A GEOMETRIC MODEL OF ARBITRARY SPIN MASSIVE PARTICLE

S.M. Kuzenko, S.L. Lyakhovich[†] and A.Yu. Segal

Department of Theoretical Physics,
Tomsk State University, 634050 Tomsk, Russia

Abstract

A new model of relativistic massive particle with arbitrary spin ((m, s)-particle) is suggested. Configuration space of the model is a product of Minkowski space and two-dimensional sphere, $\mathcal{M}^6 = \mathbb{R}^{3,1} \times S^2$. The system describes Zitterbewegung at the classical level. Together with explicitly realized Poincaré symmetry, the action functional turns out to be invariant under two types of gauge transformations having their origin in the presence of two Abelian first-class constraints in the Hamilton formalism. These constraints correspond to strong conservation for the phase-space counterparts of the Casimir operators of the Poincaré group. Canonical quantization of the model leads to equations on the wave functions which prove to be equivalent to the relativistic wave equations for the massive spin-s field.

[†]e-mail: SLL@fftgu.tomsk.su

1 Introduction

The progress in the string theories stimulated an intensive development of another, more traditional, direction in high-energy physics concerning the construction of mechanical models for point-like spinning particles and the study of their dynamical properties. First of all, the interest to spinning (super)particles is caused by the fact that the spectra of free (super)strings contain particle-like excitations of higher spins. One more important point is that the superparticle models possess a lot of features being highly similar to those arising in superstrings (infinite reducibility of gauge algebra, nontrivial mixing of first- and second-class constraints), what makes it possible to exploit these models for polishing out the string quantization methods. Finally, the systematization and study of spinning particles models are necessary for accomplishing the standard interpretation of quantum field theory as a second quantized theory of relativistic particles.

Nowadays it is too early to speak about any systematization of particle models, since new and new models appear permanently. In this regard, it is worth mentioning the models desribing spin-1/2 particle[1–4], (extended) superparticles[5–9], higher spin particles[10–13].

In most of the spinning particle models, anticommuting variables are incorporated into configuration space to describe the spin degrees of freedom. Such a pseudoclassical approach is known to be well adapted for constructing the quantum theory. It makes, however, impossible to provide an elegant geometric framework for the classical evolution of spinning particles, which could be considered as a natural generalization of that existing for the spinless particle. Supposing the existence of such a geometric formulation for spinning particles, one can expect that the corresponding configuration space should have sufficiently low dimension to describe only the space-time evolution and the spin dynamics. Equivalently, every physical observable commuting with the first-class constraints must be a function of the Poincaré generators on the phase space. This implies in turn that the spin variables have to transform nonlinearly with respect to the Lorentz group.

The Lorentz group $SO(3,1)^{\uparrow} \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ is usually defined to be the connected component of unit in the group $O(3,1)$ of all linear homogeneous transformation on $\mathbb{R}^{3,1}$ preserving the metric $\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b$, $\eta_{ab} = \mathrm{diag}\ (- + ++)$. On the other hand, $SL(2,\mathbb{C})/\mathbb{Z}_2$ naturally acts on a two-dimensional sphere S^2, by fractional linear transformations, and coincides with the group of all complex automorphisms of S^2. This fact underlies the twistor approach[11]. It is worth noting that S^2 is the unique representative in the family of $SO(3,1)^{\uparrow}$-transformation spaces in the sense that it has minimally possible dimension.

Since $SO(3,1)^{\uparrow}$ is realized as the transformation group of the manifolds $\mathbb{R}^{3,1}$ and S^2, it seems reasonable to consider the united space $\mathcal{M}^6 \equiv \mathbb{R}^{3,1} \times S^2$, being a transformation space of the Poincaré group (the space-time translations are defined to act trivially on S^2), and to try luck in constructing a Poincaré-invariant theory of particle on $\mathcal{M}^6$. Obviously, such a particle model on $\mathcal{M}^6$ will be equivalent to some model of spinning particle in Minkowski space with the spin sector to be formed by the spherical modes. We propose the requirement that the system on $\mathcal{M}^6$ must possess two strong conservation laws corresponding to the mass and spin of the particle as a basic dynamical principle which allows to choose unique action functional.

The present paper is devoted to realization of the above program. It will be shown that a free massive particle of arbitrary fixed spin can be described in the framework of special model of a particle on $\mathcal{M}^6$. The corresponding Lagrangian has a geometric origin, and its structure is determined by the parameters of mass m and spin s ((m,s)-particle model).

The paper is organized as follows. In section 2 we develop a Lagrange formalism for a Poincaré-invariant model of particle on $\mathcal{M}^6$. The Lagrangian of the model involves not only the interval along the world line in Minkowski space, but also a Poincaré-invariant interval on S^2. The spherical metric turns out to be dynamical in the sense that it depends explicitly on the tangent vector to the world line in Minkowski space. Here we show that the causality principle for massive spinning particle consists of imposing the inequality $\dot{x}^2 < 0$, for every point of the world line, together with some relativistic restriction on maximal velocity of particle on S^2. The dynamics of (m,s)-particle proves to be nontrivial and is similar to the known effect of Schrödinger vibration (Zitterbevegung) in relativistic quantum theory (analogous dynamical behaviour of classical spinning particles has been observed in more early models[14,15]). In section 3 we develop a Hamilton formalism for the model of (m,s)-particle, study the physical observables and integrate the equations of motion with arbitrary Lagrange multipliers. Section 4 is devoted to the quantization of the model of (m,s)-particle. In concluding section 5 we discuss our results and further perspectives. We also include into the paper three appendices having technical character. Appendix A contains necessary facts concerning the action of the Lorentz group on S^2. Tensor fields on $\mathcal{M}^6$ are described in appendix B. In appendix C we briefly present a relativistic harmonic analysis on S^2.

Our notations and conventions coincide mainly with those adopted in Wess and Bagger's book[16]. In particular, σ-matrices and γ-matrices are chosen in the form

$$(\sigma_a)_{\alpha\dot{\alpha}} = (\mathbb{1}, \vec{\sigma}), \qquad (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}}, \qquad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix}.$$

But in contrast with Ref. 16, it is useful for us to number two-component spinor indices by values 0, 1 ($\varepsilon^{01} = \varepsilon^{\dot{0}\dot{1}} = 1$) and to define matrices σ_{ab} and $\tilde{\sigma}_{ab}$ by the rule

$$\begin{aligned}(\sigma_{ab})_\alpha{}^\beta &= -\,\frac{1}{4}\{(\sigma_a)_{\alpha\dot{\gamma}}(\tilde{\sigma}_b)^{\dot{\gamma}\beta} - (a \leftrightarrow b)\},\\ (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} &= -\,\frac{1}{4}\{(\tilde{\sigma}_a)^{\dot{\alpha}\gamma}(\sigma_b)_{\gamma\dot{\beta}} - (a \leftrightarrow b)\}.\end{aligned}$$

2 Model of relativistic particle on $\mathcal{M}^6$

Similarly to Minkowski space $\mathbb{R}^{3,1}$, the manifold $\mathcal{M}^6 = \mathbb{R}^{3,1} \times S^2$ is a (homogeneous) transformation space for the Poincaré group $\mathcal{P}$ and, in principle, can be chosen on the role of an arena where dynamics of relativistic particles is developing.

The action of the Lorentz group $SL(2,\mathbb{C})/\mathbb{Z}_2$ on two-sphere S^2 is well known (see, for example, Refs. 11, 17) and described in appendix A in the form appropriate for our purposes. It can be continued uniquely to the action of the Poincaré group on S^2 by attaching to each space-time translation the identity map on S^2. Then, the relations (A.4) and

$$x^{\alpha\dot{\alpha}} \to x'^{\alpha\dot{\alpha}} = x^{\beta\dot{\beta}}(N^{-1})_\beta{}^\alpha(\overline{N}^{-1})_{\dot{\beta}}{}^{\dot{\alpha}} + f^{\beta\dot{\beta}} \tag{1}$$

define an action of the Poincaré group on $\mathcal{M}^6$. Here $x^a = -\,\frac{1}{2}(\sigma^a)_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}}$ are the coordinates on $\mathbb{R}^{3,1}$, $f^a = -\,\frac{1}{2}(\sigma^a)_{\alpha\dot{\alpha}}f^{\alpha\dot{\alpha}}$ are the parameters of translations, and the Lorentz transformations are associated in standard fashion with complex unimodular $2{\times}2$ matrices (A.2).

It should be remarked that the composite structure of $\mathcal{M}^6$, being the product of $\mathbb{R}^{3,1}$ and S^2, admits the action of the group $\mathcal{P} \times \mathcal{P}$ on $\mathcal{M}^6$. However, physically interesting theories on $\mathcal{M}^6$, i.e. admitting four-dimensional interpretation, must have only the diagonal of $\mathcal{P} \times \mathcal{P}$ as a symmetry group. Before describing the fulfilment of this requirement for a point particle on $\mathcal{M}^6$, we first present some more $\mathcal{M}^6$-formalism.

Let us introduce a two-component object

$$z^\alpha \equiv (z)^\alpha = (1, z), \qquad \alpha = 0, 1, \tag{2}$$

constructed in terms of the complex local coordinate z on S^2 and connected with the coordinates q^α on $U_0 \subset \mathbb{C}^2_*$ (A.1) by the rule $z^\alpha = q^\alpha/q^0$. The relations (A.2–4) imply that z^α is transformed under the action of the Lorentz group by the law

$$z^\alpha \to z'^\alpha = \left(\frac{\partial z'}{\partial z}\right)^{1/2} z^\beta (N^{-1})_\beta{}^\alpha, \tag{3}$$

that is, simultaneously, as a tensor field on S^2 of type $\{1/2,\, 0\}$ (see appendix B) and a left Weyl spinor[-1]. Let p^a be a time-like 4–vector,

$$p^2 = p^a p_a < 0, \tag{4}$$

"living" in Minkowski space. One can associate with this 4–vector a smooth positive definite metric on S^2 of the form

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(p^a\xi_a)^2} = 2g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}, \tag{5}$$

where

$$\xi_a = (\sigma_a)_{\alpha\dot{\alpha}} z^\alpha \bar{z}^{\dot{\alpha}} = \bar{\xi}_a = (1 + z\bar{z}, z + \bar{z}, \mathrm{i}z - \mathrm{i}\bar{z}, 1 - z\bar{z}), \quad \xi^a\xi_a = 0. \tag{6}$$

For the Lorentz transformation (A.4) we have, due to Eq. (3), the following relation

$$\frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{(p'_{\alpha\dot{\alpha}} z'^\alpha \bar{z}'^{\dot{\alpha}})^2} = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(p_{\alpha\dot{\alpha}} z^\alpha \bar{z}^{\dot{\alpha}})^2},$$

where

$$p'{}_{\alpha\dot{\alpha}} = N_\alpha{}^\beta\, \overline{N}_{\dot{\alpha}}{}^{\dot{\beta}}\, p_{\beta\dot{\beta}}$$

is the transformation law of a 4–vector under the Lorentz group. In this sense $\mathrm{d}s^2$ *is invariant with respect to the action of the diagonal in* $\mathcal{P} \times \mathcal{P}$ *on* $\mathcal{M}^6$. It is worth noting that the metric (5) is characterized by a constant positive curvature, namely:

$$[\nabla_z, \nabla_{\bar{z}}] = \frac{1}{2}(r - s) g_{z\bar{z}} R, \qquad R = -p^2. \tag{7}$$

Here we have assumed the commutator to be taken on the space of tensor fields on S^2 of type $\{r/2, s/2\}$[7]. In a rest reference system where $p^a = (p^0, 0, 0, 0)$, the above metric coincides, up to a constant multiplier, with the standard metric on sphere,

$$\mathrm{d}s^2 \sim \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1 + z\bar{z})^2}.$$

[-1] A smooth tensor field $\Phi^{[n/2]}$ on S^2 of type $\{n/2,\, 0\}$, which satisfies the requirement of holomorphicity (B.6), can be represented in the form (B.7), $F_{\alpha_1\ldots\alpha_n}$ being a Lorentz tensor of type $(n/2,\, 0)$.

[7] The covariant derivative ∇_z acts on a field of type $\{r/2, s/2\}$ as $\nabla_z = \partial_z + \frac{r}{2}\Gamma^z_{zz}$, $\Gamma^z_{zz} = -2\partial_z \ln(p,\xi)$, where $(p,\xi) \equiv p^a\xi_a$.

Inequality (4) proves to be equivalent to the condition that Eq. (5) defines a smooth tensor field on S^2 of type {-1, -1}. In the case of a massive particle, there is a natural candidate on the role of p^a — the derivative $\dot{x}^a$ along the space-time trajectory $x^a(\tau)$ with respect to the evolution parameter τ. As a result, the requirement of Poincaré invariance, i.e. of invariance with respect to the action of the diagonal in $\mathcal{P} \times \mathcal{P}$ only, for a theory of massive particle on $\mathcal{M}^6$ means that both Poincaré scalars related to each point on the world line,

$$\dot{x}^2 = \dot{x}^a \dot{x}_a, \qquad \frac{4\dot{z}\dot{\bar{z}}}{(\dot{x},\xi)^2},$$

should arise in the corresponding Lagrangian.

The Lagrangian of a relativistic particle on $\mathcal{M}^6$ is given by the following expression

$$\mathcal{L} = \frac{1}{2e_1}\{\dot{x}^2 - (mce_1)^2\} + \frac{1}{2e_2}\left\{\frac{4\dot{z}\dot{\bar{z}}}{(\dot{x},\xi)^2}e_1^2 + (\Delta e_2)^2\right\}. \tag{8}$$

Here $e_1(\tau)$ and $e_2(\tau)$ are Lagrange multipliers ("einbeins") associated with the particle's motion in $\mathbb{R}^{3,1}$ and S^2, respectively. The Lagrangian consists of two parts: a Minkowskian length and its spherical counterpart, the former being well known Lagrangian of a massive spinless particle. The parameter Δ can be interpreted as a spherical "mass". As will be shown below, Δ is connected with the particle's spin. The introduction of e_1 into the spherical part of $\mathcal{L}$ guarantees the invariance of the action functional under world-line reparametrizations looking infinitesimally like

$$\begin{gathered} \delta_\epsilon x^a = \dot{x}^a\epsilon, \qquad \delta_\epsilon z = \dot{z}\epsilon, \\ \delta_\epsilon e_i = \frac{\mathrm{d}}{\mathrm{d}\tau}(e_i\epsilon), \qquad i = 1, 2, \end{gathered} \tag{9}$$

the parameter $\epsilon(\tau)$ being arbitrary modulo standard boundary conditions. What is more, the action functional possesses another gauge invariance of the form

$$\delta_\mu x^a = p^a\mu, \qquad \delta_\mu e_1 = \dot{\mu}, \qquad \delta_\mu z = \delta_\mu e_2 = 0. \tag{10}$$

Here μ denotes the gauge parameter, and p^a the four-momentum of the particle,

$$p^a = \frac{\partial\mathcal{L}}{\partial\dot{x}_a} = \frac{\dot{x}^a}{e_1} - \frac{4\dot{z}\dot{\bar{z}}}{(\dot{x},\xi)^3}\,\frac{e_1^2}{e_2}\xi^a. \tag{11}$$

The presence of two independent gauge symmetries (9) and (10) in the theory with the Lagrangian (8) gives rise to two first-class constraints in Hamilton formalism.

The Lagrange multipliers e_1 and e_2 can be eliminated with the aid of their equations of motion

$$\left(\frac{\dot{x}}{e_1}\right)^2 + m^2c^2 - \frac{8\dot{z}\dot{\bar{z}}}{(\dot{x},\xi)^2}\,\frac{e_1}{e_2} = 0; \tag{12.a}$$

$$\frac{4\dot{z}\dot{\bar{z}}}{(\dot{x},\xi)^2}\left(\frac{e_1}{e_2}\right)^2 - \Delta^2 = 0. \tag{12.b}$$

Then $\mathcal{L}$ takes the form

$$\mathcal{L} = -mc\sqrt{-\dot{x}^2\left(1 - \frac{4\Delta}{m^2c^2}\,\frac{|\dot{z}|}{(\dot{x},\xi)}\right)}. \tag{13}$$

For $\Delta = 0$ this expression apparently reduces to the Lagrangian of massive spinless particle.

It is necessary to point out that the parameter Δ is dimensional. As is easily seen, it can not be made dimensionless by means of redefinitions involving only another parameter of the theory, the mass of particle, and the speed of light. Accounting, however, the Planck constant, Δ can be written on the manner

$$\Delta = \hbar mc\sqrt{s(s+1)}, \tag{14}$$

where s is a dimensionless non-negative constant. We identify s with particle's spin. Classically, s can take arbitrary values. In what follows, we are going to show that consistent quantization of the theory turns out to be possible only for (half)integer values of s. As for the appearance of $\hbar$ in the classical Lagrangian (13) of massive spinning particle ((m,s)-particle), this seems natural from the usual standpoint that spin is a quantum effect disappearing in the limit $\hbar \to 0$.

Let us discuss a global structure of the space of world lines (the space of histories) in the model of (m,s)-particle. Similarly to the spinless case, the space of histories must consist only of those world lines $\varphi^i = \{x^a(\tau), z(\tau), \bar{z}(\tau)\}$ consistent with the requirement of space-time causality:

$$\dot{x}^2 < 0, \qquad \dot{x}^0 > 0. \tag{15}$$

On the other hand, it follows from the explicit form of the Lagrangian (13) that $\mathcal{L}$ is well defined under fulfilment of the following restriction:

$$|\dot{z}|/(\dot{x},\xi) < m^2c^2/4\Delta; \tag{16.a}$$

here and below we set $\Delta \neq 0$. This restriction means that (m,s)-particle can not move with arbitrary large velocity not only in Minkowski space, but also in the internal (spinning) space. In addition, the requirement (16.a) should be supplemented by the inequality

$$\dot{z} \neq 0 \tag{16.b}$$

presenting a consistency condition for the equation (12.b). At first sight, the restrictions have a technical character. However, one readily finds that a solution φ_0^i of the dynamical equations

$$\delta\mathcal{L}/\delta x^a = 0, \qquad \delta\mathcal{L}/\delta z = 0$$

is causal (consistent with (15)) if and only if it is consistent with (16). Hence, all admissible world lines must obey the system of inequalities (15) and (16) which are to be understood as the full set of causal conditions for the massive spinning particle. In virtue of the importance of this assertion, let us give more comments.

We would like to note first that for every world line consistent with (15) the conjugate 4–momentum of particle (11) satisfies automatically the relations (4) and $p^0 > 0$. On the mass-shell for x^a, the 4–momentum is constant, $\dot{p}^a = 0$, while the equation (12.a) implies

$$p^2 + m^2c^2 = 0, \tag{17}$$

as a consequence of the identity $\xi^2 = 0$. Let us choose a constant 4–vector p^a satisfying (17) and the condition $p^0 > 0$. Let us also choose arbitrary positive-definite functions $e_1(\tau)$ and $e_2(\tau)$. Making use of the relations $(\dot{x}, \xi) = e_1(p, \xi)$ and (12.b), we rewrite the dynamical equations for the variables x^a and z in the following equivalent form:

$$\dot{x}^a = e_1 p^a + \Delta^2 e_2 \frac{\xi^a}{(p, \xi)}; \tag{18.a}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tilde{\tau}^2} + \Gamma^z_{zz}\left(\frac{\mathrm{d}z}{\mathrm{d}\tilde{\tau}}\right)^2 = 0, \qquad \tilde{\tau} = \int_{\tau_0}^{\tau} \mathrm{d}s\; e_2(s). \tag{18.b}$$

Here $\Gamma^z_{zz} = -2\partial_z\ \ln(p, \xi)$ is the Christoffel symbol for the metric (5). Now, it is easy to show that the system of inequalities (15) is satisfied for a solution $\{x^a(\tau), z(\tau), \bar{z}(\tau)\}$ of Eqs. (18) if and only if this solution is characterized by Eqs. (16).

We briefly run through discrete symmetries of the (m, s)-particle model. One can readily find that the Lagrangian (8) is invariant under the transformation of space inversion

$$x'^0 = x^0, \qquad \vec{x}' = -\vec{x}$$

acting on S^2 as the complex antiautomorphism

$$z' = -1/\bar{z}. \tag{19}$$

Associated with the space-time inversion is the identity map on S^2.

An admissible way to fix the gauge freedom (9), (10) in the model of (m, s)-particle consists in imposing the following gauge conditions:

$$\dot{x}^0 = c, \qquad e_2 = \frac{mc^2}{\Delta^2}\kappa = \text{const.} \tag{20}$$

Here κ is a dimensionless parameter. In accordance with Eqs. (18), in the gauge chosen the dynamics on S^2 is specified by the geodesic equation

$$\ddot{z} + \Gamma^z_{zz}(\dot{z})^2 = 0, \tag{21}$$

while the multiplier e_1 and the space components of the trajectory in Minkowski space are determined by the trajectory on S^2:

$$\begin{gathered} e_1 = \frac{c}{p^0}\left(1 + mc\kappa\frac{1+z\bar{z}}{(p,\xi)}\right), \\ \dot{\vec{x}} = \frac{c}{p^0}\left(1 + mc\kappa\frac{1+z\bar{z}}{(p,\xi)}\right)\vec{p} + mc^2\kappa\frac{\vec{\xi}}{(p,\xi)}. \end{gathered} \tag{22}$$

For the gauge considered, the equations of motion are integrated most simply in the case $p^a = (mc, 0, 0, 0)$. Then a particular solution is given by the expressions

$$\begin{gathered} z(\tau) = \mathrm{e}^{\imath\omega\tau}, \\ \vec{x}(\tau) = \frac{\hbar}{mc}\sqrt{s(s+1)}\begin{pmatrix} \sin\ \omega\tau \\ \cos\ \omega\tau \\ 0 \end{pmatrix} + \vec{x}(0), \end{gathered} \tag{23}$$

where the frequency of rotation ω is connected with κ by the rule

$$\omega = \frac{mc^2}{\hbar\sqrt{s(s+1)}}\kappa, \tag{24}$$

and the restrictions (16) mean

$$0 < \omega < mc^2/\hbar\sqrt{s(s+1)}. \tag{25}$$

General solution can be restored from (23) by means of applying $SO(3)$-rotations and inversions. As is seen from (25), the upper bound for the frequency of rotation is determined by spin and decreases with increasing of s.

Basing on the explicit form of the solution (23), it is easily to imagine the situation having place in the case of arbitrary 4-momentum p^a. Namely, in the space-time (m, s)-particle moves in and oscillates around a straight line to which p^a is tangent. But the amplitude of such oscillations turns out to be of the order $\hbar$ and, hence, disappears in the limit $\hbar \to 0$. Therefore, we come to the conclusion that the classical dynamics in the model of (m, s)-particle is analogous to the phenomenon of Schrödinger vibration (Zitterbevegung) in relativistic quantum theory.

To conclude this Section, we present the reformulation of the model most closely related to Penrose's treatment of S^2 as the celestial sphere[11]. It turns out that the (m, s)-particle can be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2e}\left(\dot{x}^2 - (mce)^2\right) + \frac{1}{2}\left(\frac{\dot{u}^2}{(\dot{x}, u)^3}e^2 + \Delta^2(\dot{x}, u)\right), \tag{26}$$

where $e(\tau)$ is an einbein, and $u^a(\tau)$ is constrained to be a light-like 4–vector $(u^0(\tau) < 0)$ at each point of the world-line. The action functional, constructed on the base of (26), is obviously invariant under the world-line reparametrizations

$$\delta_\epsilon x^a = \dot{x}^a\epsilon, \qquad \delta_\epsilon u^a = \dot{u}^a\epsilon, \qquad \delta_\epsilon e = \frac{\mathrm{d}}{\mathrm{d}\tau}(e\epsilon), \tag{27}$$

and possesses another local invariance

$$\delta_\mu x^a = p^a\mu, \qquad \delta_\mu u^a = -\,\frac{u^a}{e}\dot{\mu}, \qquad \delta_\mu e = \dot{\mu}, \tag{28}$$

p_a being the canonically conjugate momentum to x^a with respect to (26). The Lagrangian (26) takes the form (8) after the identification

$$e \equiv e_1, \qquad u^a \equiv \frac{\xi^a}{(\dot{x}, \xi)}e_2. \tag{29}$$

3 Hamilton formulation

We are going now to construct constrained Hamilton formulation[18] for the model.

Starting with the Lagrangian (13), let us introduce conjugate momenta for the variables x^a and z (in what follows, we set $\hbar = c = 1$):

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{\dot{x}_a(m^2 - 4\Delta\Psi) + 2\Delta\Psi\dot{x}^2\xi_a/(\dot{x}, \xi)}{\sqrt{-\dot{x}^2(m^2 - 4\Delta\Psi)}}, \tag{30}$$

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{\Delta}{(\dot{x}, \xi)} \sqrt{- \frac{\dot{\bar{z}} \dot{x}^2}{\dot{z}(m^2 - 4\Delta\Psi)}}, \qquad \Psi \equiv \frac{|\dot{z}|}{(\dot{x}, \xi)}.$$

Then one readily finds that the system possesses two primary constraints:

$$T_1 \equiv p^a p_a + m^2 = 0; \tag{31.a}$$

$$T_2 \equiv (p^a \xi_a)^2 p_z p_{\bar{z}} - \Delta^2 = 0, \qquad \Delta^2 = m^2 s(s+1). \tag{31.b}$$

These constraints are Abelian ones, $\{T_1, T_2\} = 0$, with respect to the canonical Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_a} + \frac{\partial A}{\partial z} \frac{\partial B}{\partial p_z} + \frac{\partial A}{\partial \bar{z}} \frac{\partial B}{\partial p_{\bar{z}}} - (A \leftrightarrow B), \tag{32}$$

A, B being scalar functions over the phase space. Because of reparametrization invariance, the Hamiltonian vanishes,

$$H_0 = p_a \dot{x}^a + p_z \dot{z} + p_{\dot{z}} \dot{\bar{z}} - \mathcal{L} \equiv 0,$$

hence there are no secondary constraints. As a result, the system possesses two first class constraints, and the total Hamiltonian is a linear combination of constraints,

$$H = \frac{1}{2}\lambda_1 \{p^2 + m^2\} + \frac{1}{2}\lambda_2 \{(p, \xi)^2 p_z p_{\bar{z}} - \Delta^2\}, \tag{33}$$

with arbitrary Lagrange multipliers $\lambda_1(\tau)$ and $\lambda_2(\tau)$. The canonical action looks like

$$S = \int \mathrm{d}\tau \; (p_a \dot{x}^a + p_z \dot{z} + p_{\bar{z}} \dot{\bar{z}} - H). \tag{34}$$

The constraints $T_i, i = 1, 2$, appearing in the model of (m, s)-particle, have a simple group — theoretic interpretation. The action of the Poincaré group defined on $\mathcal{M}^6$ can be lifted up on the cotangent bundle $T^*(\mathcal{M}^6)$ determining phase space of the system. Obviously, all the Poincaré transformations on the phase space are canonical. The action of $\mathcal{P}$ on $T^*(\mathcal{M}^6)$ induces special representation of this group in the space of scalar fields on $T^*(\mathcal{M}^6)$, and the corresponding infinitesimal Poincaré transformations can be written in terms of the Poisson bracket (32) as follows

$$\delta A = \{A, -f^a P_a + \frac{1}{2} K^{ab} J_{ab}\}. \tag{35}$$

Here f^a and $K^{ab} = -K^{ba}$ are the parameters of translations and Lorentz transformations, respectively, and the Hamilton generators look like

$$P_a = p_a, \qquad J_{ab} = x_a p_b - x_b p_a + M_{ab}, \tag{36}$$

where

$$M_{ab} = -(\sigma_{ab})_{\alpha\beta} z^\alpha z^\beta p_z + (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{z}^{\dot{\alpha}} \bar{z}^{\dot{\beta}} p_{\bar{z}}.$$

As is seen, the spinning part of the Lorentz generators is realized by the spherical variables only. Let us construct, using the Hamilton potentials, the classical Pauli–Lubanski vector $W^a = \frac{1}{2}\varepsilon^{abcd} P_b J_{cd}$ and the function of squared spin $W^a W_a$. Now, it is a simple exercise to check the relation

$$W^a W_a = (p^a \xi_a)^2 p_z p_{\bar{z}}. \tag{37}$$

Therefore, Eqs. (31) mean that the functions of squared momentum and spin (which can be treated as phase-space counterparts of the Casimir operators of the Poincaré group) conserve strongly on the constrained surface. Each weak physical observable $\mathcal{A}(x^a, p_b, z, p_z, \bar{z}, p_{\bar{z}})$ should meet the requirements

$$\{\mathcal{A}, T_i\}|_{T_j=0} = 0, \qquad i, j = 1, 2,$$

what provides $\mathcal{A}$ to be gauge invariant on the constrained surface. It can be shown that the most general form for physical observables reads as

$$\mathcal{A} = f(P_a, J_{bc}) + \varphi_i T_i, \tag{38}$$

φ's being arbitrary functions on the phase space. If $\mathcal{A}$ is a strong physical observable, i.e. $\{\mathcal{A}, T_i\} = 0$, then it proves to depend only on the Hamilton generators (36). So every physical observable is a function of the Hamilton generators of the Poincaré group modulo constraints.

The Hamilton formulation is convenient for solving the equations of motion in the model of (m, s)-particle. The complete integrability of the dynamical equations with arbitrary Lagrange multipliers, in spite of their nonlinearity, seems to be natural because of the fact that the theory describes a free particle. To analyze the equations of motion, it is helpful to introduce another parametrization in the phase space.

Let us consider the domain $\tilde{T}^*(S^2)$ in the cotangent bundle of sphere $T^*(S^2)$, which is selected out by the condition $p_z \neq 0$. In accordance with the restriction (31.b), the spherical part of the constrained surface in the phase space is embedded completely in $\tilde{T}^*(S^2)$. On $\tilde{T}^*(S^2)$ one can replace the phase variables z and p_z by a covariant parametrization in terms of the coordinates

$$q^\alpha = z^\alpha \sqrt{2p_z} \tag{39}$$

transforming by the spinor law (A.3) under the Lorentz group. As is easily seen, the correspondence $(z, p_z) \to q^\alpha$ defined by Eq. (39) maps the space $\tilde{T}^*(S^2)$ onto $\mathbb{C}^2_*$ (see

Appendix A); then $\tilde{T}^*(S^2)$ turns out to be the factor-space of $\mathbb{C}^2_*$ with respect to the equivalence relation $q^\alpha \sim -q^\alpha$. The local functions q^α (39) on the phase space and their conjugates $\bar{q}^{\dot\alpha}$ possess the following Poisson brackets

$$\{q^\alpha, q^\beta\} = -\varepsilon^{\alpha\beta}, \qquad \{\bar{q}^{\dot\alpha}, \bar{q}^{\dot\beta}\} = -\varepsilon^{\dot\alpha\dot\beta}, \qquad \{q^\alpha, \bar{q}^{\dot\beta}\} = 0, \tag{40}$$

$\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$ being the spinor metrics ($\varepsilon^{01} = \varepsilon^{\dot0\dot1} = 1$). These relations imply that the transformation (39) is canonical.

In terms of the variables q^α and $\bar{q}^{\dot\alpha}$, the constraint (31.b) looks like

$$T_2 = \frac{1}{4}(p^a\mathcal{F}_a)^2 - \Delta^2, \qquad \mathcal{F}_a = (\sigma_a)_{\alpha\dot\alpha} q^\alpha \bar{q}^{\dot\alpha}, \tag{41}$$

and the Hamilton actions (34) takes the explicitly Lorentz-covariant form

$$S = \int \mathrm{d}\tau (p_a \dot{x}^a + \frac{1}{2}\{q_\alpha \dot{q}^\alpha + \bar{q}_{\dot\alpha}\dot{\bar{q}}^{\dot\alpha} - \lambda_i T_i\}). \tag{42}$$

Let us introduce the 4-component Majorana spinor

$$Q = \begin{pmatrix} q_\alpha \\ \bar{q}^{\dot\alpha} \end{pmatrix}, \qquad \overline{Q} = (q^\alpha, \bar{q}_{\dot\alpha}) \tag{43}$$

and make use of the identities

$$\mathcal{F}^a = \frac{1}{2}\overline{Q}\gamma^a Q, \qquad q^\alpha \dot{q}_\alpha + \bar{q}^{\dot\alpha}\dot{\bar{q}}_{\dot\alpha} = \overline{Q}\gamma_5 \dot{Q}.$$

Then, the Hamilton equations can be written as follows:

$$\begin{gathered} \dot{p}^a = 0, \qquad \dot{x}^a - \lambda_1 p^a - \frac{1}{4}\lambda_2(p,\mathcal{F})\mathcal{F}^a = 0, \\ \dot{Q} + \frac{1}{4}\lambda_2(p,\mathcal{F})\gamma_5(p,\gamma)Q = 0. \end{gathered} \tag{44}$$

Taking into account the constraints, these equations are easily integrated for arbitrary $\lambda_1(\tau)$ and $\lambda_2(\tau)$. The general solution reads

$$\begin{gathered} Q(\tau,\tau_0) = \exp\Big[-\alpha(\tau,\tau_0)\gamma_5(p,\gamma)\Big] Q(\tau_0), \\ \alpha(\tau,\tau_0) = \frac{\Delta}{2}\int_{\tau_0}^{\tau} \mathrm{d}s\, \lambda_2(s); \end{gathered} \tag{45.a}$$

$$x^a(\tau,\tau_0) = x^a(\tau_0) + p^a \int_{\tau_0}^{\tau} \mathrm{d}s \left\{ \lambda_1(s) + \left(\frac{\Delta}{m}\right)^2 \lambda_2(s)\Big[\cos\Big(2m\alpha(s,\tau_0)\Big) - 1\Big]\right\}$$

$$+\frac{\Delta}{2}\mathcal{F}^a(\tau_0)\int_{\tau_0}^{\tau}\mathrm{d}s\ \lambda_2(s)\ \cos\Big(2m\alpha(s,\tau_0)\Big)$$

$$+\frac{\Delta}{8m}\overline{Q}(\tau_0)\gamma_5\big[\gamma^a,\gamma^b\big]Q(\tau_0)\,p_b\int_{\tau_0}^{\tau}\mathrm{d}s\,\lambda_2(s)\ \sin\Big(2m\alpha(s,\tau_0)\Big). \qquad (45.b)$$

To restore the general solution in the gauge (20), one is simply to use the correspondence $e_1 \leftrightarrow \lambda_1$, $e_2 \leftrightarrow \lambda_2$ together with the relation $z(\tau,\tau_0) = q^1(\tau,\tau_0)/q^0(\tau,\tau_0)$.

4 Quantization of the (m, s)-particle model

In the present section, a realization of operator formulation is suggested for the quantum theory of (m, s)-particle. The phase variables in this formulation are considered to be Hermitian operators defined in a Hilbert space, with Poincaré-invariant inner product, and subjected to the canonical commutation relations. Operator ordering in functions on the phase space is chosen in such a way that associated with the Hamilton generators (36) of the Poincaré group will be Hermitian operators $\mathbb{P}_a$ and $\mathbb{J}_{ab}$ satisfying the commutation relations

$$[\mathbb{P}_a,\mathbb{P}_b] = 0, \qquad [\mathbb{J}_{ab},\mathbb{P}_c] = \mathrm{i}\eta_{ac}\mathbb{P}_b - \mathrm{i}\eta_{bc}\mathbb{P}_a,$$
$$[\mathbb{J}_{ab},\mathbb{J}_{cd}] = \mathrm{i}\eta_{ac}\mathbb{J}_{bd} - \mathrm{i}\eta_{bc}\mathbb{J}_{ad} + \mathrm{i}\eta_{ad}\mathbb{J}_{cb} - \mathrm{i}\eta_{bd}\mathbb{J}_{ca}. \qquad (46)$$

Then, since the constraints (31) were expressed in terms of the Hamilton generators, owing to Eq. (37), the quantum analogs of T_i should be Hermitian operators of the form

$$\widehat{T}_1 = \mathbb{P}^a\mathbb{P}_a + m^2\mathbb{1}; \qquad (47.a)$$

$$\widehat{T}_2 = \mathbb{W}^a\mathbb{W}_a - m^2 s(s+1)\mathbb{1}, \qquad \mathbb{W}^a = \frac{1}{2}\varepsilon^{abcd}\mathbb{P}_b\mathbb{J}_{cd}. \qquad (47.b)$$

The physical states are to be subject to the conditions

$$\widehat{T}_i|\Psi_{\mathrm{phys}}>= 0, \qquad i = 1, 2, \qquad (47.c)$$

which have the meaning that the Casimir operators of the Poincaré group are multiples of unity on the space of physical states.

As we have seen, the general structure of physical observables in the model of (m, s)-particle is described by Eq. (38). It is now obvious that the quantization procedure formulated allows to assign to every classical physical observable a well defined Hermitian operator in the Hilbert space.

In the classical regime, the parameter s could take arbitrary non-negative values, and for any such value the dynamics was non-contradictory. It turns out, however, that nontrivial solutions to the equations for physical states exist only for (half)integer s. Really, the set of Eqs. (46), (47) coincides with that specifying an irreducible massive spin-s unitary representation of the Poincaré group. As is well known, such representations exist only for (half)integer spins.

The problem of constructing the operator formulation described reduces to finding an operator realization for the canonical commutation relations consistent with the conditions (46), (47). So, the problem is in finding realizations for the massive irreducible Poincaré representations in spaces of tensor fields on $\mathcal{M}^6$.

The general structure of tensor fields on $\mathcal{M}^6$ is discussed in Appendix B. Here we will be interested only in those tensor fields on $\mathcal{M}^6$ which are scalar in Minkowski space and have a spherical type $\{r/2, s/2\}$, r and s being integers. The Poincaré-generators for the fields of type $\{r/2, s/2\}$ are given by Eqs. (B.8) and (B.9).

Let us consider the space ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$ of massive positive-frequency fields on $\mathcal{M}^6$ of type $[n/2] \equiv \{n/2, 0\}$, where $n = 0, 1, 2, \ldots$. Such fields are subject to the mass-shell condition (47.a),

$$(\Box - m^2)\Phi^{[n/2]}(x, z, \bar{z}) = 0, \tag{48}$$

and possess the Fourier decomposition

$$\begin{gathered}\Phi^{[n/2]}(x, z, \bar{z}) = \int \frac{\mathrm{d}^3\vec{p}}{p^0} \mathrm{e}^{\mathrm{i}(p,x)} \Phi^{[n/2]}(p, z, \bar{z}), \\ p^2 + m^2 = 0, \qquad p^0 > 0.\end{gathered} \tag{49}$$

We define inner product for the elements from ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$ by the rule

$$\ll \Phi_1 | \Phi_2 \gg_{[n/2]} = N \int \frac{\mathrm{d}^3\vec{p}}{p^0} \, \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(p,\xi)^2} \, \frac{1}{(p,\xi)^n} \overline{\Phi_1^{[n/2]}(p, z, \bar{z})} \Phi_2^{[n/2]}(p, z, \bar{z}), \tag{50}$$

for N being some normalization constant, ξ_a given as in Eq. (6). Then ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$ becomes a Hilbert space. It should be stressed that the inner product (50) is well defined on S^2 and built upon the metric (5). We note also that the integration measure

$$\frac{\mathrm{d}^3\vec{p}}{p^0} \, \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(p,\xi)^2}$$

proves to be Poincaré invariant, and the expression

$$(p,\xi)^{-n} \overline{\Phi_1^{[n/2]}(p, z, \bar{z})} \Phi_2^{[n/2]}(p, z, \bar{z}),$$

is a scalar with respect to the Poincaré group. As a result, the Poincaré representation acting on ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$ is unitary. This representation can be readily decomposed into a direct sum of irreducible ones by accounting the fact that the spin operator $C^{[n/2]} = \mathbb{W}^a \mathbb{W}_a$ on ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$ coincides with the Laplacian $\Delta^{[n/2]}$ (B.11). The spectrum of $\Delta^{[n/2]}$ is described in Appendix C and given by Eq. (C.1), where $R = -p^2 = m^2$ is the curvature of the metric (5) building up from the 4–momentum. Thus we get the decomposition

$$ {}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m) \bigoplus_{s=\frac{n}{2},\frac{n}{2}+1,\ldots} {}^{\uparrow}\mathcal{H}_s^{[n/2]}(\mathcal{M}^6; m). \tag{51} $$

Here the invariant subspace ${}^{\uparrow}\mathcal{H}_s^{[n/2]}(\mathcal{M}^6; m)$ realizes the Poincaré representation of mass m and spin s, that is, it is characterized by the conditions (47.a) and (47.b). As is shown in Appendix C, the expansion of an arbitrary field from ${}^{\uparrow}\mathcal{H}_s^{[n/2]}(\mathcal{M}^6; m)$, which corresponds to the decomposition (51), reads as follows

$$ \Phi^{[n/2]}(p, z, \bar{z}) = \sum_{s=\frac{n}{2},\frac{n}{2}+1,\ldots} F_{\alpha_1 \ldots \alpha_{s+n/2} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-n/2}}(p) \frac{z^{\alpha_1} \ldots z^{\alpha_{s+n/2}} \bar{z}^{\dot{\alpha}_1} \ldots \bar{z}^{\dot{\alpha}_{s-n/2}}}{(p, \xi)^{s-n/2}}. \tag{52} $$

In this expansion, each coefficient $F_{\alpha(s+n/2)\dot{\alpha}(s-n/2)}(p)$ is symmetric in its undotted and, independently, in its dotted indices

$$ F_{\alpha_1 \ldots \alpha_{s+n/2} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-n/2}}(p) = F_{(\alpha_1 \ldots \alpha_{s+n/2})(\dot{\alpha}_1 \ldots \dot{\alpha}_{s-n/2})}(p), \tag{53} $$

and p-transversal,

$$ p^{\beta\dot{\beta}} F_{\beta\alpha(s+n/2-1)\dot{\beta}\dot{\alpha}(s-n/2-1)}(p) = 0. \tag{54} $$

The F's can be identified with the Fourier transforms of tensor fields on Minkowski space.

We describe in more detail the case of massive scalar fields on $\mathcal{M}^6$. As is seen from (51), the corresponding representation of the Poincaré group on ${}^{\uparrow}\mathcal{H}^{[0]}(\mathcal{M}^6; m)$ is decomposed into the direct sum of all representations with integer spins. Therefore, a massive scalar field on $\mathcal{M}^6$ generates massive fields of arbitrary integer spins in Minkowski space. For $n = 0$ the decomposition (52) can be rewritten on the manner

$$ \Phi^{[0]}(p, z, \bar{z}) = \sum_{s=0}^{\infty} F_{a_1 \ldots a_s}(p) \frac{\xi^{a_1} \ldots \xi^{a_s}}{(p, \xi)^s}, \tag{55} $$

where

$$ F_{a_1 \ldots a_s}(p) = \left(-\frac{1}{2}\right)^s (\tilde{\sigma}_{a_1})^{\dot{\alpha}_1 \alpha_1} \ldots (\tilde{\sigma}_{a_s})^{\dot{\alpha}_s \alpha_s} F_{\alpha_1 \ldots \alpha_s \dot{\alpha}_1 \ldots \dot{\alpha}_s}(p). $$

For just introduced fields with vector indices, the requirement (53) means that $F_{a(s)}$ is symmetric and traceless

$$ F_{a_1 \ldots a_s}(p) = F_{(a_1 \ldots a_s)}(p), $$

$$F^b{}_{ba_1\ldots a_{s-2}}(p) = 0, \tag{56}$$

while the condition (54) takes the form

$$p^b F_{ba_1\ldots a_{s-1}}(p) = 0. \tag{57}$$

Together with the mass-shell condition $p^2+m^2=0$, the equations (56) and (57) constitute the set of relativistic wave equations for a massive field of integer spin s [19].

It seems instructive to reexpress the scalar product (51) for $n=0$ in terms of the fields appearing in the expansion (55). Direct calculations basing on the use of the identity

$$\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(p,\xi)^2} = -\,\frac{\pi}{p^2}$$

lead to the following result

$$\ll \Phi_1|\Phi_2 \gg_{[0]} = \pi N \int \frac{\mathrm{d}^3\vec{p}}{p^0} \sum_{s=0}^{\infty} \left(\frac{2}{m^2}\right)^s \frac{(s!)^2}{(2s+1)!} \overline{\Phi_1^{a_1\ldots a_s}(p)} \Phi_{2\ a_1\ldots a_s}(p). \tag{58}$$

Now, let us consider massive spinor fields on $\mathcal{M}^6$, i.e. $n=1$. The relations (51) show that the Poincaré representation defined on ${}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6;m)$ is decomposed into the direct sum of all representations with half-integer spins. Thus, a massive spinor field on $\mathcal{M}^6$ generates massive fields with arbitrary half-integer spins. The decomposition (52) can be rewritten for $n=1$ by the rule

$$\Phi^{[1/2]}(p,z,\bar{z}) = \sum_{k=0}^{\infty} F_{a_1\ldots a_k\alpha}(p) \frac{z^\alpha \xi^{a_1}\ldots\xi^{a_k}}{(p,\xi)^k}, \tag{59}$$

where

$$F_{a_1\ldots a_k\alpha}(p) = \left(-\,\frac{1}{2}\right)^k (\tilde{\sigma}_{a_1})^{\dot{\beta}_1\beta_1} \ldots (\tilde{\sigma}_{a_k})^{\dot{\beta}_k\beta_k} F_{\alpha\beta_1\ldots\beta_k\dot{\beta}_1\ldots\dot{\beta}_k}.$$

In terms of the spin-tensors introduced, the requirements (53) and (54) are equivalent to the equations

$$\begin{gathered} F_{a_1\ldots a_k\alpha}(p) = F_{(a_1\ldots a_k)\alpha}(p), \qquad F^b{}_{ba_1\ldots a_{k-2}\alpha}(p) = 0, \\ (\tilde{\sigma}^b)^{\dot{\beta}\beta} F_{ba_1\ldots a_{k-1}\beta}(p) = p^b F_{ba_1\ldots a_{k-1}\alpha}(p) = 0, \end{gathered} \tag{60}$$

which form, together with the condition $p^2+m^2=0$, the set of relativistic wave equations determining an on-shell massive field of half-integer spin $(k+1/2)$ [20]. It is worth remarking that a massive field $\Phi^{[1/2]}$ on $\mathcal{M}^6$ turns out to describe spin 1/2 if and only if $\Phi^{[1/2]}$ is holomorphic, $\partial_{\bar{z}}\Phi^{[1/2]} = 0$.

Real massive fields on $\mathcal{M}^6$ can be realized by considering spaces

$$\mathcal{H}^{[n/2]}(\mathcal{M}^6; m) = {}^{\uparrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m) \oplus {}^{\downarrow}\mathcal{H}^{[n/2]}(\mathcal{M}^6; m)$$

spanned on fields with mixed frequency and, then, selecting out in these spaces real Poincaré-invariant subspaces. For instance, a real field $\Phi^{[1/2]}$ of spin 1/2 satisfies the equations

$$(p, \xi)\nabla_z \Phi^{[1/2]} = m\overline{\Phi^{[1/2]}}, \qquad \partial_{\bar{z}}\Phi^{[1/2]} = 0$$

in momentum space, or

$$\xi^a \overset{\leftrightarrow}{\partial_z} \partial_a \Phi^{[1/2]} = \mathrm{i}m\overline{\Phi^{[1/2]}}, \qquad \partial_{\bar{z}}\Phi^{[1/2]} = 0$$

in coordinate space. The above equations prove to be equivalent to ordinary Dirac equation on Majorana spinor field.

5 Conclusion

In the present paper we have constructed the model of relativistic massive particle of arbitrary spin with the configuration space $\mathcal{M}^6 = \mathbb{R}^{3,1} \times S^2$. The theory possesses two gauge symmetries of reparametrization type. This gauge structure induces strong conservation of the phase-space functions of squared momentum and squared Pauli–Lubanski vector, while in the quantum theory it implies that the Casimir operators of the Poincaré group are multiples of unity in the space of physical states. That is why we can identify one of the two parameters arising in the Lagrangian with the spin of particle.

The theory suggested has quite clear and simple geometric origin, and can be also considered as a minimal model of relativistic massive spinning particle. The point is that dimension of the corresponding configuration space turns out to be minimally possible to describe the joint space-time evolution and dynamics of spin.

Our model admits a number of nontrivial generalizations. In particular, we have already developed (super)particle models extending the model of (m, s)-particle to the cases when the space-time manifold $\mathbb{R}^{3,1}$ is replaced by N-extended flat global superspace, (anti) de Sitter space, anti–de Sitter superspace[20,21].

The model of this paper describes the dynamics of free spinning particle with non-vanishing mass in terms of $\mathcal{M}^6$-geometry. It is believed, however, that in the massless case total configuration space should also involve sphere as a subspace. The appearance of sphere S^{D-2} for describing massless dynamics in special dimensions $D = 3$, 4, 6, and 10 has recently been demonstrated for the case of a massless superparticle by Galperin, Howe, and Stelle[22] suggested an elegant group-theoretic interpretation for the twistor formulation of superparticle[23]. In the approach of Refs. 23, 24, there appeared a light-like D-vector as a variable representing spherical variables. This is analogous to the formulation (26) for our model. Interestingly, the Lagrangian (26) leads to a consistent spinning particle model in any dimension in the sense that the local transformations (27) and (28) leave invariant the corresponding D-dimensional action functional. But only for $D = 3$, 4 the theory describes a massive particle with arbitrary irreducible spin.

In conclusion, we would like to note an interesting relationship between a special algebra of functions on the phase space of (m, s)-particle and higher spin superalgebras[24] underlying the theories of interacting higher spin massless fields[25]. As has been shown in section 3, the spherical part of the constraint surface in the phase space can be parametrized by the spinor variables (39). The set of all regular functions of q and $\bar{q}$,

$$f(q,\bar{q}) = \sum_{s=0}^{\infty} \sum_{\substack{m,n\geq 0 \\ m+n=s}} f_{\alpha_1\ldots\alpha_m\dot{\alpha}_1\ldots\dot{\alpha}_n} q^{\alpha_1}\ldots q^{\alpha_m}\bar{q}^{\dot{\alpha}_1}\ldots\bar{q}^{\dot{\alpha}_n},$$

forms an infinite dimensional Lie algebra, with respect to the Poisson bracket (39). The quantization of this algebra, which consists in associating with the phase variables q^α and $\bar{q}^{\dot{\alpha}}$ operators $\hat{q}^\alpha$ and $\hat{\bar{q}}^{\dot{\alpha}}$ under the commutation relations

$$[\hat{q}^\alpha, \hat{q}^\beta] = \mathrm{i}\varepsilon^{\alpha\beta}, \qquad [\hat{\bar{q}}^{\dot{\alpha}}, \hat{\bar{q}}^{\dot{\beta}}] = \mathrm{i}\varepsilon^{\dot{\alpha}\dot{\beta}}, \qquad [\hat{q}^\alpha, \hat{\bar{q}}^{\dot{\alpha}}] = 0$$

leads to an associative operator algebra being naturally $\mathbb{Z}_2$-graded and, hence, possessing the structure of superalgebra. The superalgebra obtained proves to coincide with some special higher spin superalgebra[25].

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Appendix A. The Lorentz group action on S^2

A two sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is naturally endowed by the structure of a transformation space for the Lorentz group $SL(2,\mathbb{C})/\mathbb{Z}_2$ if one realizes S^2 as the complex projective space $\mathbb{C}P^1$, i.e. the factor-space of the complex space $\mathbb{C}^2_* = \mathbb{C}^2 \backslash \{(0,0)\}$, spanned by non-zero complex two-vectors $q^\alpha = (q^0, q^1)$, with respect to the equivalence relation $q^\alpha \sim \lambda q^\alpha, \forall\, \lambda \in \mathbb{C}_* = \mathbb{C}\backslash\{0\}$. Let $\pi : \mathbb{C}^2_* \to S^2$ be the canonical projection. The open cover of $\mathbb{C}^2_*$ by two charts

$$\begin{aligned} U_0 &= \{q^\alpha \in \mathbb{C}^2_*, q^0 \neq 0\}, \qquad z \equiv q^1/q^0; \\ U_1 &= \{q^\alpha \in \mathbb{C}^2_*, q^1 \neq 0\}, \qquad w \equiv -q^0/q^1 \end{aligned} \tag{A.1}$$

induces the atlas $\{\pi(U_0), \pi(U_1)\}$ on S^2; the only point of S^2 not covered by the chart $\pi(U_0)$ can be identified with infinitely removed point. On the role of local complex coordinates in the charts $\pi(U_0)$ and $\pi(U_1)$ it is useful to choose the variables z and w, respectively, connected to each other in the overlap $\pi(U_0) \cap \pi(U_1) = \mathbb{C}_*$ by the transition function $w = -1/z$.

Let us consider the spinor representation (1/2, 0) of the Lorentz group, i.e. the representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ assigning to each matrix

$$N = (N_\alpha{}^\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \qquad \alpha, \beta = 0, 1 \tag{A.2}$$

the following transformation on $\mathbb{C}^2$

$$q^\alpha \to q'^\alpha = q^\beta (N^{-1})_\beta{}^\alpha. \tag{A.3}$$

Since mutually equivalent points from $\mathbb{C}^2_*$ are moved into equivalent ones, the restriction of this representation to $\mathbb{C}^2_*$ indices some action of the group on S^2 given in the local coordinates $z, \bar{z}$ by fractional linear transformations of the form

$$z \to z' = \frac{az - b}{-cz + d}. \tag{A.4}$$

For any $N \in SL(2,\mathbb{C})$, the group elements $\pm N$ define the same transformation on S^2, therefore we result in the action of the Lorentz group on S^2.

In accordance with Eq. (A.4), the Lorentz group acts on S^2 by holomorphic fractional linear transformations. Reversally, it is easy to observe that every holomorphic one–to–one mapping of S^2 onto itself is a transformation of the form (A.4). Therefore, the Lorentz group coincides with the group of complex automorphisms of two-dimensional sphere.

Appendix B. Tensor fields on $\mathcal{M}^6$

Representations of the Poincaré group in terms of fields living on $\mathcal{M}^6$ can be obtained via tensor products of analogous representations defined on Minkowski space and on two-sphere. Let us recall that a tensor field on $\mathbb{R}^{3,1}$ of Lorentz type $(k/2, l/2),\ k, l = 0, 1, 2, \ldots,$ is given by a set of smooth components

$$F_{\alpha(k)\dot{\alpha}(l)}(x) = F_{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x), \tag{B.1}$$

symmetric independently in their undotted and dotted indices,

$$F_{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x) = F_{(\alpha_1\ldots\alpha_k)(\dot{\alpha}_1\ldots\dot{\alpha}_l)}(x) \tag{B.2}$$

and changing by the law

$$F'{}_{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x') = N_{\alpha_1}{}^{\beta_1}\ldots N_{\alpha_k}{}^{\beta_k}\ \bar{N}_{\dot{\alpha}_1}{}^{\dot{\beta}_1}\ldots\bar{N}_{\dot{\alpha}_l}{}^{\dot{\beta}_l}\ F_{\beta_1\ldots\beta_k\dot{\beta}_1\ldots\dot{\beta}_l}(x) \tag{B.3}$$

under the Poincaré transformations (1). The round brackets in (B.2) mean the symmetrization of indices. A tensor field on S^2 of type $\{r/2, s/2\}$, where $r, s = 0, \pm1, \pm2, \ldots,$ is given by a smooth function $\Phi(z, \bar{z})$ in chart $\mathbb{C}$ and by a smooth function $\Psi(w, \bar{w})$ in chart $\mathbb{C}_* \cap \{\infty\}$ such that in the overlap of charts these functions are connected as follows:

$$\Psi(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{r/2}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{s/2}\Phi(z, \bar{z}). \tag{B.4}$$

The Lorentz transformations (A.4) act on the field of type $\{r/2, s/2\}$ by the law

$$\Phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{r/2}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{s/2}\Phi(z, \bar{z}). \tag{B.5}$$

In summary, each tensor field on $\mathcal{M}^6$ is characterized by its Lorentz type $\{k/2, l/2\}$ and its spherical type $\{r/2, s/2\}$.

In the present paper, we mainly consider tensor fields on $\mathcal{M}^6$ without external Lorentz indices ($k = l = 0$). For most of applications this proves to be sufficient, because ordinary tensor fields in Minkowski space arise as coefficients in expansions of tensor fields on $\mathcal{M}^6$ over special harmonics. Let us consider, for instance, a tensor field on S^2 of type $\{n/2, 0\} \equiv [n/2]$, $\Phi^{[n/2]}$, satisfying the requirement of holomorphicity

$$\partial_{\bar{z}}\Phi^{[n/2]} = 0. \tag{B.6}$$

According to the Riemann—Roch theorem[18], Eq. (B.6) possesses nontrivial solutions on S^2 only for $n \geq 0$ (in the class of smooth tensor fields on S^2), and dimension of the corresponding space of holomorphic fields is equal to $(n+1)$. The general solution of Eq. (B.6) reads

$$\Phi^{[n/2]}(z) = F_{\alpha_1\ldots\alpha_n} z^{\alpha_1}\ldots z^{\alpha_n},$$
$$\Rightarrow \Psi^{[n/2]}(w) = F_{\alpha_1\ldots\alpha_n} w^{\alpha_1}\ldots w^{\alpha_n}, \qquad w^{\alpha} = (w,-1). \tag{B.7}$$

Here $F_{\alpha_1\ldots\alpha_n}$ is a Lorentz tensor of type $(n/2,0)$.

Representation of the Poincaré group in the space of tensor fields on $\mathcal{M}^6$ of type $\{r/2,s/2\}$ is characterized by the following generators:

$$\mathbb{P}_a = -\mathrm{i}\partial_a, \qquad \mathbb{J}_{ab} = -\mathrm{i}(x_a\partial_b - x_b\partial_a + M_{ab}), \tag{B.8}$$

where the spinning part of $\mathbb{J}_{ab}$ is realized by spherical variables on the manner

$$M_{ab} = (\sigma_{ab})_{\alpha\beta}M^{\alpha\beta} - (\tilde{\sigma}_{ab})_{\dot\alpha\dot\beta}\bar{M}^{\dot\alpha\dot\beta},$$
$$M^{\alpha\beta} = -z^{\alpha}z^{\beta}\partial_z + \frac{r}{2}\partial_z(z^{\alpha}z^{\beta}), \qquad \bar{M}^{\dot\alpha\dot\beta} = -\bar{z}^{\dot\alpha}\bar{z}^{\dot\beta}\partial_{\bar z} + \frac{s}{2}\partial_{\bar z}(\bar{z}^{\dot\alpha}\bar{z}^{\dot\beta}). \tag{B.9}$$

Then, the operator of squared spin $C^{\{r/2,s/2\}} = \mathbb{W}^a\mathbb{W}_a$, $\mathbb{W}^a = \frac{1}{2}\varepsilon^{abcd}\mathbb{P}_b\mathbb{J}_{cd}$ being the Pauli—Lubanski vector, has the form

$$C^{\{r/2,s/2\}} = -(\mathbb{P},\xi)^2\partial_z\partial_{\bar z} + r(\mathbb{P},\xi)(\mathbb{P},\partial_z\xi)\partial_{\bar z} + s(\mathbb{P},\xi)(\mathbb{P},\partial_{\bar z}\xi)\partial_z -$$
$$-rs(\mathbb{P},\partial_z\xi)(\mathbb{P},\partial_{\bar z}\xi) - \mathbb{P}^2\left\{\left(\frac{r-s}{2}\right)^2 + \frac{r+s}{2}\right\}. \tag{B.10}$$

Let us restrict this representation to the subspace of massive fields satisfying the Klein—Gordon equation $\mathbb{P}^2 + m^2\mathbb{1} = 0$ and introduce the Fourier transform. In result, the spin operator (B.10) can be expressed in terms of the covariant derivatives constructed on the base of the metric (5). In particular, for the massive tensor fields on $\mathcal{M}^6$ of type $\{n/2,0\} \equiv [n/2]$ we get

$$C^{[n/2]} \equiv C^{\{n/2,0\}} = \Delta^{[n/2]},$$
$$\Delta^{[n/2]} = -2g^{z\bar z}\nabla_z\nabla_{\bar z} + \frac{n}{2}\left(\frac{n}{2}+1\right)R. \tag{B.11}$$

As is seen, the operator of squared spin is determined by special spherical Laplacian.

Appendix C. Relativistic harmonic analysis on S^2

In this appendix we shall prove a lot of important assertions concerning spectra of the Laplacians (B.11) and generalized Fourier decompositions for tensor fields on S^2. We begin with formulating the basic statements.

I. Spectrum of the Laplacian $\Delta^{[n/2]}$, where $n = 0, 1, 2, \ldots$, is given by the following eigenvalues

$$s(s+1)R, \qquad s = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \ldots; \tag{C.1}$$

dimension of the eigenspace corresponding to an eigenvalue $s(s+1)R$ is equal to $(2s+1)$.

II. A smooth tensor field $\Phi^{[n/2]}(z,\bar{z})$ on S^2 of type $[n/2] \equiv \{n/2, 0\}, n \geq 0$, can be represented in the form

$$\Phi^{[n/2]}(z,\bar{z}) = \sum_{k=0}^{\infty} F_{\alpha_1 \ldots \alpha_{n+k} \dot{\alpha}_1 \ldots \dot{\alpha}_k} \frac{z^{\alpha_1} \ldots z^{\alpha_{n+k}} \bar{z}^{\dot{\alpha}_1} \ldots \bar{z}^{\dot{\alpha}_k}}{(p,\xi)^k}. \tag{C.2}$$

Here the expansion coefficients, being Lorentz tensors, are determined uniquely from the two requirements:

a) $F_{\alpha_1 \ldots \alpha_{n+k} \dot{\alpha}_1 \ldots \dot{\alpha}_k}$ is a tensor of type $((n+k)/2, k/2)$, i.e.

$$F_{\alpha_1 \ldots \alpha_{n+k} \dot{\alpha}_1 \ldots \dot{\alpha}_k} = F_{(\alpha_1 \ldots \alpha_{n+k})(\dot{\alpha}_1 \ldots \dot{\alpha}_k)}; \tag{C.3}$$

b) for $k \neq 0$, $F_{\alpha_{n+k} \dot{\alpha}(k)}$ is p-transversal,

$$p^{\beta\dot{\beta}} F_{\beta\alpha(n+k-1)\dot{\beta}\dot{\alpha}(k-1)} = 0. \tag{C.4}$$

III. Under the fulfilment of Eqs. (C.3) and (C.4), the k-th term in the expansion (C.2) is an eigenvector for $\Delta^{[n/2]}$ with the eigenvalue $s(s+1)R$, where $s = n/2 + k$.

To prove the assertions formulated, we associate with each non-negative integer n the Hilbert space $\mathcal{H}^{[n/2]}$ of squared integrable tensor fields on S^2 of type $[n/2]$ with respect to the inner product

$$< \Phi_1^{[n/2]} | \Phi_2^{[n/2]} > \equiv < \Phi_1 | \Phi_2 >_{[n/2]} = \int \mathrm{d}z \mathrm{d}\bar{z} (g_{z\bar{z}})^{n/2+1} \overline{\Phi_1(z,\bar{z})} \Phi_2(z,\bar{z}), \tag{C.5}$$

Φ_1 and Φ_2 being tensor fields on S^2 of type $[n/2]$. One immediately gets the following identities:

$$< \nabla_z \Phi^{[n/2+1]} | \Phi^{[n/2]} > = - < \Phi^{[n/2+1]} | \nabla^z \Phi^{[n/2]} >; \tag{C.6.a}$$

$$< \nabla^z \Phi^{[n/2]} | \Phi^{[n/2+1]} > = - < \Phi^{[n/2]} | \nabla_z \Phi^{[n/2+1]} >; \tag{C.6.b}$$

$$< \Phi | \Delta^{[n/2]} | \Phi >_{[n/2]} = 2 \| \nabla^z \Phi \|^2_{[n/2+1]} + \frac{n}{2}\left(\frac{n}{2}+1\right) \|\Phi\|^2_{[n/2]}, \tag{C.6.c}$$

where $\nabla^z \equiv g^{z\bar{z}} \nabla_{\bar{z}}$. The final identity shows that $\Delta^{[n/2]}$ is a positive operator on $\mathcal{H}^{[n/2]}$,

$$< \Phi | \Delta^{[n/2]} | \Phi >_{[n/2]} \geq \frac{n}{2}\left(\frac{n}{2}+1\right) \|\Phi\|^2_{[n/2]},$$

and the equality here proves to take place only on the subspace of holomorphic fields

$$\mathcal{H}_0^{[n/2]} = \left\{ \Phi^{[n/2]} \in \mathcal{H}^{[n/2]}, \quad \nabla^z \Phi^{[n/2]} = g^{z\bar{z}} \partial_{\bar{z}} \Phi^{[n/2]} = 0 \right\}. \tag{C.7}$$

Explicit structure of the states from $\mathcal{H}_0^{[n/2]}$ is given by Eq. (B.7), and each such field is seen to be an eigenstate for $\Delta^{[n/2]}$ with the eigenvalue $\frac{n}{2}\left(\frac{n}{2}+1\right)R$. Obviously, this is the eigenvalue $\frac{n}{2}\left(\frac{n}{2}+1\right)R$ realizing minimum in the spectrum of $\Delta^{[n/2]}$. The rest eigenvalues of $\Delta^{[n/2]}$ can be readily restored by taking into account two simple identities

$$\begin{gathered} \nabla^z \Delta^{[n/2]} = \Delta^{[n/2+1]} \nabla^z, \\ \nabla_z \Delta^{[n/2+1]} = \Delta^{[n/2]} \nabla_z \end{gathered} \tag{C.8}$$

and analysing also the following mappings

$$\nabla^z : \ \mathcal{H}^{[n/2]} \to \mathcal{H}^{[n/2+1]}, \qquad \nabla_z : \ \mathcal{H}^{[n/2+1]} \to \mathcal{H}^{[n/2]}. \tag{C.9}$$

One can check that kernel (Ker) and total image (Im) for the mappings (C.9) look like:

$$\text{Ker } \nabla^z|_{\mathcal{H}^{[n/2]}} = \mathcal{H}_0^{[n/2]}, \qquad \text{Im } \nabla^z|_{\mathcal{H}^{[n/2]}} = \mathcal{H}^{[n/2+1]}; \tag{C.10.a}$$

$$\text{Ker } \nabla_z|_{\mathcal{H}^{[n/2+1]}} = 0, \qquad \text{Im } \nabla_z|_{\mathcal{H}^{[n/2+1]}} = \mathcal{H}^{[n/2]} \backslash \mathcal{H}_0^{[n/2]}. \tag{C.10.b}$$

For example, let $\Phi_\perp^{[n/2+1]}$ be a field orthogonal to the subspace $\nabla^z \mathcal{H}^{[n/2]}$ in $\mathcal{H}^{[n/2+1]}$. Then we have

$$0 = \ < \nabla^z \Phi^{[n/2]} | \Phi_\perp^{[n/2+1]} > \ = \ - < \Phi^{[n/2]} | \nabla_z \Phi_\perp^{[n/2+1]} >,$$

for any $\Phi^{[n/2]} \in \mathcal{H}^{[n/2]}$, hence $\nabla_z \Phi_\perp^{[n/2+1]} = 0$, and therefore $\Phi_\perp^{[n/2+1]} = 0$. Now, it is easy to deduce the assertion I from the relations (C.8) and (C.10).

To check the assertions II and III, one is to account the explicit structure of states from $\mathcal{H}^{[n/2]}$ and to calculate the action of ∇_z and ∇^z on separate terms in the expansion (C.2). Making use of the identities

$$\begin{gathered} z^\alpha \partial_z z^\beta - z^\beta \partial_z z^\alpha = \epsilon^{\alpha\beta}, \\ \nabla_z z^\alpha = - \frac{p^\alpha{}_{\dot\beta} \bar{z}^{\dot\beta}}{(p,\xi)}, \qquad \nabla_z \bar{z}^{\dot\alpha} = \nabla_z \frac{1}{(p,\xi)} = 0, \end{gathered} \tag{C.11}$$

one gets

$$\nabla^z F_{\alpha_1 \ldots \alpha_{n+k} \dot\alpha_1 \ldots \dot\alpha_k} \frac{z^{\alpha_1} \ldots z^{\alpha_{n+k}} \bar{z}^{\dot\alpha_1} \ldots \bar{z}^{\dot\alpha_k}}{(p,\xi)^k} =$$

$$= F_{\alpha_1\ldots\alpha_{n+k+1}\dot\alpha_1\ldots\dot\alpha_{k-1}} \frac{z^{\alpha_1}\ldots z^{\alpha_{n+k+1}}\bar z^{\dot\alpha_1}\ldots\bar z^{\dot\alpha_{k-1}}}{(p,\xi)^{k-1}}. \tag{C.12.a}$$

$$F_{\alpha_1\ldots\alpha_{n+k+1}\dot\alpha_1\ldots\dot\alpha_{k-1}} \equiv -\frac{k}{2} p_{\alpha_{n+k+1}}{}^{\dot\beta} F_{\alpha_1\ldots\alpha_{n+k}\dot\alpha_1\ldots\dot\alpha_k\dot\beta};$$

$$\nabla_z F_{\alpha_1\ldots\alpha_{n+k+1}\dot\alpha_1\ldots\dot\alpha_{k-1}} \frac{z^{\alpha_1}\ldots z^{\alpha_{n+k+1}}\bar z^{\dot\alpha_1}\ldots\bar z^{\dot\alpha_{k-1}}}{(p,\xi)^{k-1}} =$$

$$F_{\alpha_1\ldots\alpha_{n+k}\dot\alpha_1\ldots\dot\alpha_k} \frac{z^{\alpha_1}\ldots z^{\alpha_{n+k}}\bar z^{\dot\alpha_1}\ldots\bar z^{\dot\alpha_k}}{(p,\xi)^{k}}, \tag{C.12.b}$$

$$F_{\alpha_1\ldots\alpha_{n+k}\dot\alpha_1\ldots\dot\alpha_k} \equiv -(n+k+1) p^{\beta}{}_{\dot\alpha_k} F_{\beta\alpha_1\ldots\alpha_{n+k}\dot\alpha_1\ldots\dot\alpha_{k-1}}.$$

As is clear from the above discussion, Eq. (C.2) constitutes the decomposition of a tensor field $\Phi^{[n/2]}$ with respect to the complete set of eigenfunctions associated with the elliptic operator $\Delta^{[n/2]}$ (B.11). Since this Laplacian is specified by a time-like 4-vector p^a, and the coefficients $F_{\alpha(n+k)\dot\alpha(k)}$ in (C.2) are Lorentz tensors, the above decomposition can be called an expansion over relativistic harmonics. Obviously, this expansion appears to be most adapted for making explicitly Lorentz covariant calculations. It is instructive to establish relationship between relativistic and ordinary spherical harmonics. Let us consider, for instance, a scalar field $\Phi(z,\bar z)$ on S^2, for which Eq. (C.2) reads

$$\Phi(z,\bar z) = \sum_{s=0}^{\infty} \Phi_s(z,\bar z), \tag{C.13}$$

where the s-th term

$$\Phi_s(z,\bar z) = F_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_s} \frac{z^{\alpha_1}\ldots z^{\alpha_s}\bar z^{\dot\alpha_1}\ldots\bar z^{\dot\alpha_s}}{(p,\xi)^s} \tag{C.14}$$

satisfies the equation

$$\Delta^{[0]}\Phi_s = s(s+1)R\Phi_s, \tag{C.15}$$

what implies the fulfilment of Eqs. (C.3), (C.4) for the coefficients $F_{\alpha(s)\dot\alpha(s)}$. We choose a coordinate system where the metric (5) is determined by 4–vector $p^a = (\sqrt{R},0,0,0)$ and, hence, proportional to the standard metric on S^2,

$$\mathrm{d}s^2 = R\frac{4\mathrm{d}z\mathrm{d}\bar z}{(1+z\bar z)^2}. \tag{C.16}$$

Let us also replace the variables $z,\bar z$ by standard spherical angles θ,φ. Their connection reads

$$\cos\ \theta = \frac{1-z\bar z}{1+z\bar z}, \qquad \mathrm{e}^{2\mathrm{i}\varphi} = z/\bar z, \tag{C.17}$$

and $\Delta^{[0]}$ takes the form

$$\Delta^{[0]} = -R\Delta, \qquad \Delta = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\ \theta}\frac{\partial}{\partial\theta}\left(\sin\ \theta\frac{\partial}{\partial\theta}\right).$$

As is well known, the general solution of the equation $\Delta\Phi_s = -s(s+1)\Phi_s$ can be written as follows:

$$\Phi_s(\theta, \varphi) = \sum_{m=-s}^{s} C_m \mathrm{e}^{\mathrm{i}m\varphi} P_s^{|m|}(\cos\ \theta), \tag{C.18}$$

with

$$P_s^{|m|}(\cos\ \theta) = \frac{1}{2^s s!}\sin^m\theta \frac{\mathrm{d}^{s+m}}{(\mathrm{d}\cos\ \theta)^{s+m}}(\cos^2\theta - 1)^s$$

being adjoint Legendre polinomials. Expressing θ, φ in (C.18) via $z, \bar{z}$, with the use of (C.17), we arrive at

$$\Phi_s(z, \bar{z}) = \Phi_s(\theta(z, \bar{z}), \varphi(z, \bar{z})) = \frac{\pi^s(z, \bar{z})}{(1 + z\bar{z})^s},$$

where π^s is a polinomial of z and $\bar{z}$ of general degree $2s$. The final expression is in agreement with Eq. (C.14).

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