Supplement to:

A geometric perspective on counting non-negative integer solutions and combinatorial identities

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M.J. Haines[a], S.R. Huddy[b], and M.A. Jones[c]

[a]*Augsburg College, Minneapolis, MN*; [b]*State University of New York at New Paltz, New Paltz, NY*; [c]*Mathematical Reviews, Ann Arbor, MI*

Theorem 5.1 is the result of counting the number of nonnegative integer solutions to $x + y + z = n$ for $x < i$, $y < j$, and $z < k$ in two different ways.

Theorem 5.1: For nonnegative integers i, j, k, and n such that $\max\{i, j, k\} \leq n$,

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1}$$

As a challenge, see if you can construct the figures to prove Theorem 5.1 for the following cases:

1) $k \leq n - i - j$
2) $0 < n - i - j < k$
   a) $k < \min\{n - i, n - j\}$
   b) $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$
   c) $\max\{n - i, n - j\} \leq k$
3) $n - i - j \leq 0$
   a) $k < \min\{n - i, n - j\}$
   b) $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$
   c) $\max\{n - i, n - j\} \leq k$.

To aid you in creating your figures we provide n, i, j, and k for each case. For all cases we use $n = 20$. The parameters i, j, and k can be written as (i, j, k), and we use the following parameters in our figures: Case (1): (9,5,2); Case (2a): (9,6,9); Case (2b): (9,5,13); Case (2c): (9,5,18); Case (3a): (16,10,2); Case (3b): (16,10,7); and Case (3c): (16,10,14). As a reminder, Case (1) is a degenerate case because there are no solutions satisfying the constraints.

Proof. Case (1): See Figure 1.

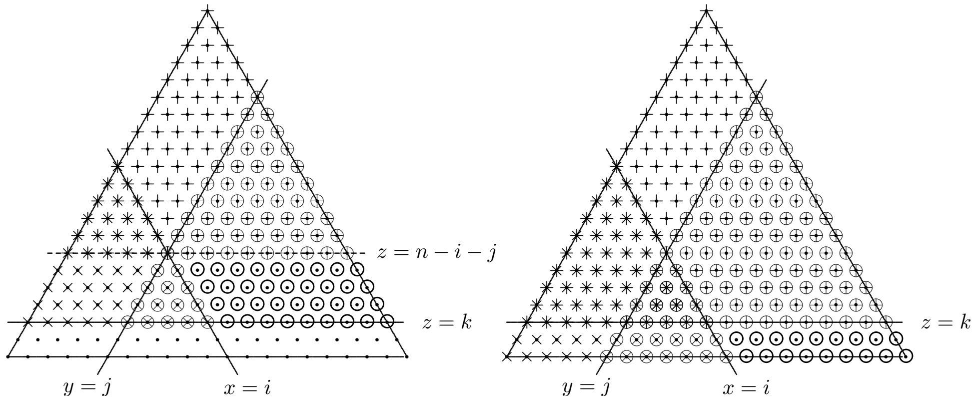

Figure 1. Case (1): $k \leq n-i-j$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1}$$

$$\begin{aligned} &|+\cup\oplus\cup*\cup\circledast| - |*\cup\circledast| - |\oplus\cup\circledast| - |all| \\ &\quad + |\times\cup\otimes\cup*\cup\circledast| + |\odot\cup\otimes\cup\oplus\cup\circledast| - |\otimes\cup\circledast| + 1 \\ &\qquad = |all| - |\times\cup\otimes\cup*\cup\circledast| - |\odot\cup\otimes\cup\oplus\cup\circledast| \\ &\qquad\quad - |+\cup\oplus\cup*\cup\circledast| + |\otimes\cup\circledast| + |*\cup\circledast| + |\oplus\cup\circledast| - |\circledast| \end{aligned}$$

Case (2a): See Figure 2.

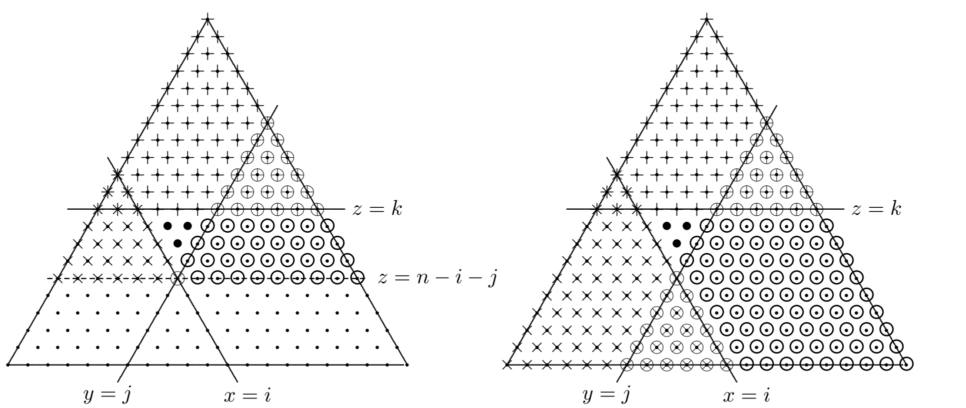

Figure 2. Case (2a): $0 < n-i-j < k$ with $k < \min\{n-i, n-j\}$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$\begin{aligned} &|\bullet\cup\odot\cup\times\cup\otimes\cup+\cup\oplus\cup*| - |\times\cup\otimes\cup*| \\ &\quad - |\odot\cup\otimes\cup\oplus| - |+\cup\oplus\cup*| + |*| + |\oplus| + 1 \\ &\qquad = |all| - |\times\cup\otimes\cup*| - |\odot\cup\otimes\cup\oplus| - |+\cup\oplus\cup*| + |\otimes| + |*| + |\oplus| \end{aligned}$$

Case (2b): See Figure 3.

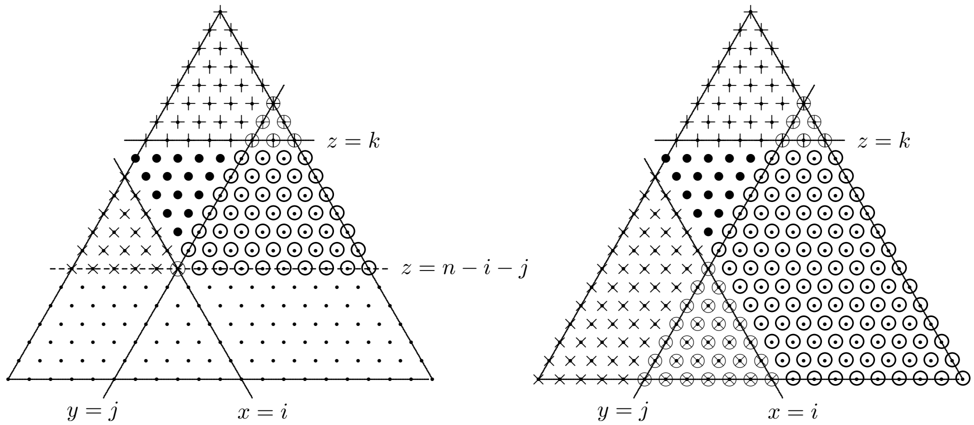

Figure 3. Case (2b): $0 < n - i - j < k$ with $\min\{n - i, n - j\} \le k < \max\{n - i, n - j\}$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$|\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| + |\oplus| + 1$$
$$= |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| + |\otimes| + |\oplus|$$

Case (2c): See Figure 4.

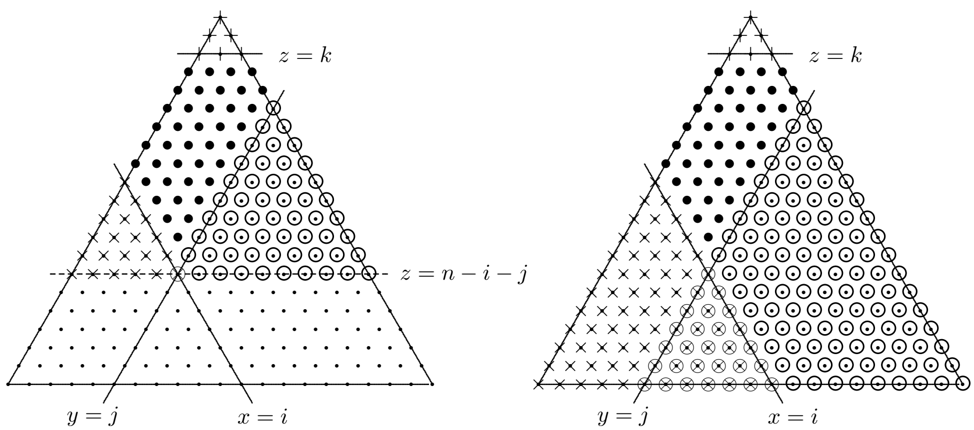

Figure 4. Case (2c): $0 < n - i - j < k$ with $\max\{n - i, n - j\} \le k$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$|\bullet \cup \odot \cup \times \cup \otimes \cup +| - |\odot \cup \otimes| - |\times \cup \otimes| - |+| + 1$$
$$= |\bullet \cup \odot \cup \times \cup \otimes \cup +| - |\times \cup \otimes| - |\odot \cup \otimes| - |+| - |\otimes|$$

Case (3a): See Figure 5.

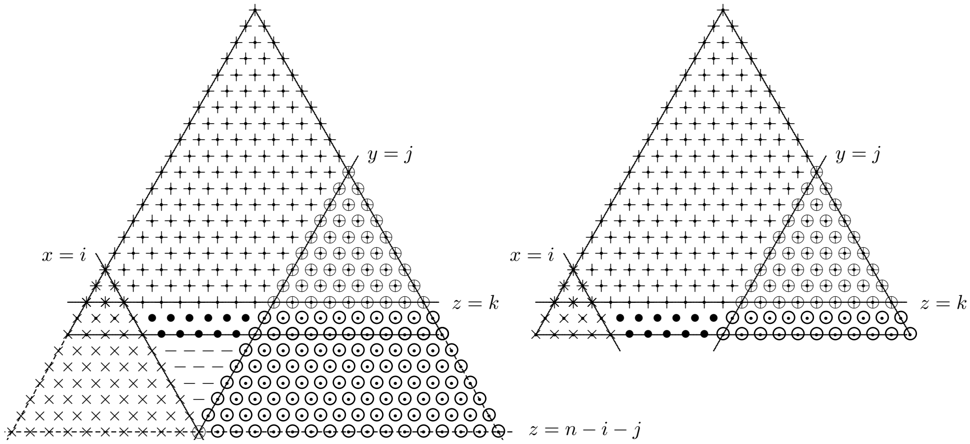

Figure 5. Case (3a): $n - i - j \leq 0$ with $k < \min\{n-i, n-j\}$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$\begin{aligned}
&|\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus \cup * \cup -| - |\times \cup \otimes \cup *| \\
&\quad -|\odot \cup \otimes \cup \oplus| - |+ \cup \oplus \cup *| - |-| + |*| + |\oplus| + 1 \\
&\qquad = |\bullet \cup \odot \cup \times \cup + \cup \oplus \cup *| - |\times \cup *| - |\odot \cup \oplus| - |+ \cup \oplus \cup *| + |*| + |\oplus|
\end{aligned}$$

Case (3b): See Figure 6.

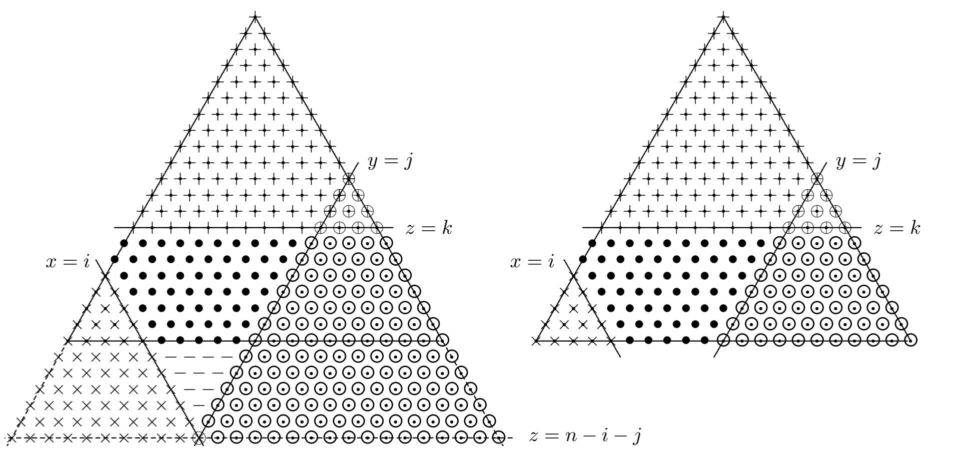

Figure 6. Case (3b): $n - i - j \leq 0$ with $\min\{n-i, n-j\} \leq k < \max\{n-i, n-j\}$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$\begin{aligned}
&|\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus \cup -| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| - |-| + |\oplus| + 1 \\
&= |\bullet \cup \odot \cup \times \cup + \cup \oplus| - |\odot \cup \oplus| - |\times| - |+ \cup \oplus| + |\oplus|
\end{aligned}$$

Case (3c): See Figure 7.

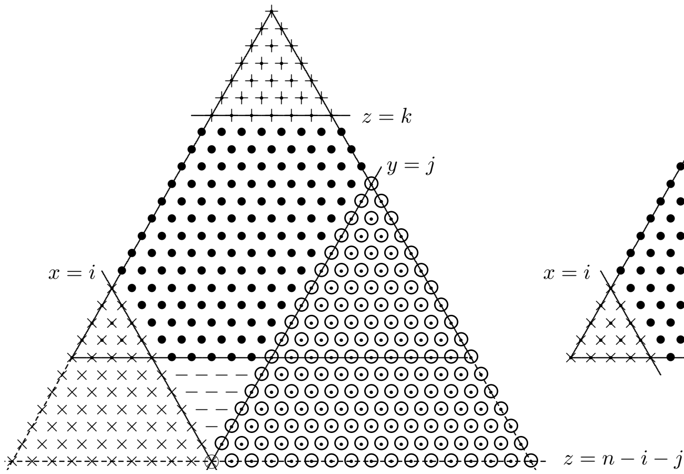

Figure 7. Case (3c): $n - i - j \le 0$ with $\max\{n-i, n-j\} \le k$.

$$T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1$$
$$= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}$$

$$|\bullet \cup \odot \cup \times \cup \otimes \cup + \cup -| - |\times \cup \otimes| - |\odot \cup \otimes| - |+| - |-| + 1$$
$$= |\bullet \cup \odot \cup \times \cup +| - |\times| - |\odot| - |+|$$

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