t)|^2 dV &= -4 \int_M |\nabla \operatorname{div}(X_t)|^2 dV \\ &\quad - 4 \int_M \operatorname{Ric}(X_t, \nabla \operatorname{div}(X_t)) dV. \end{aligned} \tag{1.17}$$

In particular, if $\operatorname{Ric} = 0$ and $\operatorname{div}(X) \equiv 0$, then $\operatorname{div}(X_t) \equiv 0$.

Proof. According to (1.4), one has

$$\begin{aligned} \partial_t \operatorname{div}(X_t) &= \nabla_i (\partial_t (X_t)^i) = \nabla_i (\Delta (X_t)^i + \nabla^i \operatorname{div}(X_t) + R^i_j (X_t)^j) \\ &= \nabla_i (\Delta (X_t)^i) + \Delta \operatorname{div}(X_t) + \nabla^i (R_{ij} (X_t)^j). \end{aligned}$$

Next, we compute the first term $\nabla_i (\Delta (X_t)^i)$ as follows:

$$\begin{aligned} \nabla_i (\Delta (X_t)^i) &= g^{pq} \nabla_i \nabla_p \nabla_q (X_t)^i \\ &= g^{pq} (\nabla_p \nabla_i \nabla_q (X_t)^i - R_{ipq}{}^r \nabla_r (X_t)^i + R_{ipr}{}^i \nabla_q (X_t)^r) \\ &= \nabla^q (\nabla_q \nabla_i (X_t)^i + R_{iqr}{}^i (X_t)^r) - R_{ir} \nabla^r (X_t)^i + R_{pr} \nabla^p (X_t)^r \\ &= \Delta \nabla_i (X_t)^i + \nabla^q (R_{qr} (X_t)^r). \end{aligned}$$

Combining those two expression gives

$$\begin{aligned} \partial_t \operatorname{div}(X_t) &= 2\Delta \operatorname{div}(X_t) + 2\nabla^i (R_{ij} (X_t)^j) \\ &= 2\Delta \operatorname{div}(X_t) + 2\nabla^i R_{ij} \cdot (X_t)^j + 2R_{ij} \nabla^i (X_t)^j \\ &= 2\Delta \operatorname{div}(X_t) + \nabla_j R \cdot (X_t)^j + 2R_{ij} \nabla^i (X_t)^j, \end{aligned}$$

proving (1.16). For (1.17), the evolution equation for $|\operatorname{div}(X_t)|^2$ is

$$\begin{aligned} \partial_t |\operatorname{div}(X_t)|^2 &= 2\operatorname{div}(X_t) \cdot \partial_t \operatorname{div}(X_t) \\ &= 2\operatorname{div}(X_t) (2\Delta \operatorname{div}(X_t) + (X_t)^i \nabla_i R + 2R_{ij} \nabla^i (X_t)^j) \\ &= 2\Delta |\operatorname{div}(X_t)|^2 - 4|\nabla \operatorname{div}(X_t)|^2 \\ &\quad + 2\operatorname{div}(X_t) \cdot (X_t)^i \nabla_i R + 4(\operatorname{div}(X_t) R_{ij}) \nabla^i (X_t)^j. \end{aligned}$$

Integrating both sides over M yields

$$\begin{aligned} \frac{d}{dt} \int_M |\operatorname{div}(X_t)|^2 dV &= -4 \int_M |\nabla \operatorname{div}(X_t)|^2 dV + 2 \int_M \operatorname{div}(X_t) ((X_t)^i \nabla_i R) dV \\ &\quad - 4 \int_M \nabla^i (\operatorname{div}(X_t) R_{ij}) (X_t)^j dV. \end{aligned}$$

Since

$$\begin{aligned} 4\nabla^i (\operatorname{div}(X_t) R_{ij}) (X_t)^j &= 4[\nabla^i \operatorname{div}(X_t) \cdot R_{ij} + \operatorname{div}(X_t) \cdot \nabla^i R_{ij}] (X_t)^j \\ &= 4R_{ij} (X_t)^j \nabla^i \operatorname{div}(X_t) + 2\nabla_j R \cdot (X_t)^j \operatorname{div}(X_t), \end{aligned}$$

it follows that (1.17) is true. When $\operatorname{Ric} = 0$, we obtain

$$\frac{d}{dt} \int_M |\operatorname{div}(X_t)|^2 dV \leq 0,$$

which means

$$\int_M |\operatorname{div}(X_t)|^2 dV \leq \int_M |\operatorname{div}(X)|^2 dV = 0$$

and therefore $|\operatorname{div}(X_t)|^2 = 0$. Thus $\operatorname{div}(X_t) \equiv 0$. □

1.4 Long-time existence

Now we can state our main result to the flow (1.4).

Theorem 1.7 (Long-time existence). *Suppose that (M, g) is a closed and orientable Riemannian manifold. Given an initial vector field, the flow (1.4) exists for all time.*

The main method on proving above theorem is the standard approach in PDEs and an application of Sobolev embedding theorem. After establishing the long-time existence, we can study the convergence problem of the flow (1.4).

Proof. We now turn to the proof of the short-time existence of the flow (1.4). Note that (1.4) can be written as

$$\begin{aligned} \partial_t(X_t)^i &= \Delta(X_t)^i + \nabla^i \operatorname{div}(X_t) + R^i_j(X_t)^j \\ &= \nabla^k \nabla_k(X_t)^i + \nabla^i \nabla_j(X_t)^j + R^i_j(X_t)^j \\ &= \sum_{j=1}^m \left(\delta_j^i \sum_{k=1}^m \nabla^k \nabla_k + \nabla^i \nabla_j \right) (X_t)^j + R^i_j(X_t)^j. \end{aligned} \tag{1.18}$$

For any $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, we have

$$\sum_{i,j=1}^m \left(\delta_j^i \sum_{k=1}^m \xi_k \xi_k + \xi_i \xi_j \right) = \sum_{i,k=1}^m \xi_k \xi_k + \sum_{i,j=1}^m \xi_i \xi_j = m|\xi|^2 + \sum_{i,j=1}^m \xi_i \xi_j, \tag{1.19}$$

where

$$|\xi| \doteq \left(\sum_{k=1}^m \xi_k^2 \right)^{1/2}$$

denotes the length of ξ in \mathbb{R}^m . On the other hand, plugging

$$\sum_{i,j=1}^m (\xi_i + \xi_j)^2 = \sum_{i,j=1}^m (\xi_i^2 + \xi_j^2 + 2\xi_i \xi_j) = 2m|\xi|^2 + 2 \sum_{i,j=1}^m \xi_i \xi_j$$

into (1.19) yields

$$\begin{aligned} \sum_{i,j=1}^m \left(\delta_j^i \sum_{k=1}^m \xi_k \xi_k + \xi_i \xi_j \right) &= \frac{1}{2} \sum_{i,j=1}^m (\xi_i + \xi_j)^2 \\ &= 2 \sum_{i=1}^m \xi_i^2 + \frac{1}{2} \sum_{i \neq j} (\xi_i + \xi_j)^2 \\ &\geq 2|\xi|^2. \end{aligned}$$

Then, by the standard theory for partial differential equations of parabolic type, we have that the flow (1.4) exists for a short time.

Since the flow equation is linear, a standard theory in PDEs implies the long-time existence. □

1.5 Convergence

In what follows, we always assume that (M, g) is a closed and oriented Riemannian manifold of dimension m . Since M is compact, we can find a constant B such that

$$R_{ij} \leq Bg_{ij}. \tag{1.20}$$

Then the energy functional $\mathcal{E}(X_t)$ satisfies

$$\int_M [|\nabla X_t|^2 + (\operatorname{div}(X_t))^2 - B|X_t|^2] dV \leq \mathcal{E}(X_t). \tag{1.21}$$

Using Proposition 1.1, we have

$$\int_M |\nabla X_t|^2 dV \leq \mathcal{E}(X_t) + B \int_M |X_t|^2 dV = \mathcal{E}(X_t) + B \cdot u(t) \leq \mathcal{E}(X) + B \cdot u(0),$$

where

$$u(t) := \int_M |X_t|^2 dV.$$

Hence $\nabla X_t \in L^2(M, TM)$. On the other hand $u(t) \leq u(0)$, we conclude that

$$\|X_t\|_{H^1(M, TM)} \leq C_1(M, g, X). \tag{1.22}$$

By the regularity of parabolic equations and the flow (1.4), we obtain

$$\|X_t\|_{H^\ell(M, TM)} \leq C_\ell = C_\ell(M, g, X)$$

for each ℓ . Therefore we can find $X_\infty \in H^\ell(M, TM)$ and a subsequence $(X_{t_i})_{i \in \mathbb{N}}$ such that $X_{t_i} \rightarrow X_\infty$ a.e. as $i \rightarrow \infty$. By Sobolev imbedding theorem, $X_\infty \in C^\infty(M, TM)$ and $X_t \rightarrow X_\infty$ as $t \rightarrow \infty$.

Corollary 1.2 implies there exists a subsequence, say, without loss of generality, $(X_{t_i})_{i \in \mathbb{N}}$, such that

$$\|\partial_t X_t|_{t=t_i}\|_{L^2(M, g)} \rightarrow 0. \tag{1.23}$$

According to (1.11) and (1.23), $\|\partial_t X_t\|_{L^2(M, g)}$ decreases and converges to 0 as $t \rightarrow \infty$. Therefore, the smooth vector field X_∞ satisfies

$$\Delta(X_\infty)^i + \nabla^i \operatorname{div}(X_\infty) + R^i_j(X_\infty)^j = 0. \tag{1.24}$$

In summary, we proved the following theorem.

Theorem 1.8 (Convergence). *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X is a vector field, there exists a unique smooth solution X_t to the flow (1.4) for all time t . As t goes to infinity, the vector field X_t converges uniformly to a Killing vector field X_∞ .*

Remark 1.9. Cliff Taubes remarked that Theorems 1.7 and 1.8 also follow from an eigenfunction expansion for the relevant linear operator that defines the flow (1.4), which gives a short proof of those two theorems.

Theorem 1.8 does not guarantee that a nontrivial Killing vector field. For example, if X is identically zero, then by the uniqueness theorem the limit vector field is also identically zero. When the Ricci curvature is negative, Bochner’s theorem implies that there is no nontrivial Killing vector field.

To obtain a nonzero Killing vector field, we have the following criterion.

Proposition 1.10. *Suppose that (M, g) is a closed and orientable Riemannian manifold and X is a vector field on M . If X_t is the solution to the flow (1.4) with the initial value X , then*

$$\int_0^\infty \mathcal{E}(X_t) dV < \infty. \tag{1.25}$$

Let

$$\operatorname{Err}(X) := \frac{1}{2} \int_M |X|^2 dV - \int_0^\infty \mathcal{E}(X_t) dt. \tag{1.26}$$

Therefore $\operatorname{Err}(X) \geq 0$ and X_∞ is nonzero if and only if $\operatorname{Err}(X) > 0$.

The higher derivatives of $\mathcal{E}(X_t)$ have explicit formulas in terms of the energy functionals of lower derivatives of X_t .

Proposition 1.11. *If X_t is the solution to the flow (1.4), then*

$$\mathcal{E}''(X_t) = 4\mathcal{E}(\partial_t X_t) \geq 0. \tag{1.27}$$

Proof. Using (1.11), we have

$$\begin{aligned}
\mathcal{E}''(X_t) &= -4 \int_M \partial_t(X_t)_i \cdot \partial_t(\partial_t(X_t)^i) dV \\
&= -4 \int_M \partial_t(X_t)_i \cdot \partial_t(\Delta_{\text{LB}}(X_t)^i + \nabla^i \text{div}(X_t) + R^i_j(X_t)^j) dV \\
&= -4 \int_M \partial_t(X_t)_i (\Delta_{\text{LB}} \partial_t(X_t)^i + \nabla^i \text{div}(\partial_t X_t) + R^i_j \partial_t(X_t)^j) dV \\
&= -4 \int_M \left(\frac{1}{2} \Delta |\partial_t X_t|^2 - |\nabla \partial_t X_t|^2 \right) dV \\
&\quad - 4 \int_M \partial_t(X_t)_i (\nabla^i \text{div}(\partial_t X_t) + R^i_j \partial_t(X_t)^j) dV \\
&= 4 \int_M [|\nabla \partial_t X_t|^2 + \nabla^i \partial_t(X_t)_i \cdot \text{div}(\partial_t X_t) - \text{Ric}(\partial_t X_t, \partial_t X_t)] dV \\
&= 4 \int_M [|\nabla \partial_t X_t|^2 + \partial_t \text{div}(X_t) \cdot \text{div}(\partial_t X_t) - \text{Ric}(\partial_t X_t, \partial_t X_t)] dV \\
&= 4 \int_M [|\nabla \partial_t X_t|^2 + (\text{div}(\partial_t X_t))^2 - \text{Ric}(\partial_t X_t, \partial_t X_t)] dV \\
&= 4\mathcal{E}(\partial_t X_t),
\end{aligned}$$

which is nonnegative according to (1.7). \square

1.6 A connection to the Navier-Stokes equations

A surprising observation is that our flow (1.4) is very close to the Navier-Stokes equations [2, 12] (without the pressure) on manifolds

$$\partial_t X_t + \nabla_{X_t} X_t = \text{div}(S_t), \quad \text{div}(X_t) = 0, \quad (1.28)$$

where $S_t := 2\text{Def}(X_t)$ is the stress tensor of X_t . By an easy computation we can write (1.28) as

$$\partial_t(X_t)^i + (\nabla_{X_t} X_t)^i = \Delta(X_t)^i + \nabla^i \text{div}(X_t) + R^i_j X^j, \quad \text{div}(X_t) = 0. \quad (1.29)$$

Compared (1.4) with (1.29), we give a geometric interpolation of the right (or the linear) part of the Navier-Stokes equations on manifolds.

When the Ricci tensor field is identically zero, our flow (1.4) keeps the property that $\text{div}(X_t) = 0$ (see (1.17)).

As a consequence of the non-negativity of \mathcal{E} we can prove the following theorem.

Theorem 1.12. *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X_t is a solution to the Navier-Stokes equations (1.29), then*

$$\frac{d}{dt} \left(\int_M |X_t|^2 dV \right) = -2\mathcal{E}(X_t) \leq 0. \quad (1.30)$$

In particular,

$$\int_M |X_t|^2 dV \leq \int_M |X_0|^2 dV. \quad (1.31)$$

Proof. By multiplying by $(X_t)_i$ the equation (1.29) equals

$$\frac{1}{2} \partial_t |X_t|^2 + \langle \nabla_{X_t} X_t, X_t \rangle = \langle \Delta X_t + \nabla \text{div}(X_t) + \text{Ric}^\sharp(X_t), X_t \rangle.$$

Integrating on both sides yields

$$\frac{1}{2} \frac{d}{dt} \int_M |X_t|^2 dV + \int_M \langle \nabla_{X_t} X_t, X_t \rangle dV = -\mathcal{E}(X_t).$$

From Lemma 1.13 below, we verify (1.30) since $\text{div}(X_t) = 0$. \square

Lemma 1.13. Suppose that (M, g) is a closed and oriented Riemannian manifold. Then for any vector field $X \in C^\infty(M, TM)$, we have

$$\int_M \langle \nabla_X X, X \rangle dV = -\frac{1}{2} \int_M \operatorname{div}(X) |X|^2 dV. \tag{1.32}$$

Proof. Indeed, using $(\nabla_X X)^j = X^i \nabla_i X^j$ we have

$$\begin{aligned} \int_M \langle \nabla_X X, X \rangle dV &= \int_M (\nabla_X X)^j X_j dV = \int_M X^i \nabla_i X^j \cdot X_j dV \\ &= \int_M \nabla_i X^j (X^i X_j) dV = - \int_M X^j \nabla_i (X^i X_j) dV \\ &= - \int_M X^j [\operatorname{div}(X) X_j + X^i \nabla_i X_j] dV \\ &= - \int_M \operatorname{div}(X) |X|^2 dV - \int_M X^i X^j \nabla_i X_j dV \\ &= - \int_M \operatorname{div}(X) |X|^2 dV - \int_M \langle \nabla_X X, X \rangle dV. \end{aligned}$$

Arranging the terms yields (1.32). □

The similar result was considered by Wilson [14] for the standard metric on \mathbb{R}^3 .

1.7 A connection to Kazdan-Warner-Bourguignon-Ezin identity

If (M, g) is a closed Riemannian manifold with $m \geq 2$ and if X is a Killing vector field, then

$$\int_M \langle \nabla R, X \rangle dV = 0, \tag{1.33}$$

where R is the scalar curvature of g . This identity (actually holds for any conformal Killing vector fields) was proved by Bourguignon and Ezin [1] and the surface case is the classical Kazdan-Warner identity [6]. For convenience, we call such an identity as KWBE *identity*. For its application to Ricci flow we refer readers to [3]. In this subsection, we study the asymptotic behavior of the KWBE identity under the flow (1.4).

For any vector field X , we define the KWBE *functional* as

$$\mathcal{I}(X) := \int_M \langle \nabla R, X \rangle dV.$$

Then, under the flow (1.4), where $X_t = X^i \frac{\partial}{\partial x^i}$,

$$\begin{aligned} \frac{d}{dt} \mathcal{I}(X_t) &= \int_M \nabla_i R (\Delta X^i + \nabla^i \operatorname{div}(X_t) + R^i_j X^j) dV \\ &= \int_M \nabla_i R \cdot \Delta X^i dV - \int_M \Delta R \cdot \operatorname{div}(X_t) dV + \int_M R_{ij} X^i \nabla^j R dV. \end{aligned}$$

Using the commutative formula $\nabla \Delta R = \Delta \nabla R - \operatorname{Ric}(\nabla R, \cdot)$ yields

$$\begin{aligned} \int_M \nabla_i R \cdot \Delta X^i dV &= \int_M \langle X_t, \Delta \nabla R \rangle dV \\ &= \int_M \langle X_t, \nabla \Delta R + \operatorname{Ric}(\nabla R, \cdot) \rangle dV \\ &= - \int_M \Delta R \cdot \operatorname{div}(X_t) dV + \int_M R_{ij} X^i \nabla^j R dV \end{aligned}$$

and therefore

$$\frac{d}{dt} \mathcal{I}(X_t) = -2 \int_M \Delta R \cdot \operatorname{div}(X_t) dV + 2 \int_M \operatorname{Ric}(X_t, \nabla R) dV. \tag{1.34}$$

The last term on the right-hand side of (1.34) can be simplified by

$$\begin{aligned} \int_M \nabla^i R(X^j R_{ij}) dV &= - \int_M R \left(\nabla^i X^j \cdot R_{ij} + X^j \cdot \frac{1}{2} \nabla_j R \right) dV \\ &= - \int_M R R_{ij} \nabla^i X^j dV - \frac{1}{2} \int_M R X^j \nabla_j R dV. \end{aligned}$$

We also have

$$\begin{aligned} \int_M R X^j \nabla_j R dV &= - \int_M \nabla_j (R X^j) R dV \\ &= \int_M R X^j \nabla_j R dV - \int_M R^2 \operatorname{div}(X_t) dV \end{aligned}$$

so that

$$\int_M R X^j \nabla_j R dV = -\frac{1}{2} \int_M R^2 \operatorname{div}(X_t) dV. \quad (1.35)$$

From (1.34), (1.35), (1.1) and Theorem 1.8, we arrive at the following proposition.

Proposition 1.14. *If (M, g) is a closed Riemannian manifold and X_t is a solution to (1.4), then*

$$\frac{d}{dt} \mathcal{I}(X_t) = 2 \int_M \left(-\Delta + \frac{R}{4} \right) R \cdot \operatorname{div}(X_t) dV - 2 \int_M R \langle \operatorname{Ric}, \operatorname{Def}(X_t) \rangle dV. \quad (1.36)$$

In particular,

$$\lim_{t \rightarrow \infty} \frac{d}{dt} \mathcal{I}(X_t) = 0. \quad (1.37)$$

This proposition gives the limiting behavior of $\frac{d}{dt} \mathcal{I}(X_t)$. In some cases, we can prove that $\frac{d}{dt} \mathcal{I}(X_t)$ is pointwisely equal to zero.

Corollary 1.15. *Suppose that (M, g) is a closed m -dimensional Einstein manifold with $m \geq 3$. When $m = 4$ or the scalar curvature of g vanishes identically, $\frac{d}{dt} \mathcal{I}(X_t) = 0$ for all t , where X_t is the solution to (1.4) with any given initial vector field X .*

Proof. By assumption we have $\operatorname{Ric} = \frac{R}{m}g$ and R is constant. Using (1.36), we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{I}(X_t) &= \int_M \frac{R^2}{2} \cdot \operatorname{div}(X_t) dV - 2 \int_M \frac{R^2}{m} \operatorname{div}(X_t) dV \\ &= \int_M \frac{m-4}{2m} R^2 \cdot \operatorname{div}(X_t) dV. \end{aligned}$$

This proves the statement. \square

2 A conjecture to the flow and its application

Before stating a conjecture to the flow (1.4), we shall look at a simple case that (M, g) is an Einstein manifold with positive sectional curvature and the solution to (1.4) is the sum of the initial vector field and a gradient vector field, i.e., we assume

$$R_{ij} = \frac{R}{m} g_{ij}, \quad m \geq 3, \quad X_t = X + \nabla f_t,$$

where f_t are some functions on M . By a theorem of Schur, the scalar curvature R must be a constant. In this case, the flow (1.4) is equivalent to

$$\nabla \left(\partial_t f_t - 2\Delta f_t - \frac{2R}{m} f_t \right) = X^\dagger, \quad (2.1)$$

where

$$X^\dagger := \Delta X + \nabla(\operatorname{div}(X)) + \operatorname{Ric}^\sharp(X) \quad (2.2)$$

is the vector field associated to X . Clearly that the operator \dagger is not self-adjoint on the space of vector fields, with respect to the L^2 -inner product with respect to (M, g) .

2.1 Einstein manifolds with positive scalar curvature

If (M, g) is an m -dimensional Einstein manifold with positive scalar curvature, then we can prove that the limit vector field converges to a nonzero Killing vector field, provided the initial vector field satisfying some conditions. We first give a L^2 -estimate for f_t .

Proposition 2.1. *Suppose that (M, g) is an m -dimensional closed and orientable Einstein manifold with positive scalar curvature R , where $m \geq 3$. Let X be a nonzero vector field satisfying $X^\dagger = \nabla\varphi_X$ for some smooth function φ_X on M . Then for any given constant c , the equation*

$$\partial_t f_t = 2\Delta f_t + \frac{2R}{m} f_t + \varphi_X, \quad f_0 = c, \tag{2.3}$$

exists for all time. Moreover,

(i) we have

$$\int_M f_t dV = \left[c \cdot \text{Vol}(M, g) + \frac{m}{2R} \int_M \varphi_X dV \right] e^{\frac{2R}{m}t} - \frac{m}{2R} \int_M \varphi_X dV. \tag{2.4}$$

Setting

$$c_X := -\frac{m}{2R \cdot \text{Vol}(M, g)} \int_M \varphi_X dV,$$

yields

$$\int_M f_t dV = -\frac{m}{2R} \int_M \varphi_X dV, \quad \text{if } c = c_X.$$

(ii) If we choose the nonzero function φ_X so that its integral over M is zero and $f_0 = 0$, then

$$\int_M f_t dV = 0$$

and the L^2 -norm of f_t is bounded by

$$\|f_t\|_2 \leq \frac{\|\varphi_X\|_2}{2(\lambda_1 - \frac{R}{m})} - \frac{\|\varphi_X\|_2}{2(\lambda_1 - \frac{R}{m})} e^{-2(\lambda_1 - \frac{R}{m})t}, \tag{2.5}$$

where $\|\cdot\|_2$ means $\|\cdot\|_{L^2(M, g)}$ the L^2 -norm with respect to (M, g) , and λ_1 stands for the first nonzero eigenvalue of (M, g) .

For a moment, we put

$$a(t) := \int_M f_t dV, \quad b(t) := \int_M |f_t|^2 dV.$$

Then, the equation (2.3) implies that

$$a'(t) = \frac{2R}{m} a(t) + \int_M \varphi_X dV,$$

and

$$\begin{aligned} b'(t) &= -4 \int_M |\nabla f_t|^2 dV + \frac{4R}{m} b(t) + 2 \int_M f_t \varphi_X dV \\ &\leq -4 \left(\lambda_1 - \frac{R}{m} \right) b(t) + 2b^{1/2}(t) \|\varphi_X\|_2. \end{aligned}$$

By a theorem of Lichnerowicz, we have that

$$\lambda_1 \geq \frac{R}{m-1} > \frac{R}{m}.$$

Hence, (2.4) and (2.5) follow immediately.

Consequently, we have the following theorem.

Theorem 2.2. *Suppose that (M, g) is an m -dimensional closed and orientable Einstein manifold with positive scalar curvature R , where $m \geq 3$. If X is a nonzero vector field satisfying the following two conditions:*

- (i) X^\dagger is a gradient vector field, and
- (ii) X is not a gradient vector field.

Then the flow (1.4) with initial value X converges uniformly to a nonzero Killing vector field.

2.2 A conjecture and its applications

By Bochner's theorem, any Killing vector field on a closed and orientable Riemannian manifold with negative Ricci curvature is trivial. Hence, based on a result in the Einstein case, we propose the following conjecture.

Conjecture 2.3. *Suppose that M is a closed Riemannian manifold with positive sectional curvature. For some initial vector field and a certain Riemannian metric g of positive sectional curvature, the flow (1.4) converges uniformly to a nonzero Killing vector field with respect to g .*

Our study shows that we may need to change to a new metric, which still has positive sectional curvature, to get the nonzero limit which is a Killing vector field with respect to this new metric. For this purpose we have computed variations of the functional \mathfrak{L} or \mathcal{E} relative to the new metric, as well as the Perelman-type functional for our flow.

Obviously a solution to this conjecture immediately answers the following long-standing question of Yau [11].

Question 2.4. *Does there exist an effective \mathbb{S}^1 -action on a closed manifold with positive sectional curvature?*

Assuming Conjecture 2.3, we can deduce several important corollaries. We first recall the well-known Hopf's conjectures.

Conjecture 2.5. *If M is a closed and even dimensional Riemannian manifold with positive sectional curvature, then the Euler characteristic number of M is positive, i.e., $\chi(M) > 0$.*

Conjecture 2.6. *On $\mathbb{S}^2 \times \mathbb{S}^2$ there is no Riemannian metric with positive sectional curvature.*

For the recent development of Hopf's conjectures, we refer to [10, 11]. A simple argument shows that Conjectures 2.5 and 2.6 follow from Conjecture 2.3.

Corollary 2.7. *Conjecture 2.3 implies Conjecture 2.5.*

Proof. From [7] we know that the Killing vector field X must have zero, and the zero sets consist of finite number of totally geodesic submanifolds $\{M_i\}$ of M with the induced Riemannian metrics. Moreover, each M_i is even dimensional and has positive sectional curvature. Hence we have $\chi(M) = \sum_i \chi(M_i)$. By induction, we obtain $\chi(M) > 0$. \square

Hsiang and Kleiner [5] showed that if M is a 4-dimensional closed Riemannian manifold with positive sectional curvature, admitting a nonzero Killing vector field, then M is homeomorphic to \mathbb{S}^4 or $\mathbb{C}\mathbb{P}^2$. Consequently, $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a Riemannian metric, whose sectional curvature is positive, with a nontrivial Killing vector field. Therefore, we obtain the following corollary.

Corollary 2.8. *Conjecture 2.3 implies Conjecture 2.6.*

3 Variants geometric flows

In Section 3, we discuss several new geometric flows whose fixed points give Killing vector fields. Recall the notions in [9]. Let (M, g) be a closed and orientable Riemannian manifold of dimension m and ϕ a positive smooth function on M . Define

$$\widetilde{\text{Ric}}_\infty := \text{Ric} - \text{Hess}(\ln \phi) \tag{3.1}$$

the Bakry-Émery Ricci tensor field. For any smooth tensor field T on M consider the weighted L^2 -inner product given by

$$\langle T, T \rangle_\phi := \int_M (T, T)\phi dV \tag{3.2}$$

and let us denote $\tilde{\delta}$ the formal adjoint of d with respect to this inner product. Then

$$\tilde{\delta} = \delta - i_{(d \ln \phi)^\#}, \tag{3.3}$$

where δ is the usual formal adjoint of d and $(d \ln \phi)^\#$ stands for the corresponding vector field of the 1-form $d \ln \phi$.

Lott [9] obtained the following Bochner formula (where ω is a 1-form):

$$\langle d\omega, d\omega \rangle_\phi + \langle \tilde{\delta}\omega, \tilde{\delta}\omega \rangle_\phi - \langle \nabla\omega, \nabla\omega \rangle_\phi = \langle \widetilde{\text{Ric}}_\infty\omega, \omega \rangle_\phi \tag{3.4}$$

or

$$\langle \nabla\omega, \nabla\omega \rangle_\phi + \langle \tilde{\delta}\omega, \tilde{\delta}\omega \rangle_\phi - \langle \omega, \widetilde{\text{Ric}}_\infty\omega \rangle_\phi = \langle \mathcal{L}_{\omega^\#}g, \mathcal{L}_{\omega^\#}g \rangle_\phi, \tag{3.5}$$

where \mathcal{L} means the Lie derivative. Let $X := \omega^\#$ or $X_b = \omega$ in (3.5) we obtain

$$\int_M |\mathcal{L}_X g|^2 \phi dV = \int_M [|\nabla X|^2 + |\tilde{\delta}X_b|^2 - \widetilde{\text{Ric}}_\infty(X, X)]\phi dV. \tag{3.6}$$

3.1 New criterion: I

Given a smooth function f on M , set

$$\phi := e^f, \quad \ln \phi = f \tag{3.7}$$

and define

$$\begin{aligned} \text{Ric}_f &:= \widetilde{\text{Ric}}_\infty = \text{Ric} - \text{Hess}(f), \\ \text{div}_f &:= -\tilde{\delta} = -\delta + i_{\nabla f} = \text{div} + i_{\nabla f}. \end{aligned}$$

For any smooth vector field X , we have

$$e^{-f} \text{div}(e^f X) = e^{-f} (e^f \text{div}(X) + e^f \langle \nabla f, X \rangle) = \text{div}(X) + \langle \nabla f, X \rangle,$$

which implies that

$$\text{div}_f = \frac{1}{e^f} \text{div}(e^f), \tag{3.8}$$

a weighted divergence in the sense of [4]. Therefore the identity (3.6) can be rewritten as

$$\int_M |\mathcal{L}_X g|^2 e^f dV = \int_M [|\nabla X|^2 + |\text{div}_f(X)|^2 - \text{Ric}_f(X, X)]e^f dV. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \int_M |\nabla X|^2 e^f dV &= \int_M \nabla_i X_j (e^f \nabla^i X^j) dV \\ &= - \int_M X_j (\nabla_i f \nabla^i X^j + \Delta X^j) e^f dV \\ &= - \int_M \langle X, \Delta_f X \rangle e^f dV, \end{aligned}$$

where

$$\Delta_f X^j := \Delta X^j + \nabla_i f \nabla^i X^j.$$

Similarly,

$$\int_M |\text{div}_f(X)|^2 e^f dV = \int_M \text{div}_f(X) (e^f \text{div}_f(X)) dV$$

$$\begin{aligned}
&= \int_M e^{-f} \operatorname{div}(e^f X)(e^f \operatorname{div}_f(X)) dV \\
&= - \int_M \langle X, \nabla \operatorname{div}_f(X) \rangle e^f dV.
\end{aligned}$$

Hence, the identity (3.9) implies

$$\int_M |L_X g|^2 e^f dV = - \int_M \langle X, \Delta_f X + \nabla \operatorname{div}_f(X) + \operatorname{Ric}_f(X) \rangle e^f dV. \quad (3.10)$$

The above identity shows that the Euler-Lagrange equation for the functional $X \mapsto \int_M |L_X g|^2 e^f dV$ is

$$\Delta_f X + \nabla \operatorname{div}_f(X) + \operatorname{Ric}_f(X) = 0. \quad (3.11)$$

We now simplify the equation (3.11). Compute

$$\begin{aligned}
\Delta_f X^i &= \Delta X^i + \nabla_j f \nabla^j X^i, \\
\nabla^i \operatorname{div}_f(X) &= \nabla^i (e^{-f} \operatorname{div}(e^f X)) \\
&= \nabla^i (\operatorname{div}(X) + \langle \nabla f, X \rangle) \\
&= \nabla^i \operatorname{div}(X) + \nabla^i (X^j \nabla_j f) \\
&= \nabla^i \operatorname{div}(X) + \nabla^i X^j \nabla_j f + X^j \nabla^i \nabla_j f.
\end{aligned}$$

Consequently,

$$\Delta_f X^i + \nabla^i \operatorname{div}_f(X) = \Delta X^i + \nabla^i \operatorname{div}(X) + \nabla_j f (L_X g)^{ij} + X_j \nabla^i \nabla^j f. \quad (3.12)$$

Plugging (3.12) into (3.11) and noting the definition of Ric_f yields

$$0 = \Delta X^i + \nabla^i \operatorname{div}(X) + R^i_j X^j + \nabla_j f (L_X g)^{ij}. \quad (3.13)$$

As in [15], we can prove the following theorem.

Theorem 3.1. *Given any smooth function f on a closed orientable Riemannian manifold (M, g) . A smooth vector field X is Killing if and only if it satisfies (3.13). When $f \equiv 0$, it reduces to the classical criterion of Yano.*

Proof. Suppose X is Killing. Then $L_X g = 0$ and $\Delta X + \nabla \operatorname{div}(X) + \operatorname{Ric}(X) = 0$ by Yano's theorem. These two equations immediately imply (3.13). Conversely, if X is a smooth vector field satisfying (3.13), then it also satisfies (3.11) and then

$$\int_M |L_X g|^2 e^f dV = 0$$

according to (3.10). Hence $L_X g \equiv 0$ and X is Killing. \square

The above theorem suggests us to consider the following flow:

$$\partial_t X^i = \Delta X^i + \nabla^i \operatorname{div}(X) + R^i_j X^j + \nabla_j f (L_X g)^{ij} \quad (3.14)$$

for a given smooth function $f \in C^\infty(M)$, or consider a nonlinear flow

$$\partial_t X^i = \Delta X^i + \nabla^i \operatorname{div}(X) + R^i_j X^j + \nabla_j \operatorname{div}(X) (L_X g)^{ij}. \quad (3.15)$$

As in the proof of Theorem 1.8, we have the following theorem.

Theorem 3.2. *Let (M, g) be a closed orientable Riemannian manifold, f a smooth function on M , and X a smooth vector field on M . Then the flow (3.14) starting with the initial data X smoothly converges to a Killing vector field X_∞ .*

Proof. By replacing $\operatorname{div}, \Delta, dV, \operatorname{Ric}$ by $\operatorname{div}_f, \Delta_f, e^f dV, \operatorname{Ric}_f$ in the argument of Theorem 1.8, we can show that $\int_M |X_t|^2 e^f dV$ is decreasing, $\int_M |\partial_t X_t|^2 e^f dV \rightarrow 0$, and then by the same method X_t smoothly converges to a smooth vector field X_∞ satisfying (3.13). By Theorem 3.1, X_∞ must be Killing. \square

3.2 New criterion: II

The second new criterion is based on the following identity:

$$\int_M \left[(\mathcal{L}_X g)(X, X) + \frac{1}{2} \operatorname{div}(X)|X|^2 \right] dV = 0, \tag{3.16}$$

for any smooth vector field X on M . Since

$$2(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

to prove (3.16), it suffices to show that

$$\int_M \left[X^i X^j \nabla_i X_j + \frac{1}{2} \operatorname{div}(X)|X|^2 \right] dV = 0.$$

Actually,

$$\begin{aligned} \int_M X^i X^j \nabla_i X_j dV &= - \int_M X_j \nabla_i (X^i X^j) dV \\ &= - \int_M X_j [\operatorname{div}(X) X^j + X^i \nabla_i X^j] dV \\ &= - \int_M \operatorname{div}(X)|X|^2 dV - \int_M X^i X_j \nabla_i X^j dV, \end{aligned}$$

which yields

$$\int_M X^i X^j \nabla_i X_j dV = -\frac{1}{2} \int_M \operatorname{div}(X)|X|^2 dV.$$

The second new criterion can be stated as follows:

Theorem 3.3. *A smooth vector field X on a closed orientable Riemannian manifold (M, g) is a Killing vector field if and only if it satisfies*

$$0 = \Delta X + \nabla \operatorname{div}(X) + \operatorname{Ric}(X, \cdot) + (\mathcal{L}_X g)(X, \cdot) + \frac{1}{2} \operatorname{div}(X)X. \tag{3.17}$$

Proof. If X is Killing, then $\operatorname{div}(X) = \mathcal{L}_X g = 0$ and hence (3.17) reduces to Yano’s classical result. Conversely, suppose a smooth vector field X satisfies (3.17). Multiplying (3.17) by X and integrating over M , we obtain

$$\begin{aligned} 0 &= - \int_M [|\nabla X|^2 + |\operatorname{div}(X)|^2 - \operatorname{Ric}(X, X)] dV \\ &\quad + \int_M \left[(\mathcal{L}_X g)_{ij} X^i X^j + \frac{1}{2} \operatorname{div}(X)|X|^2 \right] dV. \end{aligned} \tag{3.18}$$

The second integral on the right-hand side equals zero by the identity (3.16), and consequently, (3.18) is equivalent to $\mathcal{E}(X) = 0$, where $\mathcal{E}(X)$ was defined in (1.8). By a result of Watanabe [13], X must be Killing. □

Theorem 3.3 also suggests a nonlinear equation

$$\partial_t X_t = \Delta X_t + \nabla \operatorname{div}(X_t) + \operatorname{Ric}(X_t, \cdot) + (\mathcal{L}_{X_t} g)(X_t, \cdot) + \frac{1}{2} \operatorname{div}(X_t)X_t. \tag{3.19}$$

We note that the flows (1.4) and (3.14) are linear, while the flows (3.15) and (3.19) are nonlinear. We will later study those flows and applications to geometry.

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