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Let C_N be a cube of volume one centered at the origin in $\boldsymbol{R}^N$ and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. We prove that $C_N \cap P_K$ has K-dimensional volume greater than or equal to one. As an application of this inequality we obtain a precise version of Minkowski's linear forms theorem. We also state a conjecture which would allow our method to be generalized.

1. Introduction.

Let $C_N = [-1/2, 1/2]^N$ be the N-dimensional cube of volume one centered at the origin in $\boldsymbol{R}^N$ and suppose that P_K is a K-dimensional linear subspace of $\boldsymbol{R}^N$. Dr. Anton Good has conjectured that the K-dimensional volume of $P_K \cap C_N$ is always greater than or equal to one. In case $K = N - 1$ this has recently been proved by Hensley [6], who also obtained upper bounds for this volume. Our purpose in this paper is to prove the conjecture for arbitrary K and to give some applications to Minkowski's theorem on linear forms. In fact we prove a more general inequality for the product of spheres of various dimensions which contains the conjecture as a special case.

We write $\bar{x}$ for the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ in $\boldsymbol{R}^n$ and

$$|\bar{x}| = \left(\sum_{j=1}^{n} (x_j)^2\right)^{1/2}$$

for its length. We define the sphere S_n by

$$S_n = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| \leq \rho_n\}$$

where $\rho_n = \pi^{-1/2}\{\Gamma(n/2 + 1)\}^{1/n}$. It follows that $\mu_n(S_n) = 1$ where μ_n is Lebesgue measure on $\boldsymbol{R}^n$. Also we let $\chi_U(\bar{x})$ denote the characteristic function of a subset U in $\boldsymbol{R}^n$.

Our first main result is contained in the following theorem.

THEOREM 1. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers, $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$, and A is a real $N \times K$ matrix,* rank$(A) = K$. *Then*

$$|\det A^T A|^{-1/2} \leq \int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}) , \tag{1.1}$$

where A^T is the transpose of A.

We note that if $\operatorname{rank}(A) < K$ then each side of (1.1) is infinite. From Theorem 1 we easily deduce a lower bound for $\mu_K(Q_N \cap P_K)$.

COROLLARY. *Let Q_N be as in Theorem 1 and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. Then $\mu_K(Q_N \cap P_K) \geqq 1$.*

Proof. Choose A in Theorem 1 so that the columns of A form an orthonormal basis for P_K in $\boldsymbol{R}^N$. Then the left hand side of (1.1) is 1 while the right hand side is $\mu_K(Q_N \cap P_K)$.

The corollary clearly contains Good's conjecture since $Q_N = C_N$ if $n_j = 1$ and $J = N$.

Next we suppose that $L_j(\bar{x})$, $j = 1, 2, \cdots, N$ are N linear forms in K variables,

$$L_j(\bar{x}) = \sum_{k=1}^{K} a_{jk} x_k ,$$

so that $A = (a_{jk})$ is an $N \times K$ matrix. We assume that the forms L_j are real for $j = 1, 2, \cdots, r$ and that the remaining forms consist of s pairs of complex conjugate forms arranged so that $L_{r+2j-1} = \bar{L}_{r+2j}$ for $j = 1, 2, \cdots, s$. Thus $N = r + 2s$. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N$ be positive with $\varepsilon_{r+2j-1} = \varepsilon_{r+2j}$ for $j = 1, 2, \cdots, s$. We define the $N \times N$ diagonal matrix E by $E = (c_j \delta_{jk})$ where $c_j = \varepsilon_j^{-1}$ if $j = 1, 2, \cdots, r$, $c_j = (2/\pi)^{1/2} \varepsilon_j^{-1}$ if $j = r + 1, r + 2, \cdots, N$ and δ_{jk} is the Kronecker delta. Theorem 1 allows us to prove the following precise version of Minkowski's classical result on linear forms.

THEOREM 2. *Let M be a positive integer and suppose that*

$$M |\det A^* E^2 A|^{1/2} \leqq 1 , \tag{1.2}$$

where A^* is the complex conjugate transpose of the matrix A. Then there exist at least M distinct pairs of nonzero lattice points $\pm \bar{v}_m$, $m = 1, 2, \cdots, M$, such that

$$|L_j(\pm \bar{v}_m)| \leqq \varepsilon_j \tag{1.3}$$

for each j and each m. In particular if $|\det A^* A| > 0$ then there exists a pair of nonzero lattice points $\pm \bar{v}$ such that

$$|L_j(\pm \bar{v})| \leqq |\det A^* A|^{1/2K} \tag{1.4}$$

for $j = 1, 2, \cdots, r$, and

$$|L_j(\pm \bar{v}) \leqq \left(\frac{2}{\pi}\right)^{1/2} |\det A^* A|^{1/2K} \tag{1.5}$$

for $j = r + 1, r + 2, \cdots, N$.

Theorem 2 was first proved in the case $N \leqq K$ and $M = 1$ by Minkowski [8, p. 104]. Subsequently the extension of Minkowski's convex body theorem by van der Corput [5] allowed Theorem 2 to be proved for $N \leqq K$ and arbitrary M. Of course if $N = K$ then (1.2) becomes the more familiar condition

$$M\left(\frac{2}{\pi}\right)^s |\det A| \leqq \varepsilon_1\varepsilon_2 \cdots \varepsilon_N ,$$

and if $N < K$ then (1.2) is trivially satisfied since the left hand side is zero. The novelty in our result is that Theorem 2 now holds for $1 \leqq K < N$. Previously in the case $1 \leqq K < N$ we knew only that (1.3) held if

$$2^K M \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K : |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{1.6}$$

We prove Theorem 2 by showing that the right hand side of (1.6) is bounded from below by $2^K |\det A^* E^2 A|^{-1/2}$. As will be clear from the proof, Theorem 2 could be generalized to include linear forms with values in $\boldsymbol{R}^n$ for various n.

In §5 we state a conjecture which would allow us to obtain a significant improvement in Theorem 1. Specifically, we deduce from this conjecture an analogue of Theorem 1 in which Q_N is replaced by an arbitrary closed, convex, symmetric subset of $\boldsymbol{R}^N$ having N-dimensional volume equal to one.

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2. Preliminary results. In this section we briefly summerize some facts about logarithmically concave measures and functions. A more detailed discription can be found in the papers of Kanter [7] and Prékopa [9].

A function $f: \boldsymbol{R}^n \to [0, \infty)$ is said to be *log-concave* if for every pair of vectors $\bar{x}_1, \bar{x}_2$ in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$f(\lambda\bar{x}_1 + (1 - \lambda)\bar{x}_2) \geqq (f(\bar{x}_1))^\lambda (f(\bar{x}_2))^{1-\lambda} .$$

A probability measure ν defined on the measurable subsets of $\boldsymbol{R}^n$ is *log-concave* if for every pair of open convex sets U_1 and U_2 in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$\nu(\lambda U_1 + (1 - \lambda)U_2) \geqq (\nu(U_1))^\lambda (\nu(U_2))^{1-\lambda} , \tag{2.1}$$

where $+$ on the left hand side of (2.1) indicates Minkowski addition of sets. Clearly (2.1) holds for all open convex sets U_1 and U_2 if and only if it holds for all closed convex sets U_1 and U_2. The relationship between log-concave measures and log-concave functions is contained in the following lemma.

LEMMA 3. *Let ν be a log-concave probability measure on $\boldsymbol{R}^n$ and suppose that the support of ν spans the k-dimensional subspace P_k in $\boldsymbol{R}^n$. Then there is a log-concave probability density function f defined on P_k such that $d\nu = fd\mu_k$, where μ_k is k-dimensional Lebesgue measure on P_k. Conversely for any log-concave probability density function f defined on a k-dimensional subspace P_k in $\boldsymbol{R}^n$, the probability measure defined by $d\nu = fd\mu_k$ is log-concave, where μ_k is Lebesgue measure on P_k.*

The first part of Lemma 3 is a result of Borell [2, p. 123] while the converse was proved by Prékopa [9], (see also Kanter [7, Lemma 2.1]).

Let ν_1 and ν_2 be probability measures on $\boldsymbol{R}^n$. We say that ν_2 is *more peaked* than ν_1 if

$$\nu_1(U) \leqq \nu_2(U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. (We recall that $U \subseteq \boldsymbol{R}^n$ is symmetric if $U = -U$.) If f_1 and f_2 are probability density functions on $\boldsymbol{R}^n$ we say that f_2 is *more peaked* than f_1 if the measure $f_2 d\mu_n$ is more peaked than the measure $f_1 d\mu_n$. The notion of peakedness was introduced by Birnbaum [1] and Sherman [10]. A complementary relation is that of symmetric dominance in the sense of Kanter [7]. If ν_3 and ν_4 are measures on $\boldsymbol{R}^n$ then ν_3 symmetrically dominates ν_4 if

$$\nu_3(\boldsymbol{R}^n \backslash U) \geqq \nu_4(\boldsymbol{R}^n \backslash U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. It is clear that if ν_3 and ν_4 are both probability measures then ν_3 symmetrically dominates ν_4 if and only if ν_4 is more peaked than ν_3. For our purposes it is more convenient to work with the relation of peakedness.

If ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^n$ then the convolution $\nu_1 * \nu_2$ is also log-concave on $\boldsymbol{R}^n$ (Kanter [7, Lemma 2.3]). It follows that if ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^{n_1}$ and $\boldsymbol{R}^{n_2}$ respectively then the product measure $\nu_1 \times \nu_2$ is log-concave on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$. Forming product measures also preserves the peakedness relation.

LEMMA 4. *Suppose that ν_1, ν_2, ν_1' and ν_2' are all log-concave probability measures such that ν_1 is mome peaked than ν_1' on $\boldsymbol{R}^{n_1}$ and*

ν_2 is more peaked than ν_2' on $\boldsymbol{R}^{n_2}$. Then $\nu_1 \times \nu_2$ is more peaked than $\nu_1' \times \nu_2'$ on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$.

For the proof of Lemma 4 we refer to Kanter [7, Corollary 3.2] where the result is obtained for the more general class of unimodal measures.

3. Proof of Theorem 1. We begin by proving the following lemma.

LEMMA 5. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers and $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$. Then $\chi_{Q_N}(\bar{x})$ is more peaked than the normal density function* $\exp\{-\pi|\bar{x}|^2\}$ *on $\boldsymbol{R}^N$.*

Proof. Since the measures $\chi_{Q_N}(\bar{x})d\mu_N(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}d\mu_N(\bar{x})$ are both product measures which factor in $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \cdots \times \boldsymbol{R}^{n_J}$ it suffices to prove the peakedness relation in each factor space and then apply Lemma 4. Thus we need only show that for each positive integer n, $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on $\boldsymbol{R}^n$. Of course it is trivial to verify that both of the density functions $\chi_{S_n}(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}$ are log-concave on $\boldsymbol{R}^n$.

Let $\sum_{n-1} = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| = 1\}$ so that for each $\bar{x} \neq \bar{0}$ in $\boldsymbol{R}^n$ we have the unique polar decomposition $\bar{x} = r\bar{x}'$ where $r = |\bar{x}|$ and $\bar{x}' \in \sum_{n-1}$. *If* U is a closed, convex, symmetric subset of $\boldsymbol{R}^n$ then it follows that

$$(3.1) \qquad \int_U \exp\{-\pi|\bar{x}|^2\}d\mu_n(\bar{x}) = \int_{\Sigma_{n-1}} \int_0^\infty \chi_U(r\bar{x}') \exp\{-\pi r^2\} r^{n-1} dr d\bar{x}' ,$$

where $d\bar{x}'$ is the induced Lebesgue measure on $\sum_{n-1}$. Now for each fixed $\bar{x}' \in \sum_{n-1}$ we have either

$$(3.2) \qquad \chi_U(r\bar{x}') \leqq \chi_{S_n}(r\bar{x}') , \quad 0 \leqq r < \infty$$

or

$$(3.3) \qquad \chi_{S_n}(r\bar{x}') \leqq \chi_U(r\bar{x}') , \quad 0 \leqq r < \infty ,$$

since S_n and U are convex. If (3.2) holds at $\bar{x}'$ then

$$(3.4) \qquad \begin{aligned} &\int_0^\infty \chi_U(r\bar{x}') \exp\{-\pi r^2\} r^{n-1} dr \\ &\leqq \int_0^\infty \chi_U(r\bar{x}') r^{n-1} dr = \int_0^\infty \chi_U(r\bar{x}') \chi_{S_n}(r\bar{x}') r^{n-1} dr . \end{aligned}$$

If (3.3) holds at $\bar{x}'$ then

$$\begin{aligned}
&\int_0^\infty \chi_U(r\bar{x}') \exp\{-\pi r^2\} r^{n-1} dr \\
(3.5) \qquad &\leqq \int_0^\infty \exp\{-\pi r^2\} r^{n-1} dr = n^{-1}\pi^{-n/2}\Gamma\left(\frac{n}{2}+1\right) \\
&= \int_0^\infty \chi_{S_n}(r\bar{x}') r^{n-1} dr \\
&= \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}') r^{n-1} dr\ .
\end{aligned}$$

Combining (3.1), (3.4) and (3.5) we obtain

$$\int_U \exp\{-\pi|\bar{x}|^2\} d\mu_n(\bar{x}) \leqq \int_{\Sigma_{n-1}} \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}') r^{n-1} dr d\bar{x}' = \int_U \chi_{S_n}(\bar{x}) d\mu_n(\bar{x})\ .$$

Thus $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on R^n and the lemma is proved.

We now prove Theorem 1. If $N = K$ then (1.1) is trivial so we may suppose that $K' = N - K$ is positive. Let P_K be the K-dimensional subspace of R^N spanned by the columns of A. Next let W be an $N \times N$ matrix whose first K columns are the columns of A and whose next K' columns are the columns of an $N \times K'$ matrix B. We choose the columns of B so that they form an orthonormal basis in $\boldsymbol{R}^N$ of the K'-dimensional subspace which is orthogonal to P_K. Identifying $\boldsymbol{R}^N$ with $\boldsymbol{R}^K \times \boldsymbol{R}^{K'}$ we may write each $\bar{z} \in \boldsymbol{R}^N$ as $\bar{z} = (\bar{x}/\bar{y})$ where $\bar{x} \in \boldsymbol{R}^K$ and $\bar{y} \in \boldsymbol{R}^{K'}$. For each ε, $0 < \varepsilon \leqq 1$ we define

$$H_\varepsilon = \left\{\bar{z} \in \boldsymbol{R}^N : z = \left(\frac{\bar{x}}{\bar{y}}\right),\ \max_{1 \leqq j \leqq K'} |y_j| \leqq \frac{\varepsilon}{2}\right\}$$

and

$$H'_\varepsilon = \left\{\bar{y} \in \boldsymbol{R}^{K'} : \max_{1 \leqq j \leqq K'} |y_j| \leqq \frac{\varepsilon}{2}\right\}\ .$$

Clearly H_ε is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ and so is the image of H_ε under the nonsingular linear transformation determined by W. Thus by Lemma 5,

$$(3.6) \qquad \int_{H_\varepsilon} \exp\{-\pi|W\bar{z}|^2\} d\mu_N(\bar{z}) \leqq \int_{H_\varepsilon} \chi_{Q_N}(W\bar{z}) d\mu_N(\bar{z})\ .$$

Multiplying each side of (3.6) by $\{\mu_{K'}(H'_\varepsilon)\}^{-1} = \varepsilon^{-K'}$ and factoring H_ε into $\boldsymbol{R}^K \times H'_\varepsilon$ we find that

$$\begin{aligned}
(3.7) \qquad &\varepsilon^{-K'} \int_{\boldsymbol{R}^K} \int_{H'_\varepsilon} \exp\{-\pi|A\bar{x} + B\bar{y}|^2\} d\mu_{K'}(\bar{y}) d\mu_K(\bar{x}) \\
&\leqq \varepsilon^{-K'} \int_{\boldsymbol{R}^K} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) d\mu_K(\bar{x})\ .
\end{aligned}$$

By the orthogonality condition $|A\bar{x} + B\bar{y}|^2 = |A\bar{x}|^2 + |B\bar{y}|^2$ and so as $\varepsilon \to 0+$ the left hand side of (3.7) clearly converges to

$$\int_{R^K} \exp\{-\pi |A\bar{x}|^2\} d\mu_K(\bar{x}) = |\det A^T A|^{-1/2} .$$

To evaluate the corresponding limit on the right hand side of (3.7) we observe that for $0 < \varepsilon \leqq 1$ and each $\bar{x} \in \boldsymbol{R}^K$,

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \leqq 1 .$$

Since Q_N and H'_ε are both bounded we have

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = 0$$

for sufficiently large $|\bar{x}|$ independent of ε. Thus by dominated convergence the limit on the right of (3.7) as $\varepsilon \to 0+$ is

$$\int_{R^K} \left\{ \lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \right\} d\mu_K(\bar{x}) . \tag{3.8}$$

Clearly

$$\lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = \chi_{Q_N}(A\bar{x})$$

except possibly when $A\bar{x}$ is a boundary point of $Q_N \cap P_K$, Since this boundary has K-dimensional measure zero we see that (3.8) is equal to

$$\int_{R^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}) .$$

We have now shown that as $\varepsilon \to 0+$ on each side of (3.7) we obtain (1.1) and this proves the theorem.

4. Proof of Theorem 2. By van der Corput's extension of Minkowski's convex body theorem [5] (see also Cassels [4, Chapter III, Theorem II]) the condition (1.6) implies that there exist at least M distinct pairs $\pm\bar{v}_m$, $m = 1, 2, \cdots, M$, of nonzero lattice points such that (1.3) holds. If $\operatorname{rank}(A) < K$ then (1.2) and (1.6) are both trivially satisfied. Thus to eatablish the first part of Theorem 2 it suffices to show that if $\operatorname{rank}(A) = K$ then

$$2^K |\det A^* E^2 A|^{-1/2} \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K : |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{4.1}$$

Let $G_j(\bar{x})$, $j = 1, 2, \cdots, N$ be linear forms defined by $G_j(\bar{x}) = L_j(\bar{x})$ for $j = 1, 2, \cdots, r$ and

$$G_{r+2j-1}(\bar{x}) = \sqrt{2}\,\mathrm{Re}\{L_{r+2j-1}(\bar{x})\}\ ,$$
$$G_{r+2j}(\bar{x}) = \sqrt{2}\,\mathrm{Im}\{L_{r+2j-1}(\bar{x})\}$$

for $j = 1, 2, \cdots, s$. We write $B = (b_{jk})$ for the corresponding real $N \times K$ matrix so that

$$G_j(\bar{x}) = \sum_{k=1}^{K} b_{jk} x_k\ .$$

Next we let $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_{r+s}}$ where $n_j = 1$ for $j = 1, 2, \cdots, r$ and $n_j = 2$ for $j = r+1, r+2, \cdots, r+s$. It follows that $|L_j(\bar{x})| \leqq \varepsilon_j$ if and only if $1/2\varepsilon_j^{-1} G_j(\bar{x}) \in S_{n_j}$, $j = 1, 2, \cdots, r$, and

$$|L_{r+2j-1}(\bar{x})| = |L_{r+2j}(\bar{x})| \leqq \varepsilon_{r+2j}$$

if and only if

$$(2\pi)^{-1/2}\varepsilon_{r+2j}^{-1}\begin{pmatrix} G_{r+2j-1}(\bar{x}) \\ G_{r+2j}(\bar{x}) \end{pmatrix} \in S_{n_{r+j}}\ ,$$

$j = 1, 2, \cdots, s$. Therefore

$$\begin{aligned}
&\mu_K(\{\bar{x} \in \boldsymbol{R}^K \colon |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) \\
&\quad = \mu_K\Big(\Big\{\bar{x} \in \boldsymbol{R}^K \colon \frac{1}{2}EB\bar{x} \in Q_N\Big\}\Big) = \int_{\boldsymbol{R}^K} \chi_{Q_N}\Big(\frac{1}{2}EB\bar{x}\Big) d\mu_K(\bar{x}) \\
&\quad \geqq \Big|\det\Big(\frac{1}{2}EB\Big)^T\Big(\frac{1}{2}EB\Big)\Big|^{-1/2} = 2^K|\det B^T E^2 B|^{-1/2}\ .
\end{aligned}$$

An easy computation shows that $B^T E^2 B = A^* E^2 A$ and so completes the proof of (4.1).

To prove the second part of Theorem 2 we choose $\varepsilon_j = |\det A^* A|^{1/2K}$ for $j = 1, 2, \cdots, r$ and $\varepsilon_j = (2/\pi)^{1/2}|\det A^* A|^{1/2K}$ for $j = r+1, r+2, \cdots, N$. Then

$$|\det A^* E^2 A| = 1$$

and so (1.4) and (1.5) follow from the first part of the theorem.

5. Lower bounds for arbitrary convex bodies. In this section we suppose that Q_N is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ with $\mu_N(Q_N) = 1$. If A is an $N \times K$ matrix, rank$(A) = K$, we will be interested in the problem of finding a lower bound for

$$\int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})\, d\mu_K(\bar{x})\ . \tag{5.1}$$

The method used to deduce Theorem 1 from Lemma 5 will also lead to a lower bound in this more general situation, provided that we can find a suitable normal density function on $\boldsymbol{R}^N$ which is less peaked than $\chi_{Q_N}(\bar{x})$. We succeeded in proving Lemma 5 because the special structure imposed on Q_N allowed us to appeal to Lemma 4. We now describe an alternative method which leads to a conjectured lower bound for (5.1).

We write Q for Q_N and we assume that Q is a fixed, closed, convex, symmetric subset of $\boldsymbol{R}^N$, $\mu_N(Q) = 1$. For each positive integer m let

$$\chi_Q^{(m)}(\bar{x}) = \chi_Q^* \chi_Q^* \cdots \chi_Q(\bar{x})$$

be the m-fold convolution of χ_Q. We define the dilation operator D_λ for $\lambda > 0$ and for integrable real valued functions f on $\boldsymbol{R}^N$ by

$$D_\lambda(f)(\bar{x}) = \lambda^N f(\lambda \bar{x}) .$$

Next we define a sequence of positive numbers λ_m, $m = 1, 2, \cdots$ by

$$(\lambda_m)^N \chi_Q^{(m)}(\bar{0}) = 1 .$$

With this notation we have the following

CONJECTURE 6. *For each positive integer m, $\chi_Q(\bar{x})$ is more peaked than $D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})$.*

Now let Ω be the $N \times N$ covariance matrix determined by a random vector which is uniformly distributed on the convex body Q. That is $\Omega = (\omega_{rs})$ is the $N \times N$ matrix defined by

$$\omega_{rs} = \int_{R^N} y_r y_s \chi_Q(\bar{y}) d\mu_N(\bar{y}) ,$$

where y_r and y_s are the rth and sth co-ordinate functions of $\bar{y}$, $r = 1, 2, \cdots, N$, and $s = 1, 2, \cdots, N$. It is clear that Ω is symmetric and nonsingular since Q has a nonempty interior. By the Central Limit Theorem (Breiman [3, Theorem 11.10]) we have

$$\lim_{m\to\infty} D_{\sqrt{m}}(\chi_Q^{(m)})(\bar{x}) = (2\pi)^{-N/2}(\det \Omega)^{-1/2} \exp\left\{-\frac{1}{2}\bar{x}^T \Omega^{-1}\bar{x}\right\}$$

uniformly for $x \in \boldsymbol{R}^N$. It follows that

$$\lim_{m\to\infty} \frac{\lambda_m}{\sqrt{m}} = (2\pi)^{1/2}(\det \Omega)^{1/2N}$$

and hence

$$\lim_{m\to\infty} D_{\lambda_m}(\chi_Q^{(m)})(\bar{x}) = \exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T \Omega^{-1}\bar{x}\}$$

uniformly for $x \in \boldsymbol{R}^N$. If the Conjecture 6 is true then for each positive integer m and each closed, convex, symmetric subset U of $\boldsymbol{R}^N$

$$\int_U D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})d\mu_N(\bar{x}) \leqq \int_U \chi_Q(\bar{x})d\mu_N(\bar{x}) \ . \tag{5.2}$$

Letting $m \to \infty$ on the left hand side of (5.2) and we have proved that $\chi_Q(\bar{x})$ is more peaked than $\exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T\Omega^{-1}\bar{x}\}$ on $\boldsymbol{R}^N$. By the same method used to prove Theorem 1 we obtain

THEOREM 7. *Assume that the Conjecture* 6 *holds and let* A *be a real* $N \times K$ *matrix,* $\operatorname{rank}(A) = K$. *Then*

$$(\det \Omega)^{-K/2N}|\det A^T\Omega^{-1}A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_Q(A\bar{x})d\mu_K(\bar{x}) \ . \tag{5.3}$$

If the set Q in Theorem 7 is such that Ω is a constant multiple of the identity matrix then the left hand side of (5.3) is simply $|\det A^TA|^{-1/2}$. Just as in our proof of the corollary to Theorem 1, we deduce that in this case $\mu_K(Q \cap P_K) \geqq 1$, where P_K is a K-dimensional subspace of $\boldsymbol{R}^N$. There is also an application of Theorem 7 to linear forms. If $L_j(\bar{x})$, $j = 1, 2, \cdots, N$, are N linear forms in K-variables we could determine precise conditions under which

$$\left(\sum_{j=1}^N |L_j(\bar{v})|^p\right)^{1/p} \leqq \varepsilon$$

at a nonzero lattice point $\bar{v}$ for any $p \geqq 1$ and $\varepsilon > 0$. At present, however, these results remain hypothetical since they depend on the open problem stated in Conjecture 6.

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